การแก้ไขปัญหาสมการการเคลื่อนที่ของไฮเซนเบิร์กของอนุภาคที่ถูกบังคับด้วยแรงฮาร์โมนิกออสซิเลเตอร์ที่ขึ้นอยู่กับเวลาโดยใช้ตัวดำเนินการสร้างและทำลายสถานะ

SOLVING THE HEISENBERG EQUATION OF MOTION OF THE TIME-DEPENDENT FORCED HARMONIC OSCILLATOR USING TIME-DEPENDENT ANNIHILATION AND CREATION OPERATORS

ปิยรัตน์ มูลศรี[1], มัณฑนา ดีนา[2] และ อาทิตย์ หู้เต็ม[2, *]
Piyarut Moonsri[1], Mantana Deena[2] and Artit Hutem[2, *]

1สาขาวิชาวิทยาศาสตร์ศึกษา คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยราชภัฏเพชรบูรณ์
2สาขาวิชาฟิสิกส์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยราชภัฏเพชรบูรณ์
[1]Chemistry Division, Faculty of Science and Technology, Phetchabun Rajabhat University
[2]Physics Division, Faculty of Science and Technology, Phetchabun Rajabhat University

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บทคัดย่อ

งานวิจัยนี้เราต้องการคำนวณหาสมการการเคลื่อนที่สำหรับตัวดำเนินการสร้างและทำลายสถานะของระบบภายใต้แรงที่มีการสั่นอย่างง่ายฮาร์โมนิกออสซิเลเตอร์โดยใช้วิธีสมการเชิงอนุพันธ์เชิงเส้นอันดับที่ 1 แบบไม่เอกพันธุ์โดยการแก้ปัญหาสมการการเคลื่อนที่ของไฮเซนเบิร์กจะได้ตัวดำเนินการสร้างและทำลายสถานะของระบบที่เป็นฟังก์ชันเวลา ซึ่งตัวดำเนินการสร้างและทำลายสถานะของอนุภาคที่ถูกบังคับให้เคลื่อนที่ภายในบ่อศักย์แบบฮาร์โมนิกออสซิเลเตอร์มีลักษณะเป็นกลุ่มคลื่น แต่ถ้าจำนวนค่าของพารามิเตอร์ α เพิ่มขึ้นจะส่งผลทำลายการเป็นกลุ่มคลื่นของตัวดำเนินการสร้างและทำลายสถานะของอนุภาค

คำสำคัญ: ตัวดำเนินการสร้างที่เป็นฟังก์ชันของเวลา, ตัวดำเนินการทำลายที่เป็นฟังก์ชันของเวลา, แรงที่มีการสั่นแบบฮาร์โมนิก

* ผู้ประสานงาน: อาทิตย์ หู้เต็ม
อีเมล์: artithutem@pcru.ac.th

Abstract

For this study, the researchers purposed to evaluate the Heisenberg equation of motion for the creation and annihilation operator under the force vibration simple harmonic oscillation. Instead of using the first-order ordinary differential equation to solve the Heisenberg equation of motion, the researchers got the time-dependent creation and annihilation operator under the time-dependent force vibration simple harmonic oscillation. The behavior of wave group in the creation and annihilation operator for image part is destroyed by increasing of the parameter value of α.

Keywords: time-dependent creation operator, time-dependent annihilation operator, force vibration harmonic oscillator

Introduction

The quantum mechanics of the time-dependent forced harmonic oscillator is fundamental to several branches of physics: for example, quantum gases, quantum field theory, quantum electrodynamics. In this paper, we can be shown creation of the quantum mechanics model for a time-dependent force vibration harmonic oscillator in particle bound harmonic oscillator potential. Gilbey & Goodman (1996) show the analysis method of the operator and the time-dependent force mechanics for the simple harmonic oscillator quantum theory. Glauber & Man'ko (1984) represent evaluation of the damping and equilibrium state due to random fluctuations for the time-dependent external force in quantum mechanics system. Balcou et al. (1996) present the quantum mechanics theoretical study of high-order harmonic oscillator generation via a slowly driven "Duffing" anharmonic vibration. Rigo et al. (1997) use quantum-state diffusion theory to describe of the time-dependent damped anharmonic oscillator system. Chung-In & Kyu-Hwang (2002) show that the path integral method of evaluation the propagator for the damped harmonic oscillator via the Caldirola-Kanai Hamiltonian.

