ข้อสังเกตหนึ่งของสมการเชิงฟังก์ชันที่เกี่ยวกับการกรองดิจิทัล

A NOTE ON A FUNCTIONAL EQUATION RELATED TO DIGITAL FILTERING

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บทคัดย่อ

วิจัยนี้เป็นการวิจัยเพื่อหาสมการเชิงฟังก์ชันที่สมนัยกับสมการเชิงฟังก์ชันต่อไปนี้

$$f(x+s,y+t)+f(x-s,y)+f(x,y-t)=f(x-s,y-t)+f(x+s,y)+f(x,y+t) \qquad \text{(M)}$$

สำหรับทุก $x,y,t,s\in G$ และ $f: G\times G\to\mathbb{C}$ โดยที่ G เป็น 2-divisible abelian group และ $\mathbb{C}$ เป็นเซตของจำนวนเชิงซ้อน ผลการวิจัย พบว่า สามารถหาสมการเชิงฟังก์ชันที่สมนัยได้ 2 สมการต่อไปนี้

$$f(x+2s,y+2t)+f(x-2s,y-t)+f(x-s,y-2t)+f(x,y+t)+f(x+s,y)$$
$$=f(x-2s,y-2t)+f(x+2s,y+t)+f(x+s,y+2t)+f(x,y-t)+f(x-s,y) \qquad \text{(M1)}$$

และ

$$f(x+2s,y)+f(x,y+2t)+f(x-s,y)+f(x,y-t)+f(x-2s,y+t)+f(x+s,y-2t)$$
$$=f(x-2s,y)+f(x,y-2t)+f(x+s,y)+f(x,y+t)+f(x+2s,y-t)+f(x-s,y+2t). \qquad \text{(M2)}$$

คำสำคัญ: สมการเชิงฟังก์ชัน, ตัวดำเนินการการแปลง shift, สมการเชิงฟังก์ชันที่เกี่ยวกับการกรองดิจิทัล, สมการคลื่น

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Abstract

This research was a study to determine the functional equation corresponding to the following functional equation:

$$f(x+s, y+t) + f(x-s, y) + f(x, y-t) = f(x-s, y-t) + f(x+s, y) + f(x, y+t) \qquad \text{(M)}$$

for all $x, y, t, s \in G$, and $f : G \times G \to \mathbb{C}$, where a 2-divisible abelian group and $\mathbb{C}$ were a set of complex numbers. The research results revealed that there were two corresponding functional equations could be obtained as the followings:

$$f(x+2s, y+2t) + f(x-2s, y-t) + f(x-s, y-2t) + f(x, y+t) + f(x+s, y)$$
$$= f(x-2s, y-2t) + f(x+2s, y+t) + f(x+s, y+2t) + f(x, y-t) + f(x-s, y) \qquad \text{(M1)}$$

and

$$f(x+2s, y) + f(x, y+2t) + f(x-s, y) + f(x, y-t) + f(x-2s, y+t) + f(x+s, y-2t)$$
$$= f(x-2s, y) + f(x, y-2t) + f(x+s, y) + f(x, y+t) + f(x+2s, y-t) + f(x-s, y+2t). \qquad \text{(M2)}$$

Keywords: functional equation, translation (shift) operators, functional equation related to digital filtering, wave equation

Introduction

A functional equation is an equation where the unknowns are functions such as the following functional equation

$$f(x+y) = f(x) + f(y), \qquad (1.1)$$

is a well-known Cauchy functional equation, whose additive form, referred to as the additive Cauchy functional equation. In this paper, we study the geometric functional equation which is one of the most important functional equation and has also useful applications in many fields.

