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Solutions of the Diophantine Equation $x^2 + py^2 = z^2$

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Abstract

This paper is to identify the Diophantine equation $x^2 + py^2 = z^2$ where p is a prime number and x, y and z are integers satisfies; case 1: $p = 2$, has no integer solution if y is odd, and have integer solutions (x, y, z) is $\left(\pm(2\alpha - \beta), \pm 2\sqrt{\alpha\beta}, \pm(2\alpha + \beta)\right)$ where $\alpha\beta$ is a square number if y is even, case 2: $p \neq 2$, have integer solutions (x, y, z) is $\left(\pm\frac{p\alpha - \beta}{2}, \pm\sqrt{\alpha\beta}, \pm\frac{p\alpha + \beta}{2}\right)$ where $\alpha\beta$ is an odd square number if y is odd, and $\left(\pm(p\alpha - \beta), \pm 2\sqrt{\alpha\beta}, \pm(p\alpha + \beta)\right)$ where $\alpha\beta$ is a square number if y is even.

Keywords: Diophantine equation; Congruence; Integer solutions

1. Introduction

In 1995, Wiles [1] showed that the Diophantine equation $x^n + y^n = z^n, xyz \neq 0$ has not had integer solutions when $n \geq 3$. In 1999, Bruin [2] studied the Diophantine equation $x^2 \pm y^4 = \pm z^6$ and $x^2 + y^8 = z^3$. In 2004, Bennett [3] found the solution of the Diophantine equation $x^{2n} + y^{2n} = z^5$. In 2014, Abdelalim and Dyani [4] searched the solution of the Diophantine equation $x^2 + 3y^2 = z^2$. In 2015, Abdelalim and Diany [5] characterized the solutions of the Diophantine equation $x^2 + y^2 = 2z^2$.

Thus, this paper aims to study the Diophantine equation $x^2 + py^2 = z^2$ where p is a prime number and x, y and z are integers.

2. Materials and Methods

Lemma 1: The Diophantine equation $x^2+2y^2=z^2$ has no integer solution where x, y and z are integers with y is odd.

Proof:

Suppose that (x,y,z) is a solution of the Diophantine equation $x^2+2y^2=z^2$ with y is odd.

Since $y^2\equiv 1(\mathrm{mod}\,4)$ then $2y^2\equiv 2(\mathrm{mod}\,4)$, these consider into 2 cases as follow:

Case 1: If x is odd then z is odd.

Thus, $x^2\equiv 1(\mathrm{mod}\,4)$ then $x^2+2y^2\equiv 3(\mathrm{mod}\,4)$. This is a contradiction with $z^2\equiv 1(\mathrm{mod}\,4)$.

Case 2: If x is even then z is even.

Thus, $x^2\equiv 0(\mathrm{mod}\,4)$ then $x^2+2y^2\equiv 2(\mathrm{mod}\,4)$. This is a contradiction with $z^2\equiv 0(\mathrm{mod}\,4)$.

Therefore, by case 1 and 2, the Diophantine equation $x^2+2y^2=z^2$ has no integer solution where x, y and z are integers with y is odd.

Lemma 2: The Diophantine equation $x^2+2y^2=z^2$ has the solutions in the form

$$(x,y,z)=\left(\pm(2\alpha-\beta),\pm 2\sqrt{\alpha\beta},\pm(2\alpha+\beta)\right) \tag{1}$$

where x, y and z are integers with y is even, $\alpha\beta$ is a square number.

Proof:

Let (x,y,z) be a solution of the Diophantine equation $x^2+2y^2=z^2$ with y is even.

So, we have

$$\begin{aligned}x^2+2y^2&=z^2\\2y^2&=z^2-x^2\\2y^2&=\left(|z|-|x|\right)\left(|z|+|x|\right).\end{aligned}$$

So that, $2\,\big|\left(|z|-|x|\right)\left(|z|+|x|\right)$ hence, $2\,\big|\left(|z|-|x|\right)$ or $2\,\big|\left(|z|+|x|\right)$, we have x, z are even or x, z are odd, then $|z|-|x|$ and $|z|+|x|$ are even, we have $\frac{|z|-|x|}{2},\frac{|z|+|x|}{2}$ are integers. It follows that,

$$\begin{aligned}\frac{2y^2}{4}&=\frac{\left(|z|-|x|\right)\left(|z|+|x|\right)}{4}\\2\left(\frac{y}{2}\right)^2&=\frac{|z|-|x|}{2}\frac{|z|+|x|}{2}.\end{aligned}$$

Hence, $2\,\Big|\frac{|z|-|x|}{2}\frac{|z|+|x|}{2}$ then $2\,\Big|\frac{|z|-|x|}{2}$ or $2\,\Big|\frac{|z|+|x|}{2}$, these consider into 2 cases as follow:

Case 1: If $2\left|\dfrac{|z|-|x|}{2}\right.$ so that, there exists an integer α such that $2\alpha=\dfrac{|z|-|x|}{2}$ and let $\beta=\dfrac{|z|+|x|}{2}$. It follows that,

$$2\left(\frac{y}{2}\right)^2=(2\alpha)\beta$$

$$\left(\frac{y}{2}\right)^2=\alpha\beta$$

$$\frac{y}{2}=\pm\sqrt{\alpha\beta}\ ,\quad \text{where } \alpha\beta \text{ is a square number;}$$

$$y=\pm 2\sqrt{\alpha\beta}$$

Since $2\alpha=\dfrac{|z|-|x|}{2}$ and $\beta=\dfrac{|z|+|x|}{2}$ then $|x|=-(2\alpha-\beta)$ and $|z|=2\alpha+\beta$ that is, $x=\pm(2\alpha-\beta)$ and $z=\pm(2\alpha+\beta)$.

Case 2: If $2\left|\dfrac{|z|+|x|}{2}\right.$ so that, there exists an integer α such that $2\alpha=\dfrac{|z|+|x|}{2}$ and let $\beta=\dfrac{|z|-|x|}{2}$. It follows that,

$$2\left(\frac{y}{2}\right)^2=\beta(2\alpha)$$

$$\left(\frac{y}{2}\right)^2=\alpha\beta$$

$$\frac{y}{2}=\pm\sqrt{\alpha\beta}\ , \text{ where } \alpha\beta \text{ is a square number;}$$

$$y=\pm 2\sqrt{\alpha\beta}\ .$$

Since $2\alpha=\dfrac{|z|+|x|}{2}$ and $\beta=\dfrac{|z|-|x|}{2}$ then $|x|=2\alpha-\beta$ and $|z|=2\alpha+\beta$. That are $x=\pm(2\alpha-\beta)$ and $z=\pm(2\alpha+\beta)$.

Therefore, by case 1 and 2, the solutions of the Diophantine equation are

$$(x,y,z)=\left(\pm(2\alpha-\beta),\pm 2\sqrt{\alpha\beta},\pm(2\alpha+\beta)\right)$$

where $\alpha\beta$ is a square number.

3. Results and Discussion

Theorem 3: The Diophantine equation $x^2 + py^2 = z^2$ has the solutions in the form

$$\left(x,y,z\right)=\left(\pm\frac{p\alpha-\beta}{2},\pm\sqrt{\alpha\beta},\pm\frac{p\alpha+\beta}{2}\right) \qquad (2)$$

where p is an odd prime number, x, y, z are integers with y is odd and $\alpha\beta$ is an odd square number.

Proof:

Let (x, y, z) be a solution of the Diophantine equation $x^2 + py^2 = z^2$ with y is odd.

We have,

$$\begin{aligned} x^2 + py^2 &= z^2 \\ py^2 &= z^2 - x^2 \\ py^2 &= (|z|-|x|)(|z|+|x|). \end{aligned}$$

Hence, $p \mid (|z|-|x|)(|z|+|x|)$ then $p \mid (|z|-|x|)$ or $p \mid (|z|+|x|)$, these consider into 2 cases as follow:

Case 1: If $p \mid (|z|-|x|)$ so that, there exists an integer α such that $p\alpha = |z|-|x|$ and let $\beta = |z|+|x|$. It follows that,

$$\begin{aligned} py^2 &= \left(p\alpha\right)\beta \\ y^2 &= \alpha\beta \\ y &= \pm\sqrt{\alpha\beta}, \text{ where } \alpha\beta \text{ is an odd square number.} \end{aligned}$$

Since $p\alpha = |z|-|x|$ and $\beta = |z|+|x|$ then $|x| = -\frac{p\alpha-\beta}{2}$ and $|z| = \frac{p\alpha+\beta}{2}$.

That are $x = \pm\frac{p\alpha-\beta}{2}$ and $z = \pm\frac{p\alpha+\beta}{2}$.

Case 2: If $p \mid (|z|+|x|)$ so that, there exists an integer α such that $p\alpha = |z|+|x|$ and let $\beta = |z|-|x|$. It follows that,

$$\begin{aligned} py^2 &= \beta\left(p\alpha\right) \\ y^2 &= \alpha\beta \\ y &= \pm\sqrt{\alpha\beta}, \text{ where } \alpha\beta \text{ is an odd square number.} \end{aligned}$$

Since $p\alpha = |z|+|x|$ and $\beta = |z|-|x|$ then $|x| = \frac{p\alpha-\beta}{2}$ and $|z| = \frac{p\alpha+\beta}{2}$.

That are $x = \pm\frac{p\alpha-\beta}{2}$ and $z = \pm\frac{p\alpha+\beta}{2}$.

