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On the diophantine equation $p^x+5^y=z^2$

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Abstract

The aim of this work is to show that the Diophantine equations $p^x + 5^y = z^2$ where *p* is prime number and *p* satisfies; case 1: $p \equiv 1 \pmod 4$ or case 2: $p \equiv 3 \pmod 4$ and $p \equiv 2 \pmod 5$ or case 3: $p \equiv 3 \pmod 4$ and $p \equiv 3 \pmod 5$, has no non-negative integer solution. In addition, the Diophantine equations $p^x + 5^y = z^2$ have some non-negative integer solutions (*p, x, y, z*) if $p \equiv 3 \pmod 4$ and $p \equiv 1 \pmod 5$, such as (11, 1, 2, 6), (11, 1, 1, 4), (31, 1, 1, 6), (71, 1, 3, 4), (131, 1, 3, 16) and (191, 1, 1, 14), etc. Moreover, if $p \equiv 3 \pmod 4$ and $p \equiv 4 \pmod 5$, such as (19, 1, 3, 12), (139, 1, 1, 12) and (199, 1, 3, 18).

Keywords: Exponential Diophantine equation, Integer solutions, Congruent

1. Introduction

In 2007, Acu [1] showed that the Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions in non-negative integers. The solutions are (3, 0, 3) and (2, 1, 3). In 2011, Suvarnamani's study [2] showed that the Diophantine equation of type $2^x + p^y = z^2$ where *p* is prime number. He found that there are infinitely many solutions for the equation where *p* = 2. In 2012, Sroysang [3] found that the Diophantine equation $3^x + 5^y = z^2$ has a unique non-negative integer solution. The solutions are (1, 0, 2). In 2013, Rabago [4] revealed that the Diophantine equations $p^y + 1 = z^2$ has no positive integer solution for prime *p* > 3.

This research aims to study the Diophantine equation of form $p^x + 5^y = z^2$ where p is prime number and *x, y* and *z* are non-negative integers.

2. Materials and Methods

Preliminaries

Proposition 2.1 [5] (the Catalan's conjecture) (3, 2, 2, 3) is a unique solution (*a, b, x, y*) for the Diophantine equation $a^x - b^y = 1$ where *a, b, x* and *y* are integers with min{*a, b, x, y*} > 1.

Lemma 2.2 [4] The Diophantine equations $p^x + 1 = z^2$ has no positive integer solution for prime *p* > 3.

Lemma 2.3 [1] The Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions in non-negative integers which are (3, 0, 3) and (2, 1, 3).

3. Results and Discussion

Throughout this paper, *x, y* and *z* are non-negative integers, and *p* is a prime number. We begin this section by the following Theorem.

Theorem 3.1 The Diophantine equations $p^x + 5^y = z^2$, where $p \equiv 1 (\bmod 4)$, has no non-negative integer solution.

Proof. Let *p* be prime number and *x, y* and *z* be non-negative integer solutions of $p^x + 5^y = z^2$.

Suppose that $p \equiv 1 (\bmod 4)$ then $p^x \equiv 1 (\bmod 4)$.

Case 1 : If *x* or *y* is zero, then by Lemma 2.2 the Diophantine equations $p^x + 5^y = z^2$ has no positive integer solution. On the other hand, the case *z* = 0 is obviously not possible. Hence, these equations have no non-negative integer solution.

Case 2 : Let $x \geq 1$ and $y \geq 1$. Note that *z* is even. Then $z^2 \equiv 0 (\bmod 4)$. Since $5 \equiv 1 (\bmod 4)$, it follows that $p^x \equiv 3 (\bmod 4)$. This is a contradiction with $p \equiv 1 (\bmod 4)$.

Therefore, by case 1 and 2, the Diophantine equations $p^x + 5^y = z^2$ has no non-negative integer solution if $p \equiv 1 (\bmod 4)$.

Theorem 3.2 The Diophantine equations $p^x + 5^y = z^2$, where $p \equiv 2 (\bmod 4)$, has exactly two non-negative integer solutions. The solutions are (3, 0, 3) and (2, 1, 3).