Currently, Antia et al. (2010), show that evaluation of the Caldirola-Kanai Hamiltonian model about the damped harmonic oscillator system in the under-damped regime. Segovia-Chaves (2018) represent the solution of the Hamilton-Jacobi equation in damped harmonic oscillator problem. The scheme of the paper is as follows. In section materials and method, we write the method of Calculation of the Heisenberg equation of motion for the time-dependent annihilation operator $\left(\hat{b}(t)\right)$ and the time-dependent creation operator $\left(\hat{b}^{\dagger}(t)\right)$ in forced vibration (Castanos & Zuniga-Segundo, 2019). Next, we show the result of plot graph for the annihilation operator and the creation operator in the section of result. Finally, we can be representation of the summary.

Materials and Methods

Calculation of the Heisenberg equation of motion for the time-dependent annihilation operator $\left(\hat{b}(t)\right)$ and the time-dependent creation operator $\left(\hat{b}^{\dagger}(t)\right)$ in forced vibration

We are now in a position and linear momentum to apply the Heisenberg equation of motion to a particle of mass (μ) bound in the force simple harmonic oscillator. The time-dependent Hamiltonian of the force simple harmonic oscillator system is

$$\hat{H}(t)=\frac{\hat{p}^{2}}{2\mu}+\frac{1}{2}\mu\omega^{2}\hat{x}^{2}-\left(\hat{x}F(t)+\hat{p}R(t)\right), \quad (1)$$

where $\omega=\sqrt{k/\mu}$ is the angular frequency of the classical mechanics oscillator related to the spring constant k in Hooke's law, $F(t)$ is the time-dependent external force cosine function, $R(t)$ is the time-dependent external force sine function. We have chosen a sinusoidal form for the time-dependent driving force $\left(F(t), R(t)\right)$ because it is the type of forcing usually considered in quantum opto-mechanics, cavity quantum electrodynamics, and quantum optics. The linear momentum operator and position operator are, of course, Hermitian operators (Kim et al., 2003). It is convenient to define two non-Hermitian operators,

$$\hat{b}(t)=\sqrt{\frac{\mu\omega}{2\hbar}}\left(\hat{x}+i\frac{\hat{p}}{\mu\omega}\right),$$
$$\hat{b}^{\dagger}(t)=\sqrt{\frac{\mu\omega}{2\hbar}}\left(\hat{x}-i\frac{\hat{p}}{\mu\omega}\right). \qquad (2)$$

We know as the annihilation operator and the creation operator, respectively, for reasons that will become evident shortly. $\hat{x}$ is the position operator, $\hat{p}$ is the linear momentum operator. $F(t)$ and $R(t)$ is the time-dependent external force simple harmonic oscillation. Substitution Equation (2) into Equation (1) to yield

$$\hat{H}(t)=\frac{\hbar\omega}{2}\left(\hat{b}^{\dagger}\hat{b}+\hat{b}\hat{b}^{\dagger}\right)-f(t)\hat{b}^{\dagger}\hbar-f^{*}(t)\hat{b}\,\hbar, \qquad (3)$$

where we have defined

$$f(t)=\sqrt{\frac{1}{2\mu\hbar\omega}}F(t)+i\sqrt{\frac{\mu\omega}{2\hbar}}R(t)$$
$$f^{*}(t)=\sqrt{\frac{1}{2\mu\hbar\omega}}F(t)-i\sqrt{\frac{\mu\omega}{2\hbar}}R(t) \qquad (4)$$

To demonstrate this Heisenberg equation of motion (Li, 2008) approach we consider once more the time-dependent external force vibration simple harmonic oscillator in Equation (4) (Lopez & Suslov, 2009). In the Heisenberg equation of motion the case of the time-dependent annihilation operator are

$$\frac{d\hat{b}(t)}{dt} = \frac{\left[\hat{b}, \hat{H}(t)\right]}{i\hbar}$$

$$= \frac{1}{i\hbar}\left[\hat{b}, \left(\frac{\hbar\omega}{2}\left(\hat{b}^{\dagger}\hat{b} + \hat{b}\hat{b}^{\dagger}\right) - f(t)\hat{b}^{\dagger}\hbar - f^{*}(t)\hat{b}\,\hbar\right)\right]$$

$$= \frac{1}{i\hbar}\left(\frac{\hbar\omega}{2}\left(\left[\hat{b}, \hat{b}^{\dagger}\hat{b}\right] + \left[\hat{b}, \hat{b}\hat{b}^{\dagger}\right]\right) - f(t)\left[\hat{b}, \hat{b}^{\dagger}\right]\hbar\right)$$