Therefore, by case 1 and 2, the solutions of this equation are in the form

$$\left(x,y,z\right)=\left(\pm\frac{p\alpha-\beta}{2},\pm\sqrt{\alpha\beta},\pm\frac{p\alpha+\beta}{2}\right)$$

where $\alpha\beta$ is an odd square number.

Theorem 4: The Diophantine equation $x^2 + py^2 = z^2$ has the solutions in the form

$$(x,y,z) = \left(\pm(p\alpha - \beta), \pm 2\sqrt{\alpha\beta}, \pm(p\alpha + \beta)\right) \quad (3)$$

where p is a prime number, x, y, z are integers with y is even and $\alpha\beta$ is a square number.

Proof:

Let (x, y, z) be a solution of the Diophantine equation $x^2 + py^2 = z^2$ with y is even.

We have,

$$x^2 + py^2 = z^2$$
$$py^2 = z^2 - x^2$$
$$py^2 = (|z| - |x|)(|z| + |x|).$$

Since y is even then y^2 is divisible by 4 so, $4\,|\,(|z| - |x|)(|z| + |x|)$ hence, $|z| - |x|$ and $|z| + |x|$ are even, we have

$$\frac{py^2}{4} = \frac{(|z| - |x|)(|z| + |x|)}{4}$$
$$p\left(\frac{y}{2}\right)^2 = \frac{|z| - |x|}{2}\frac{|z| + |x|}{2}$$

Hence, $p\,\Big|\,\frac{|z| - |x|}{2}\frac{|z| + |x|}{2}$ then $p\,\Big|\,\frac{|z| - |x|}{2}$ or $p\,\Big|\,\frac{|z| + |x|}{2}$, these consider into 2 cases as follow:

Case 1: If $p\,\Big|\,\frac{|z| - |x|}{2}$ so that, there exists an integer α such that $p\alpha = \frac{|z| - |x|}{2}$ and let $\beta = \frac{|z| + |x|}{2}$. It follows that,

$$p\left(\frac{y}{2}\right)^2 = (p\alpha)\beta$$
$$\left(\frac{y}{2}\right)^2 = \alpha\beta$$
$$\frac{y}{2} = \pm\sqrt{\alpha\beta}\text{, where } \alpha\beta \text{ is a square number;}$$
$$y = \pm 2\sqrt{\alpha\beta}$$

Since $p\alpha = \frac{|z| - |x|}{2}$ and $\beta = \frac{|z| + |x|}{2}$ then $|x| = -(p\alpha - \beta)$ and $|z| = p\alpha + \beta$. That are $x = \pm(p\alpha - \beta)$ and $z = \pm(p\alpha + \beta)$.

Case 2: If $p\,\Big|\,\frac{|z| + |x|}{2}$ setting $p\alpha = \frac{|z| + |x|}{2}$ and $\beta = \frac{|z| - |x|}{2}$. It follows that,

$$p\left(\frac{y}{2}\right)^2 = \beta(p\alpha)$$

$$\left(\frac{y}{2}\right)^2 = \alpha\beta$$

$$\frac{y}{2} = \pm\sqrt{\alpha\beta}$$, where $\alpha\beta$ is a square number;

$$y = \pm 2\sqrt{\alpha\beta}$$

Since $p\alpha = \frac{|z|+|x|}{2}$ and $\beta = \frac{|z|-|x|}{2}$ then $|x| = p\alpha - \beta$ and $|z| = p\alpha + \beta$.

That are $x = \pm(p\alpha - \beta)$ and $z = \pm(p\alpha + \beta)$.

Therefore, by case 1 and 2, the Diophantine equation has the solutions in the form

$$(x, y, z) = \left(\pm(p\alpha - \beta), \pm 2\sqrt{\alpha\beta}, \pm(p\alpha + \beta)\right)$$

where $\alpha\beta$ is a square number.

4. Conclusion

It can be seen that this paper shown the solution of the Diophantine equation $x^2 + py^2 = z^2$ has no integer solution if p is an even prime number, y is odd and have integer solutions $(x, y, z) = \left(\pm\frac{p\alpha - \beta}{2}, \pm\sqrt{\alpha\beta}, \pm\frac{p\alpha + \beta}{2}\right)$ where $\alpha\beta$ is an odd square number, p is an odd prime number and x, y and z are integers with y is odd. Also, we have seen that the solution of the Diophantine equation $x^2 + py^2 = z^2$ is $(x, y, z) = \left(\pm(p\alpha - \beta), \pm 2\sqrt{\alpha\beta}, \pm(p\alpha + \beta)\right)$ where $\alpha\beta$ is a square number, p is a prime number and x, y and z are integers with y is even.

In the future, we may extend this work by extending the value of p is a prime number to the value of n is a natural number.

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