Firstly, Aczél et al. (1968) studied the general solution of the functional equation

$$f(x+t,y+t)+f(x+t,y-t)+f(x-t,y+t)+f(x-t,y-t)=4f(x,y), \quad (1.2)$$

where $f:\mathbb{R}^2\to\mathbb{R}$ and x,y,t are real variables. The general solution of (1.2) is given in terms of arbitrary symmetric multi-additive functions of four variables. The few years later, Haruki (1970) (see also Kannappan, 2009), considered the wave equation,

$$f(x+t,y)+f(x-t,y)=f(x,y+t)+f(x,y-t), \quad (1.3)$$

where $f:\mathbb{R}^2\to\mathbb{R}$ and the general solution of (1.3) is given by

$$f(x,y)=\alpha(x+y)+\beta(x-y)+B(x,y)$$

where $\alpha,\beta:\mathbb{R}\to\mathbb{R}$ are arbitrary functions and $B:\mathbb{R}^2\to\mathbb{R}$ is biadditive and skew-symmetric.

Sahoo & Székelyhidi (2001) determined the general complex-valued solution of the functional equation related to digital filtering,

$$f(x+t,y+t)+f(x-t,y)+f(x,y-t)=f(x-t,y-t)+f(x,y+t)+f(x+t,y), \quad \text{(W1)}$$

for all $x,y,t\in G$ and $f:G\times G\to\mathbb{C}$ where G is a 2-divisible abelian group and $\mathbb{C}$ is the set of complex numbers. The general solution is given by

$$f(x,y)=B(x,y)+\phi(x)+\psi(y)+\chi(x-y),$$

for all $x,y\in G$, where $B:G\times G\to\mathbb{C}$ is biaddtive and $\phi,\psi,\chi:G\to\mathbb{C}$ are arbitrary functions. Next, Haruki & Nakagiri (2007) considered the pexider type generalization of the wave equation (1.3), that is the following equation,

$$f_1(x+t,y)+f_2(x-t,y)=f_3(x,y+t)+f_4(x,y-t), \quad (1.4)$$

for all $x,y,t\in G$ and $f_1,f_2,f_3,f_4:G\times G\to\mathbb{C}$ where G is a 2-divisible abelian group and $\mathbb{C}$ the set of complex numbers. Their result is:

Theorem 1.1 (Haruki & Nakagiri, 2007) If $f_1, f_2, f_3, f_4 : G \times G \to \mathbb{C}$ satisfy (1.3) for all $x, y, t \in G$, then there exist

(i) a skew-symmetric biadditive function $A : G \times G \to \mathbb{C}$

(ii) biadditive functions $B_1, B_2 : G \times G \to \mathbb{C}$ and

(iii) arbitrary functions $\alpha, \beta, \phi_1, \phi_2, \psi_1, \psi_2, \chi_1, \chi_2 : G \to \mathbb{C}$ such that

$$\left.\begin{aligned} f_1(x,y) &= A(x,y) + B_1(x+y, x-y) + \alpha(x+y) + \beta(x-y) \\ &\quad + \phi_1(x+y) + \psi_1(x-y) + \chi_1(2y), \\ f_2(x,y) &= A(x,y) - B_1(x+y, x-y) + \alpha(x+y) + \beta(x-y) \\ &\quad - \phi_1(x+y) - \psi_1(x-y) - \chi_1(2y), \\ f_3(x,y) &= A(x,y) + B_2(x+y, 2x) + \alpha(x+y) + \beta(x-y) \\ &\quad + \phi_2(x+y) + \psi_2(2x) + \chi_2(-x+y), \\ f_4(x,y) &= A(x,y) - B_2(x+y, 2x) + \alpha(x+y) + \beta(x-y) \\ &\quad - \phi_2(x+y) - \psi_2(2x) - \chi_2(-x+y). \end{aligned}\right\} \quad (1.5)$$

However, for their work, they considered (1.3) under the assumption that $f : G \times G \to \mathbb{C}$ and they proved that the equation (1.3) is equivalent to each one of the following two equations:

$$\begin{aligned} f(x+t, y+t) + f(x+t, y-t) + f(x-t, y+t) + f(x-t, y-t) \\ = f(x, y+2t) + f(x, y-2t) + 2f(x,y), \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} f(x+t, y+t) + f(x+t, y-t) + f(x-t, y+t) + f(x-t, y-t) \\ = f(x+2t, y) + f(x-2t, y) + 2f(x,y). \end{aligned} \quad (1.7)$$

