Received: October 15, 2024 | Revised: January 20, 2025 | Accepted: January 22, 2025

JOURNAL OF APPLIED RESEARCH ON SCIENCE AND TECHNOLOGY
JARST
ISSN ONLINE: 2773-9473
https://ph01.tci-thaijo.org/index.php/rmutt-journal
INSTITUTE OF RESEARCH AND DEVELOPMENT
RAJAMANGALA UNIVERSITY OF TECHNOLOGY THANYABURI

Behavioral analysis of two-dimensional difference equations in the third quadrant

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ABSTRACT

Piecewise linear systems of difference equations have gained significant attention for their ability to model complex behavior in population dynamics, economics, and electronics fields. Despite their simple structure, these systems can exhibit diverse behaviors, including convergence to equilibrium points, periodic solutions, and chaotic outcomes under specific conditions. This paper investigates the long-term behavior of a specific piecewise linear system of difference equations. The primary goal is understanding how initial conditions and parameter values influence the system's behavior. The research focuses on identifying and analyzing equilibrium points, periodic solutions (cycles), and the conditions under which these behaviors occur. Building on previous work by Grove et al., we study a family of two-dimensional difference equations containing absolute value terms. The analysis focuses on initial conditions in the third quadrant, divided into three distinct regions: A, B, and C. The behaviors within each region are explored to characterize the system's outcomes. Region A: The system converges to an equilibrium point. The number of iterations required for convergence varies depending on the sub-region. Region B: The system converges to an equilibrium point in exactly two iterations. Region C: The system exhibits more complex behaviors, with potential outcomes including convergence to a 4-cycle or an equilibrium point. Behavior in this region suggests that initial conditions may lead to one of two prime period-4 cycles. Regions A and B consistently lead to equilibrium points, while Region C displays more varied outcomes, including periodic cycles. These findings emphasize the complexity of piecewise linear systems and stress the need for further research to fully understand the behaviors in Region C.

Keywords: Piecewise linear system, Equilibrium point, Difference equations

INTRODUCTION

The study of piecewise linear systems of difference equations has garnered attention for their ability to model complex dynamics in various fields such as population dynamics [1], economics [2] and electronics [3]. Despite their simplicity, these systems exhibit a wide range of behaviors, including convergence to equilibrium points, periodic solutions, and sometimes chaotic behavior [4, 5, 6]. Researchers aim to understand the long-term behavior of these systems and how initial conditions and parameters influence their behaviors. One of the key research drivers is the open problem posed by Grove et al. [7], which introduced a family of systems defined by the general form

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases} \tag{1}$$

with parameters a, b, c and d belonging to $\{-1, 0, 1\}$ and initial condition $(x_0, y_0) \in \mathbb{R}^2$.

Various studies have analyzed how different parameter values affect the system's long-term behaviors. For instance, in article [8] demonstrated that certain parameter sets could result in either a unique equilibrium or periodic behavior with a prime period of 5.

Two primary types of attractors are commonly identified: equilibrium points and periodic solutions (cycles). Equilibrium points represent stable states where the system remains unchanged after iterations. These points are of particular interest because, under certain conditions, the system always converges to this stable state, regardless of initial conditions. Researchers explore whether solutions converge to a single equilibrium or multiple equilibrium points depending on the system's parameters. Periodic solutions, or cycles, are repeating patterns where the system's state follows a cycle with a fixed prime period.

Parameters and initial conditions influence the existence and stability of these periodic solutions. Researchers have identified conditions where the system exhibits periodic behavior with various prime periods. The article [9] found that in system (1), with a = b = d = -1 and c = 1, when the initial condition is an element of the closed second or fourth quadrant, the solution to the system is either a prime period-3 solution or one of two prime period-4 solutions.

DOI: 10.60101/jarst.2024.258602

One key contribution to this research is the work in [10], which studies a special case of system (1) by setting $b = c = -1$ and $d = 1$ but generalizes the parameter b into the following system:

$$\begin{cases} x_{n+1} = |x_n| - y_n - b \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \quad (2)$$

where $b \geq 4$ and found that for $b = 4$, solutions converge to equilibrium, while for $b \geq 6$, the system exhibits periodic behavior with a prime period of 5 [11]. Their findings illustrate the importance of parameter thresholds in determining whether the system converges to an equilibrium or follows periodic behavior. In article [12], the study the system (2) by setting $b = 3$:

$$\begin{cases} x_{n+1} = |x_n| - y_n - 3 \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \quad (3)$$

finding that the solutions can converge to a periodic cycle with a prime period of 4. Their investigation focuses on a specific region in the first quadrant. In [13] and [14], the behaviors of System (3) in the second quadrant were investigated, and 4-cycles and equilibrium points were identified. Building on this, we now examine System (3), considering initial conditions located in a specific region of the third quadrant.