$$\frac{d\hat{b}(t)}{dt} = \frac{1}{i\hbar}\left(\frac{\hbar\omega}{2}\left(2\hat{b}\right) - f(t)\hbar\right) = \left(\frac{\omega}{i}\cdot\frac{i}{i}\right)\hat{b} - \frac{f(t)}{i}\cdot\frac{i}{i}$$

or

$$\frac{d\hat{b}(t)}{dt} + i\omega\hat{b}(t) = i\,f(t) \tag{5}$$

These are the famous the first-order linear non-homogeneous differential equation. In the Heisenberg equation of motion the case of the time-dependent creation operator are

$$\frac{d\hat{b}^{\dagger}(t)}{dt} = \frac{1}{i\hbar}\left[\hat{b}^{\dagger}, \hat{H}(t)\right]$$

$$= \frac{1}{i\hbar}\left[\hat{b}^{\dagger}, \left(\frac{\hbar\omega}{2}\left(\hat{b}^{\dagger}\hat{b} + \hat{b}\hat{b}^{\dagger}\right) - f(t)\hat{b}^{\dagger}\hbar - f^{*}(t)\hat{b}\,\hbar\right)\right],$$

$$= \frac{1}{i\hbar}\left(\frac{\hbar\omega}{2}\left(\left[\hat{b}^{\dagger}, \hat{b}^{\dagger}\hat{b}\right] + \left[\hat{b}^{\dagger}, \hat{b}\hat{b}^{\dagger}\right]\right) - f^{*}(t)\left[\hat{b}^{\dagger}, \hat{b}\right]\hbar\right)$$

$$\frac{d\hat{b}^{\dagger}(t)}{dt} = \frac{1}{i\hbar}\left(\frac{\hbar\omega}{2}\left(-2\hat{b}^{\dagger}\right) + f^{*}(t)\hbar\right) = -\frac{\omega}{i}\cdot\frac{i}{i}\hat{b}^{\dagger} + \frac{f^{*}(t)}{i}\cdot\frac{i}{i}$$

or

$$\frac{d\hat{b}^{\dagger}(t)}{dt} - i\omega\hat{b}^{\dagger}(t) = -i\,f^{*}(t) \tag{6}$$

We want to solve Equation (5), with, written as an equation, is

$$\frac{d\hat{b}(t)}{dt} + i\omega\hat{b}(t) = i\,f(t)$$

$$\frac{d\hat{b}(t)}{dt} + i\omega\hat{b}(t) = i\sqrt{\frac{1}{2m\hbar\omega}}F(t) + i\sqrt{\frac{m\omega}{2\hbar}}R(t)$$

$$\frac{d\hat{b}(t)}{dt}+i\omega\hat{b}(t)=i\sqrt{\frac{1}{2m\hbar\omega}}F_0\cos(\alpha t)-\sqrt{\frac{m\omega}{2\hbar}}F_0\sin(\alpha t), \qquad (7)$$

where $F_0 > 0$ has units of force (F_0 is initial force), $\alpha > 0$ is an angular frequency for the time-dependent external force, ω is an angular frequency for particle bound under harmonic oscillator potential, we can use the condition $(\alpha \neq \omega)$ evaluation the time-dependent annihilation operator for image part and real part in section result. Using technique of integration by part of this Equation with respect to time, we have

$$\hat{b}(t)=i\sqrt{\frac{F_0^2}{2m\hbar\omega}}\left(\frac{\left(i\omega\cos(\alpha t)+\alpha\sin(\alpha t)\right)}{\left(\alpha^2-\omega^2\right)}\right)+\sqrt{\frac{F_0^2}{2m\hbar\omega}}\frac{\omega e^{-i\omega t}}{\left(\alpha^2-\omega^2\right)}-\sqrt{\frac{m\omega F_0^2}{2\hbar}}\frac{\alpha e^{-i\omega t}}{\left(\alpha^2-\omega^2\right)}$$
$$-\sqrt{\frac{m\omega F_0^2}{2\hbar}}\left(\frac{\left(i\omega\sin(\alpha t)-\alpha\cos(\alpha t)\right)}{\left(\alpha^2-\omega^2\right)}\right)+\hat{b}e^{-i\omega t}. \qquad (8)$$