Then, Naenudorn & Hengkrawit (2013) showed that the functional equation (W1) is equivalent to each one of the following two equations:

$$\begin{aligned} &f(x+2t, y+t) + f(x, y+t) + f(x-2t, y) + f(x+t, y-t) + f(x-t, y-t) \\ &\quad = f(x-2t, y-t) + f(x, y-t) + f(x+2t, y) + f(x-t, y+t) + f(x+t, y+t), \end{aligned} \quad \text{(W2)}$$

and

$$f(x+t, y+2t) + f(x-t, y+t) + f(x-t, y-t) + f(x+t, y) + f(x, y-2t)$$

$$= f(x-t,y-2t)+f(x+t,y-t)+f(x+t,y+t)+f(x-t,y)+f(x,y+2t) \qquad \text{(W3)}$$

for all $x,y,t \in G$ are equivalent to each other under the assumption $f : G \times G \to \mathbb{C}$.

Lastly, Hengkrawit et al. (2016) considered the pexiderized digital filtering functional equation

$$f_1(x+t,y+t)+f_2(x-t,y)+f_3(x,y-t)=f_4(x-t,y-t)+f_5(x,y+t)+f_6(x+t,y).$$

They determined three kinds of solutions, namely, biadditive, symmetric and skew-symmetric solution function, subject to different sets of conditions on the functions involved.

The aim of this paper, by using the same technique of Haruki & Nakagiri (2007), is to show that the functional equation:

$$f(x+s,y+t)+f(x-s,y)+f(x,y-t)=f(x-s,y-t)+f(x+s,y)+f(x,y+t) \qquad \text{(M)}$$

is equivalent to each one of the following two equations:

$$\begin{aligned} &f(x+2s,y+2t)+f(x-2s,y-t)+f(x-s,y-2t)+f(x,y+t)+f(x+s,y) \\ &\quad = f(x-2s,y-2t)+f(x+2s,y+t)+f(x+s,y+2t)+f(x,y-t)+f(x-s,y) \qquad \text{(M1)} \end{aligned}$$

and

$$\begin{aligned} &f(x+2s,y)+f(x,y+2t)+f(x-s,y)+f(x,y-t)+f(x-2s,y+t)+f(x+s,y-2t) \\ &= f(x-2s,y)+f(x,y-2t)+f(x+s,y)+f(x,y+t)+f(x+2s,y-t)+f(x-s,y+2t). \end{aligned} \qquad \text{(M2)}$$

Equivalence of equations (M), (M1) and (M2)

Throughout this paper $(G,+)$ and $\mathbb{C}$ denote a 2-divisible abelian group and the field of all complex numbers, respectively. It is convenient to introduce translation (shift) operators X^t and Y^t for $t \in G$ defined by $X^t f(x,y) = f(x+t,y)$

and $Y^t f(x,y)=f(x,y+t)$ for all $x,y\in G.$ In particular $1=X^0=Y^0$ denote the identity operators. Our main result is:

Theorem 2.1 The functional equations (M), $(\mathrm{M1})$ and $(\mathrm{M2})$ for all $x,y,t,s\in G$ are equivalent to each other under the assumption $f: G\times G\to\mathbb{C}$

Proof We will show that Equation (M) is equivalent to $(\mathrm{M1})$ and Equation (M) is equivalent to $(\mathrm{M2})$. Firstly, we can write (M) in the following operator form

$$\left[X^sY^t+X^{-s}+Y^{-t}\right]f=\left[X^{-s}Y^{-t}+X^s+Y^t\right]f. \tag{2.1}$$

Multiplying (2.1) by X^sY^t, we obtain that

$$\left[X^{2s}Y^{2t}+Y^t+X^s\right]f=\left[1+X^{2s}Y^t+X^sY^{2t}\right]f. \tag{2.2}$$

Again, multiplying (2.1) by $X^{-s}Y^{-t}$, we get

$$\left[1+X^{-2s}Y^{-t}+X^{-s}Y^{-2t}\right]f=\left[X^{-2s}Y^{-2t}+Y^{-t}+X^{-s}\right]f. \tag{2.3}$$