Proof. Suppose that *p* = 2, by Lemma 2.3 the Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions in non-negative integers which are (3, 0, 3) and (2, 1, 3).

Theorem 3.3 The Diophantine equations $p^x + 5^y = z^2$, where $p \equiv 3 (\bmod 4)$ and $p \equiv 2 (\bmod 5)$, has no non-negative integer solution.

Proof. Let *p* be prime number and *x, y*, and *z* be non-negative integer solutions of $p^x + 5^y = z^2$.

Suppose that $p \equiv 3 (\bmod 4)$ and $p \equiv 2 (\bmod 5)$.

Case 1: If *x* or *y* is zero, then by Lemma 2.2 the Diophantine equations $p^x + 5^y = z^2$ has no positive integer solution. On the other hand, the case $z = 0$ is obviously not possible. Hence, these equations have no non-negative integer solution.

Case 2: Let $x \geq 1$ and $y \geq 1$. Note that *z* is even, then $z^2 \equiv 0 (\bmod 4)$. By the mathematical induction, we have $z^2 \equiv 0 (\bmod 5)$ or $z^2 \equiv 1 (\bmod 5)$ or $z^2 \equiv 4 (\bmod 5)$. Since $5 \equiv 1 (\bmod 4)$, it follows that

$p^x \equiv 3(\text{mod}\,4)$. Thus, *x* is odd, then $p^x \equiv 2(\text{mod}\,5)$ or $p^x \equiv 3(\text{mod}\,5)$. Since $5^y \equiv 0(\text{mod}\,5)$, it follows that $z^2 \equiv 2(\text{mod}\,5)$ or $z^2 \equiv 3(\text{mod}\,5)$. This is a contradiction.

Therefore, by case 1 and 2, the Diophantine equations $p^x + 5^y = z^2$ has no non-negative integer solution if $p \equiv 3(\text{mod}\,4)$ and $p \equiv 2(\text{mod}\,5)$.

Theorem 3.4 The Diophantine equations $p^x + 5^y = z^2$ where $p \equiv 3(\text{mod}\,4)$ and $p \equiv 3(\text{mod}\,5)$, has no non-negative integer solution.

Proof. Let *p* be prime number and *x, y* and *z* be non-negative integer solutions of $p^x + 5^y = z^2$

Suppose $p \equiv 3(\text{mod}\,4)$ and $p \equiv 3(\text{mod}\,5)$.

Case 1: If *x* or *y* is zero, then by Lemma 2.2 the Diophantine equations $p^x + 5^y = z^2$ has no positive integer solution. On the other hand, the case $z = 0$ is obviously not possible. Hence, these equations have no non-negative integer solution.

Case 2: Let $x \geq 1$ and $y \geq 1$. Similarly prove to the case 2 in Theorem 3.2 we conclude that the Diophantine equations $p^x + 5^y = z^2$ has no non-negative integer solution if $p \equiv 3(\text{mod}\,4)$ and $p \equiv 3(\text{mod}\,5)$

Therefore, by case 1 and 2, the Diophantine equations $p^x + 5^y = z^2$ has no non-negative integer solution if $p \equiv 3(\text{mod}\,4)$ and $p \equiv 3(\text{mod}\,5)$.

4. Remark

The Diophantine equations $p^x + 5^y = z^2$, where *p* is prime number and *x, y* and *z* are non-negative integers. For *p* < 200, there are some non-negative integer solutions as following:

1) For $p \equiv 3(\text{mod}\,4)$ and $p \equiv 1(\text{mod}\,5)$, the non-negative integer solutions (p, x, y, z) are (11, 1, 2, 6), (11, 1, 1, 4), (31, 1, 1, 6), (71, 1, 3, 4), (131, 1, 3, 16), and (191, 1, 1, 14)

(2) For $p \equiv 3(\text{mod}\,4)$ and $p \equiv 4(\text{mod}\,5)$,the non-negative integer solutions (p, x, y, z) are (19, 1, 3, 12), (139, 1, 1, 12) and (199, 1, 3, 18).

5. References

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