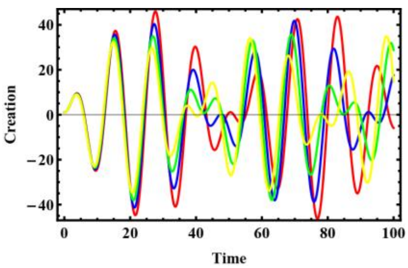
The time-dependent annihilation operator is illustrated in section result. Next, solving Equation (6) for $\hat{b}^\dagger(t)$, we obtain

$$\frac{d\hat{b}^\dagger(t)}{dt}-i\omega\hat{b}^\dagger(t)=-i f^*(t)$$
$$\frac{d\hat{b}^\dagger(t)}{dt}-i\omega\hat{b}^\dagger(t)=-i\left(\sqrt{\frac{1}{2m\hbar\omega}}F_0\cos(\alpha t)-i\sqrt{\frac{m\omega}{2\hbar}}F_0\sin(\alpha t)\right)$$
$$\hat{b}^\dagger(t)=e^{i\omega t}\left(\int_0^t e^{-i\omega t}\left(-i\sqrt{\frac{F_0^2}{2m\hbar\omega}}\cos(\alpha t)-\sqrt{\frac{m\omega F_0^2}{2\hbar}}\sin(\alpha t)\right)dt\right)+\hat{b}^\dagger e^{i\omega t}$$
$$\hat{b}^\dagger(t)=\sqrt{\frac{m\omega F_0^2}{2\hbar}}\left(\frac{\left(\alpha\cos(\alpha t)+i\omega\sin(\alpha t)\right)}{\left(\alpha^2-\omega^2\right)}\right)-\sqrt{\frac{m\omega F_0^2}{2\hbar}}\frac{\alpha e^{-i\omega t}}{\left(\alpha^2-\omega^2\right)}+\sqrt{\frac{F_0^2}{2\hbar m\omega}}\frac{\omega e^{-i\omega t}}{\left(\alpha^2-\omega^2\right)}$$
$$=\sqrt{\frac{F_0^2}{2\hbar m\omega}}\left(\frac{\left(\omega\cos(\alpha t)+i\alpha\sin(\alpha t)\right)}{\left(\alpha^2-\omega^2\right)}\right)+\hat{b}^\dagger e^{i\omega t}, \qquad (9)$$

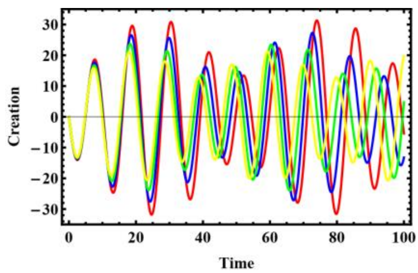
where $F_0 > 0$ has units of force, $\alpha > 0$ is an angular frequency for the time-dependent external force, ω is an angular frequency for particle bound under harmonic oscillator potential, we can use the condition $(\alpha \neq \omega)$ evaluation the time-dependent creation operator for image part and real part in section result. This is known as the time-dependent creation operator. These creation operator function of time in Equation (9) are plotted in section result. Putting this Equation (8) and Equation (9) into program mathematica for plot graph (Philbin, 2012).

Results

These time-dependent annihilation operator and creation operator are illustrated in Figure 1 to Figure 4. From Equation (8) and Equation (9) are illustrated behavior of oscillation imagine part and real part for annihilation operator and creation operator respectively. Next, we can be plot graph relation between of creation operator and time in the Figure 1, where we can vary the parameter f_{qc} (frequency). The time-dependent creation operator for image part and real part are plotted in Figure 1.



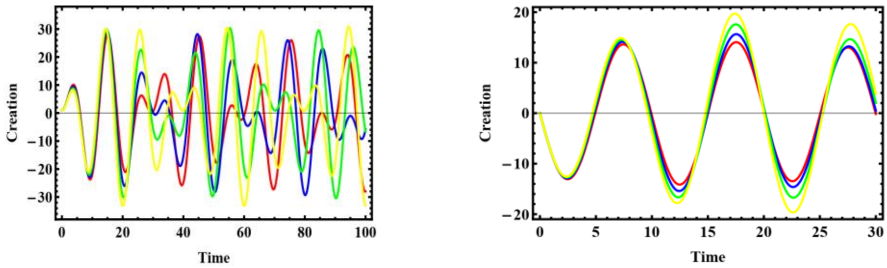
(a) Real part (f_{qc})



(b) Image part (f_{qc})

Figure1 Sketch of the time-dependent creation operator in real part (a) and image part (b) versus frequency f_{qc}.