Then, adding (2.2) and (2.3), we get

$$\left[X^{2s}Y^{2t}+X^{-2s}Y^{-t}+X^{-s}Y^{-2t}+Y^t+X^s\right]f=\left[X^{-2s}Y^{-2t}+X^{2s}Y^t+X^sY^{2t}+Y^{-t}+X^{-s}\right]f. \tag{2.4}$$

which is the operator form of $(\mathrm{M1})$. Thus, (M) implies $(\mathrm{M1})$. Conversely, squaring both sides of (2.4), we have

$$\begin{aligned}&\begin{bmatrix}X^{4s}Y^{4t}+2Y^t+2X^s+2X^{2s}Y^{3t}+2X^{3s}Y^{2t}+X^{-4s}Y^{-2t}+2X^{-3s}Y^{-3t}+X^{-2s}Y^{-4t}\\+X^{-2s}+2X^{-s}Y^{-t}+Y^{-2t}\end{bmatrix}f\\&=\begin{bmatrix}X^{-4s}Y^{-4t}+2Y^{-t}+2X^{-s}+2X^{-2s}Y^{-3t}+2X^{-3s}Y^{-2t}+X^{4s}Y^{2t}+2X^{3s}Y^{3t}+X^{2s}Y^{4t}\\+X^{2s}+2X^sY^t+Y^{2t}\end{bmatrix}f.\end{aligned} \tag{2.5}$$

Replacing t by $2t$ and s by $2s$ in (2.4), we obtain that

$$\left[X^{4s}Y^{4t}+X^{-4s}Y^{-2t}+X^{-2s}Y^{-4t}+Y^{2t}+X^{2s}\right]f$$

$$=\left[X^{-4s}Y^{-4t}+X^{4s}Y^{2t}+X^{2s}Y^{4t}+Y^{-2t}+X^{-2s}\right]f. \qquad (2.6)$$

If we subtract (2.6) from (2.5), then

$$\left[2Y^{t}+2X^{s}+2X^{2s}Y^{3t}+2X^{3s}Y^{2t}+2X^{-3s}Y^{-3t}+2X^{-s}Y^{-t}+2X^{-2s}+2Y^{-2t}\right]f$$
$$=\left[2Y^{-t}+2X^{-s}+2X^{-2s}Y^{-3t}+2X^{-3s}Y^{-2t}+2X^{3s}Y^{3t}+2X^{s}Y^{t}+2X^{2s}+2Y^{2t}\right]f. \qquad (2.7)$$

Multiplying (2.4) by $X^{s}Y^{t}$, we obtain that

$$\left[X^{3s}Y^{3t}+X^{-s}+Y^{-t}+X^{s}Y^{2t}+X^{2s}Y^{t}\right]f=\left[X^{-s}Y^{-t}+X^{3s}Y^{2t}+X^{2s}Y^{3t}+X^{s}+Y^{t}\right]f. \qquad (2.8)$$

Multiplying (2.4) by $X^{-s}Y^{-t}$, we obtain that

$$\left[X^{s}Y^{t}+X^{-3s}Y^{-2t}+X^{-2s}Y^{-3t}+X^{-s}+Y^{-t}\right]f$$
$$=\left[X^{-3s}Y^{-3t}+X^{s}+Y^{t}+X^{-s}Y^{-2t}+X^{-2s}Y^{-t}\right]f. \qquad (2.9)$$

Then, adding (2.8) and (2.9), we get

$$\left[X^{3s}Y^{3t}+X^{s}Y^{2t}+X^{2s}Y^{t}+X^{s}Y^{t}+X^{-3s}Y^{-2t}+X^{-2s}Y^{-3t}+2X^{-s}+2Y^{-t}\right]f$$
$$=\left[X^{-3s}Y^{-3t}+X^{-s}Y^{-2t}+X^{-2s}Y^{-t}+X^{-s}Y^{-t}+X^{3s}Y^{2t}+X^{2s}Y^{3t}+2X^{s}+2Y^{t}\right]f. \qquad (2.10)$$

Multiplying (2.10) by 2, we obtain that

$$\left[2X^{3s}Y^{3t}+2X^{s}Y^{2t}+2X^{2s}Y^{t}+2X^{s}Y^{t}+2X^{-3s}Y^{-2t}+2X^{-2s}Y^{-3t}+4X^{-s}+4Y^{-t}\right]f$$
$$=\left[2X^{-3s}Y^{-3t}+2X^{-s}Y^{-2t}+2X^{-2s}Y^{-t}+2X^{-s}Y^{-t}+2X^{3s}Y^{2t}+2X^{2s}Y^{3t}+4X^{s}+4Y^{t}\right]f. \qquad (2.11)$$

Then, adding (2.7) and (2.11), we get

$$\left[2X^{s}Y^{2t}+2X^{2s}Y^{t}+2X^{-2s}+2Y^{-2t}+2X^{-s}+2Y^{-t}\right]f$$
$$=\left[2X^{-s}Y^{-2t}+2X^{-2s}Y^{-t}+2X^{2s}+2Y^{2t}+2X^{s}+2Y^{t}\right]f. \qquad (2.12)$$

Multiplying (2.4) by 2, we obtain that

$$\left[2X^{2s}Y^{2t}+2X^{-2s}Y^{-t}+2X^{-s}Y^{-2t}+2Y^{t}+2X^{s}\right]f$$
$$=\left[2X^{-2s}Y^{-2t}+2X^{2s}Y^{t}+2X^{s}Y^{2t}+2Y^{-t}+2X^{-s}\right]f. \quad (2.13)$$

Then, adding (2.12) and (2.13), we get

$$\left[2X^{s}Y^{2t}+2X^{2s}Y^{t}+2X^{-2s}+2Y^{-2t}+2X^{2s}Y^{2t}+2X^{-2s}Y^{-t}+2X^{-s}Y^{-2t}\right]f$$
$$=\left[2X^{-s}Y^{-2t}+2X^{-2s}Y^{-t}+2X^{2s}+2Y^{2t}+2X^{-2s}Y^{-2t}+2X^{2s}Y^{t}+2X^{s}Y^{2t}\right]f. \quad (2.14)$$

Multiplying (2.14) by $X^{s}Y^{t}$, we obtain that

$$\left[2X^{-s}Y^{t}+2X^{s}Y^{-t}+2X^{3s}Y^{3t}\right]f=\left[2X^{3s}Y^{t}+2X^{s}Y^{3t}+2X^{-s}Y^{-t}\right]f. \quad (2.15)$$

Multiplying (2.14) by $X^{-s}Y^{-t}$, we obtain that

$$\left[2X^{-3s}Y^{-t}+2X^{-s}Y^{-3t}+2X^{s}Y^{t}\right]f=\left[2X^{s}Y^{-t}+2X^{-s}Y^{t}+2X^{-3s}Y^{-3t}\right]f. \quad (2.16)$$

Then, adding (2.15) and (2.16), we get

$$\left[2X^{3s}Y^{3t}+2X^{-3s}Y^{-t}+2X^{-s}Y^{-3t}+2X^{s}Y^{t}\right]f$$
$$=\left[2X^{-3s}Y^{-3t}+2X^{3s}Y^{t}+2X^{s}Y^{3t}+2X^{-s}Y^{-t}\right]f. \quad (2.17)$$

Multiplying (2.17) by $X^{s}Y^{t}$, we obtain that

$$\left[2X^{4s}Y^{4t}+2X^{-2s}+2Y^{-2t}+2X^{2s}Y^{2t}\right]f=\left[2X^{-2s}Y^{-2t}+2X^{4s}Y^{2t}+2X^{2s}Y^{4t}+2\right]f. \quad (2.18)$$