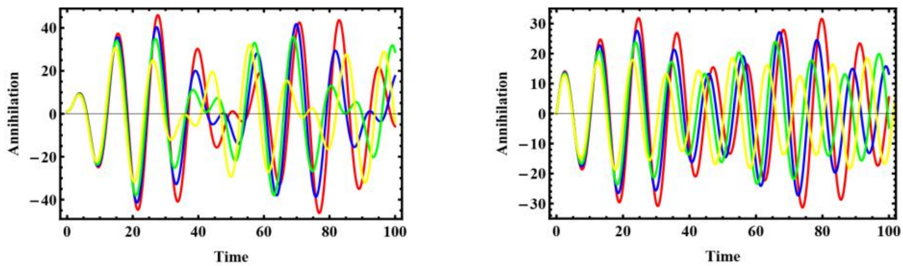
We can be shown diagrammatic sketch of time-dependent creation operator for image part and real part, where we must be vary the parameter α (angular frequency for the time-dependent external force) in the Figure 2.



(a) Real part α (b) Image part α

Figure 2 The time-dependent creation operator in real part (a) and image part (b) of a simple parameter α value plotted as a function of time.

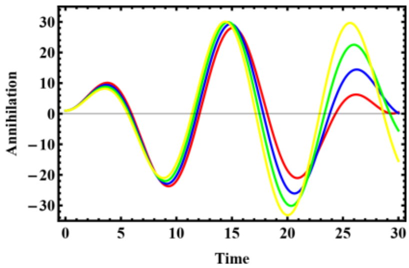
We can be representation plot graph of the time-dependent annihilation operator, where we must be vary the parameter f_{qc} (frequency) in the Figure 3.



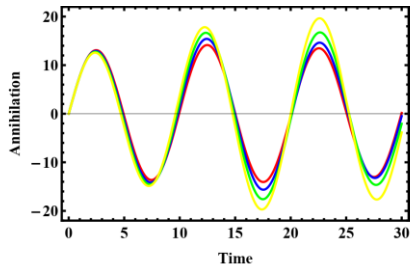
(a) Real part (f_{qc}) (b) Image part (f_{qc})

Figure 3 Calculated the time-dependent annihilation operator in real part (a) and image part (b) of a simple parameter frequency plotted as a function of time.

Finally, we can be diagrammatic sketch of the annihilation operator in real part and image part, where we must be vary the parameter α (angular frequency for the time-dependent external force) in the Figure 4.



(a) Real part α



(b) Image part α

Figure 4 Evaluated the time-dependent annihilation operator in real part (a) and image part (b) of a parameter α value plotted as a function of time.

Discussion

The time-dependent creation operator for real part and imagine part are plotted in Figure 1. From Figure 1(a), the red solid line oscillation for real part is $f_{qc} = 0.091$ Hz. The blue solid line oscillation for real part is $f_{qc} = 0.093$ Hz. The green solid line oscillation for real part is $f_{qc} = 0.095$ Hz. The yellow solid line oscillation for real part is $f_{qc} = 0.097$ Hz. From Figure 1(a), if higher the parameter frequency (f_{qc}) impinge upon decreasing value amplitude and wavelength of the time-dependent creation operator real part. We can be used the condition $\alpha < \omega$ calculation the time-dependent creation operator in Figure 1 and Figure 3. The behavior of the time-dependent creation operator for real part is wave group. From Figure 1(b), the red solid line oscillation for imagine part is $f_{qc} = 0.091$ Hz. The blue solid line oscillation for imagine part is $f_{qc} = 0.093$ Hz. The green solid line oscillation for imagine part is $f_{qc} = 0.095$ Hz. The yellow solid line oscillation for imagine part is $f_{qc} = 0.097$ Hz. From Figure 1(b), if higher the parameter frequency (f_{qc}) have an effect decreasing value amplitude and wavelength of the time-dependent creation operator for imagine part. The behavior of the time-dependent creation operator for imagine part is wave group.