Multiplying (2.17) by $X^{-s}Y^{-t}$, we obtain that

$$\left[2X^{2s}Y^{2t}+2X^{-4s}Y^{-2t}+2X^{-2s}Y^{-4t}+2\right]f=\left[2X^{-4s}Y^{-4t}+2X^{2s}+2Y^{2t}+2X^{-2s}Y^{-2t}\right]f. \quad (2.19)$$

Then, adding (2.18) and (2.19), we get

$$\left[2X^{4s}Y^{4t}+2X^{-2s}+2Y^{-2t}+4X^{2s}Y^{2t}+2X^{-4s}Y^{-2t}+2X^{-2s}Y^{-4t}\right]f$$
$$=\left[2X^{-4s}Y^{-4t}+2X^{2s}+2Y^{2t}+4X^{-2s}Y^{-2t}+2X^{4s}Y^{2t}+2X^{2s}Y^{4t}\right]f. \quad (2.20)$$

Multiplying (2.6) by 2, we obtain that

$$\begin{aligned}&\left[2X^{4s}Y^{4t}+2X^{-4s}Y^{-2t}+2X^{-2s}Y^{-4t}+2Y^{2t}+2X^{2s}\right]f\\&=\left[2X^{-4s}Y^{-4t}+2X^{4s}Y^{2t}+2X^{2s}Y^{4t}+2Y^{-2t}+2X^{-2s}\right]f.\end{aligned} \tag{2.21}$$

If we subtract (2.21) from (2.20), then

$$\left[4X^{2s}Y^{2t}+4X^{-2s}+4Y^{-2t}\right]f=\left[4X^{-2s}Y^{-2t}+4X^{2s}+4Y^{2t}\right]f. \tag{2.22}$$

Multiplying (2.22) by $\frac{1}{4}$, we obtain that

$$\left[X^{2s}Y^{2t}+X^{-2s}+Y^{-2t}\right]f=\left[X^{-2s}Y^{-2t}+X^{2s}+Y^{2t}\right]f. \tag{2.23}$$

Replacing $2t$ by t and $2s$ by s in (2.23), we obtain that

$$\left[X^{s}Y^{t}+X^{-s}+Y^{-t}\right]f=\left[X^{-s}Y^{-t}+X^{s}+Y^{t}\right]f, \tag{2.24}$$

which is (M). Hence, (M) is equivalent to $(\mathrm{M1})$.

Next, we will show that (M) is equivalent to $(\mathrm{M2})$. Multiplying (2.1) by $X^{-s}Y^{t}$, we obtain that

$$\left[Y^{2t}+X^{-2s}Y^{t}+X^{-s}\right]f=\left[X^{-2s}+Y^{t}+X^{-s}Y^{2t}\right]f. \tag{2.25}$$

Multiplying (2.1) by $X^{s}Y^{-t}$, we obtain that

$$\left[X^{2s}+Y^{-t}+X^{s}Y^{-2t}\right]f=\left[Y^{-2t}+X^{2s}Y^{-t}+X^{s}\right]f. \tag{2.26}$$

Then, adding (2.25) and (2.26), we get

$$\begin{aligned}&\left[X^{2s}+Y^{2t}+X^{-s}+Y^{-t}+X^{-2s}Y^{t}+X^{s}Y^{-2t}\right]f\\&=\left[X^{-2s}+Y^{-2t}+X^{s}+Y^{t}+X^{2s}Y^{-t}+X^{-s}Y^{2t}\right]f,\end{aligned} \tag{2.27}$$

which is the operator form of $(\mathrm{M2})$. Thus, (M) implies $(\mathrm{M2})$.

On the other hand, squaring both sides of (2.27), we have

$$\begin{bmatrix} X^{4s}+2X^{2s}Y^{2t}+4X^{s}+2X^{2s}Y^{-t}+4Y^{t}+2X^{3s}Y^{-2t}+Y^{4t}+2X^{-s}Y^{2t}+2X^{-2s}Y^{3t} \\ +3X^{-2s}+4X^{-s}Y^{-t}+2X^{-3s}Y^{t}+2X^{s}Y^{-3t}+3Y^{-2t}+X^{-4s}Y^{2t}+X^{2s}Y^{-4t} \end{bmatrix} f$$
$$=\begin{bmatrix} X^{-4s}+2X^{-2s}Y^{-2t}+4X^{-s}+2X^{-2s}Y^{t}+4Y^{-t}+2X^{-3s}Y^{2t}+Y^{-4t}+2X^{s}Y^{-2t} \\ +2X^{2s}Y^{-3t}+3X^{2s}+4X^{s}Y^{t}+2X^{3s}Y^{-t}+2X^{-s}Y^{3t}+3Y^{2t}+X^{4s}Y^{-2t}+X^{-2s}Y^{4t} \end{bmatrix} f. \tag{2.28}$$

Replacing t by $2t$ and s by $2s$ in (2.27), we obtain that

$$\left[X^{4s}+Y^{4t}+X^{-2s}+Y^{-2t}+X^{-4s}Y^{2t}+X^{2s}Y^{-4t}\right]f$$
$$=\left[X^{-4s}+Y^{-4t}+X^{2s}+Y^{2t}+X^{4s}Y^{-2t}+X^{-2s}Y^{4t}\right]f. \tag{2.29}$$

Adding (2.28) and (2.29), we get

$$\begin{bmatrix} 2X^{2s}Y^{2t}+2X^{2s}Y^{-t}+2X^{3s}Y^{-2t}+2X^{-s}Y^{2t}+2X^{-2s}Y^{3t}+2X^{-3s}Y^{t} \\ +2X^{s}Y^{-3t}+2X^{-2s}+2Y^{-2t}+4X^{s}+4Y^{t}+4X^{-s}Y^{-t} \end{bmatrix} f$$
$$=\begin{bmatrix} 2X^{-2s}Y^{-2t}+2X^{-2s}Y^{t}+2X^{-3s}Y^{2t}+2X^{s}Y^{-2t}+2X^{2s}Y^{-3t}+2X^{3s}Y^{-t} \\ +2X^{-s}Y^{3t}+2X^{2s}+2Y^{2t}+4X^{-s}+4Y^{-t}+4X^{s}Y^{t} \end{bmatrix} f. \tag{2.30}$$

Multiplying (2.30) by $\frac{1}{2}$, we obtain that

$$\begin{bmatrix} X^{2s}Y^{2t}+X^{2s}Y^{-t}+X^{3s}Y^{-2t}+X^{-s}Y^{2t}+X^{-2s}Y^{3t}+X^{-3s}Y^{t} \\ +X^{s}Y^{-3t}+X^{-2s}+Y^{-2t}+2X^{s}+2Y^{t}+2X^{-s}Y^{-t} \end{bmatrix} f$$
$$=\begin{bmatrix} X^{-2s}Y^{-2t}+X^{-2s}Y^{t}+X^{-3s}Y^{2t}+X^{s}Y^{-2t}+X^{2s}Y^{-3t}+X^{3s}Y^{-t} \\ +X^{-s}Y^{3t}+X^{2s}+Y^{2t}+2X^{-s}+2Y^{-t}+2X^{s}Y^{t} \end{bmatrix} f. \tag{2.31}$$

Adding (2.31) and (2.27), we get

$$\left[X^{2s}Y^{2t}+X^{3s}Y^{-2t}+X^{-2s}Y^{3t}+X^{-3s}Y^{t}+X^{s}Y^{-3t}+X^{s}+Y^{t}+2X^{-s}Y^{-t}\right]f$$
$$=\left[X^{-2s}Y^{-2t}+X^{-3s}Y^{2t}+X^{2s}Y^{-3t}+X^{3s}Y^{-t}+X^{-s}Y^{3t}+X^{-s}+Y^{-t}+2X^{s}Y^{t}\right]f. \tag{2.32}$$