Next, we show the characteristics of the time-dependent creation operator for real part and imagine part by vary the parameter α in the Figure 2(a) and Figure 2(b). From Figure 2(a), the red solid line oscillation for real part is $\alpha = 0.41\ rad/s$. The blue solid line oscillation for real part is $\alpha = 0.43\ rad/s$. The green solid line oscillation for real part is $\alpha = 0.45\ rad/s$. The yellow solid line oscillation for real part is $\alpha = 0.47\ rad/s$. From Figure 2(a), if higher the parameter (α) affect increasing value amplitude of the time-dependent creation operator for real part. The behavior of the time-dependent creation operator for real part is wave group. From Figure 2(b), the red solid line oscillation for imagine part is $\alpha = 0.41\ rad/s$. The blue solid line oscillation for imagine part is $\alpha = 0.43\ rad/s$. The green solid line oscillation for imagine part is $\alpha = 0.45\ rad/s$. The yellow solid line oscillation for imagine part is $\alpha = 0.47\ rad/s$. From Figure 2(b), if higher the parameter (α) affect increasing value amplitude of the time-dependent creation operator for imagine part. The behavior of the time-dependent creation operator for imagine part is not wave group (Rekhviashvili et al., 2019).

Next, we can be representation unique of the time-dependent annihilation operator for real part and imagine part via the parameter frequency in the Figure 3(a) and Figure 3(b). From Figure 3(a), the red solid line oscillation for real part is $f_{qc} = 0.091\ \text{Hz}$. The blue solid line oscillation for real part is $f_{qc} = 0.093\ \text{Hz}$. The green solid line oscillation for real part is $f_{qc} = 0.095\ \text{Hz}$. The yellow solid line oscillation for real part is $f_{qc} = 0.097\ \text{Hz}$. From Figure 3(a), if higher the parameter frequency (f_{qc}) impinge upon decreasing value amplitude and wavelength of the time-dependent annihilation operator real part. The behavior of the time-dependent annihilation operator for real part is wave group. From Figure 3(b), the red solid line oscillation for imagine part is $f_{qc} = 0.091\ \text{Hz}$. The blue solid line oscillation for imagine part is $f_{qc} = 0.093\ \text{Hz}$. The green solid line oscillation for imagine part is $f_{qc} = 0.095\ \text{Hz}$. The yellow solid line oscillation for imagine part is $f_{qc} = 0.097\ \mathrm{Hz}$. From Figure 3(b), if higher the parameter frequency $\left(f_{qc}\right)$ have an effect decreasing value amplitude and wavelength of the time-dependent annihilation operator for imagine part. The behavior of the time-dependent annihilation operator for imagine part is wave group.

Finally, we must be explain the characteristics of the time-dependent annihilation operator for real part and imagine part under vary the parameter α in the Figure 4(a) and Figure 4(b). From Figure 4(a), the red solid line oscillation for real part is $\alpha = 0.41\ rad/s$. The blue solid line oscillation for real part is $\alpha = 0.43\ rad/s$. The green solid line oscillation for real part is $\alpha = 0.45\ rad/s$. The yellow solid line oscillation for real part is $\alpha = 0.47\ rad/s$. From Figure 4(a), if higher the parameter (α) affect increasing value amplitude of the time-dependent annihilation operator for real part. The behavior of the time-dependent annihilation operator for real part is wave group. From Figure 4(b), the red solid line oscillation for imagine part is $\alpha = 0.41\ rad/s$. The blue solid line oscillation for imagine part is $\alpha = 0.43\ rad/s$. The green solid line oscillation for imagine part is $\alpha = 0.45\ rad/s$. The yellow solid line oscillation for imagine part is $\alpha = 0.47\ rad/s$. From Figure 4(b), if higher the parameter (α) affect increasing value amplitude of the time-dependent annihilation operator for imagine part. The behavior of the time-dependent annihilation operator for imagine part is not wave group.

Conclusions

As a representation of the evaluations of the time-dependent creation operator and annihilation operator under the time-dependent force vibration simple harmonic oscillation as a function of time, Figure 1 to Figure 4. From Figure 2(b) and Figure 4(b), if higher the parameter α value affect increasing value amplitude of the time-dependent creation operator and annihilation operator for imagine part. The behavior of the creation operator and annihilation operator for imagine part is not wave group. We can be used the time-dependent annihilation operator and creation operator calculation of the time-dependent position operator and the time-dependent linear momentum operator of particle bound under harmonic oscillator potential. The time-dependent annihilation operator and creation operator is important for find the eigenvalue of the Hamiltonian of a one-dimensional harmonic oscillator potential. We can utilize the annihilation operator and creation operator result to the individual case of a particle of mass moving in the Gaussian potential and other potential.

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