Multiplying (2.27) by $X^{-s}Y^{t}$, we obtain that

$$\left[X^{s}Y^{t}+X^{-s}Y^{3t}+X^{-2s}Y^{t}+X^{-s}+X^{-3s}Y^{2t}+Y^{-t}\right]f$$
$$=\left[X^{-3s}Y^{t}+X^{-s}Y^{-t}+Y^{t}+X^{-s}Y^{2t}+X^{s}+X^{-2s}Y^{3t}\right]f. \tag{2.33}$$

Multiplying (2.27) by X^sY^{-t}, we obtain that

$$\begin{aligned}&\left[X^{3s}Y^{-t}+X^sY^t+Y^{-t}+X^sY^{-2t}+X^{-s}+X^{2s}Y^{-3t}\right]f\\&=\left[X^{-s}Y^{-t}+X^sY^{-3t}+X^{2s}Y^{-t}+X^s+X^{3s}Y^{-2t}+Y^t\right]f.\end{aligned}\tag{2.34}$$

Adding (2.33) and (2.34), we get

$$\begin{aligned}&\left[2X^sY^t+2Y^{-t}+2X^{-s}+X^{-s}Y^{3t}+X^{-2s}Y^t+X^{-3s}Y^{2t}+X^{3s}Y^{-t}+X^sY^{-2t}+X^{2s}Y^{-3t}\right]f\\&=\left[2X^{-s}Y^{-t}+2Y^t+2X^s+X^sY^{-3t}+X^{2s}Y^{-t}+X^{3s}Y^{-2t}+X^{-3s}Y^t+X^{-s}Y^{2t}+X^{-2s}Y^{3t}\right]f.\end{aligned}\tag{2.35}$$

Adding (2.32) and (2.35), we get

$$\left[X^{2s}Y^{2t}+X^{-2s}Y^t+X^sY^{-2t}+X^{-s}+Y^{-t}\right]f=\left[X^{-2s}Y^{-2t}+X^{2s}Y^{-t}+X^{-s}Y^{2t}+X^s+Y^t\right]f.\tag{2.36}$$

Adding (2.27) and (2.36), we get

$$\left[X^{2s}Y^{2t}+X^{-2s}+Y^{-2t}\right]f=\left[X^{-2s}Y^{-2t}+X^{2s}+Y^{2t}\right]f.\tag{2.37}$$

Replacing $2t$ by t and $2s$ by s in (2.37), we obtain that

$$\left[X^sY^t+X^{-s}+Y^{-t}\right]f=\left[X^{-s}Y^{-t}+X^s+Y^t\right]f,$$

which is (M). Hence, (M) is equivalent to (M2). This completes the proof. ❒

Conclusion

Using the technique of Haruki and Nakagiri (2007), the equivalence of three functional equations (M), (M1) and (M2) are investigated as follows: The functional equation

$$f(x+s,y+t)+f(x-s,y)+f(x,y-t)=f(x-s,y-t)+f(x+s,y)+f(x,y+t)\tag{M}$$

is equivalent to each one of the following two functional equations:

$$f(x+2s,y+2t)+f(x-2s,y-t)+f(x-s,y-2t)+f(x,y+t)+f(x+s,y)$$
$$=f(x-2s,y-2t)+f(x+2s,y+t)+f(x+s,y+2t)+f(x,y-t)+f(x-s,y) \qquad \text{(M1)}$$

and

$$f(x+2s,y)+f(x,y+2t)+f(x-s,y)+f(x,y-t)+f(x-2s,y+t)+f(x+s,y-2t)$$
$$=f(x-2s,y)+f(x,y-2t)+f(x+s,y)+f(x,y+t)+f(x+2s,y-t)+f(x-s,y+2t) \qquad \text{(M2)}$$

for all $x,y,t,s\in G$ under the assumption $f:G\times G\to\mathbb{C}$.

Discussions

In this paper, we study the functional equation related to the digital filtering $(\mathrm{W1})$ (Sahoo & Székelyhidi, 2001). By using the same technique of Haruki & Nakagiri (2007), we obtained two equations corresponding to the functional equation (M) which is consistent with Haruki & Nakagiri (2007) and Naenudorn & Hengkrawit (2013). Our solutions of the functional equation (M) will be useful for the work related to the digital filtering.

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