MATERIALS AND METHODS

We divide the third quadrant into subregions and investigate each one by substituting specific initial conditions into System (3) and performing direct calculations. Moreover, in some regions, we identified patterns in the solutions, which we used to formulate inductive statements to explain the behavior of the solutions.

We applied the same methods to investigate the behavior of solutions as described in articles [12-14]. The following definitions [15], will be applied in this paper. A two-dimensional first-order system of difference equations takes the form: $x_{n+1} = f(x_n, y_n)$ and $y_{n+1} = g(x_n, y_n)$ where f and g are continuous functions mapping R^2 to R and $n \geq 0$. A solution to this system is a sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ that satisfies the system for all $n \geq 0$. If an initial condition $(x_0, y_0) \in R^2$ is given, then the subsequent solutions are determined as: $(x_1, y_1) = (f(x_0, y_0), g(x_0, y_0)), (x_2, y_2) = (f(x_1, y_1), g(x_1, y_1)), \cdots$.

A solution that remains constant for all $n \geq 0$ is referred to as an equilibrium solution. If $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 0$, it represents an equilibrium solution, and $(\bar{x}, \bar{y})$ is known as an equilibrium point of the system.

A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is called an eventual equilibrium point if there exists an integer $N > 0$ such that for all $n \geq$ N, $(x_n, y_n) = (\bar{x}, \bar{y})$.

A solution is called eventually periodic with a period p, or an eventually period p-cycle, if there exists an integer $N > 0$ such that the solution is periodic with period p, meaning $(x_{n+p}, y_{n+p}) = (x_n, y_n)$ for all $n \geq$ N.

The set of periodic point is prime period p (or p-cycle) if p the smallest positive integer such that the set member is periodic with period p. We define a 4-cycle as the set $\{((a,b),(c,d),(e,f),(g,h))\}$, consisting of four consecutive points $(a,b),(c,d),(e,f)$, (g,h) in the xy-plane.

A solution is considered eventually periodic with period p (eventually a 4-cycle) if the orbit, through forward iterations, passes through one of the cycle's points. We define the region of the initial condition in the third quadrant as the set $Q_3 = \{(x,y) \in R^2 | x < 0 \; and \; y < 0\}$.

RESULTS AND DISCUSSION

We will examine the behaviors of System (3) when the initial condition is located in the set $A = \{(x,y) \in Q_3 | -x-y-3 < 0$ and $x+y+1 \geq 0\}$, $B = \{(x,y) \in Q_3 | -x-y-3 < 0$ and $x+y+1 < 0\}$ and $C = \{(x,y) \in Q_3 | -x-y-3 \geq 0$ and $x+y+1 < 0\}$. We assert that System (3) has an equilibrium point at $(-1,-1)$, determined by solving equations $\bar{x} = -\bar{x} - \bar{y} - 3$ and $\bar{y} = \bar{x} + \bar{y} + 1$. Additionally, we identified two prime period-4 solutions (or 4-cycles): $P_{4.1} = \{((-5,-1),(3,-5),(5,-1),(3,5))\}$ and $P_{4.2} = \{((1,-1),(-1,1),(-3,-1),(1,-3))\}$ for System (3). We will start our investigation by calculating the first iteration for $(x_0, y_0) \in A$, which corresponds to the green region in Figure 1. In Figure 1, the third quadrant is separated into three sub-regions by the lines $f(x)$ and $g(x)$. The red point is the equilibrium point of the system (3). The green region represents the points in the set A. The pink region represents the points in set B. The remaining white region in the third quadrant represents the points in set C.

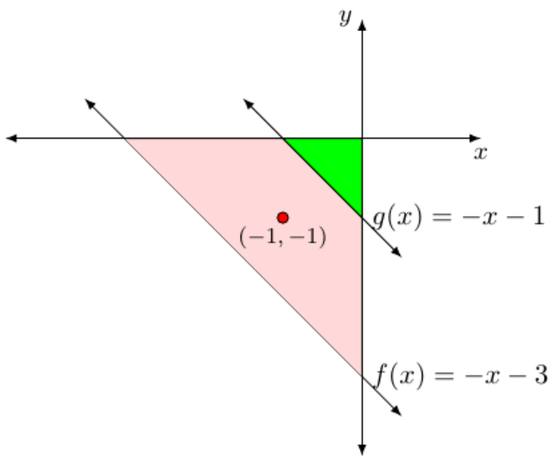

Figure 1 The third quadrant is separated into three sub-regions by the lines $f(x)$ and $g(x)$.

Thus, we obtain the following solutions:

$$\begin{cases} x_1 = |x_0| - y_0 - 3 = -x_0 - y_0 - 3 < 0 \\ y_1 = x_0 - |y_0| + 1 = x_0 + y_0 + 1 \geq 0 \end{cases} \quad (4)$$

DOI: 10.60101/jarst.2024.258602

Journal of Applied Research on Science and Technology

Because the initial condition $(x_0, y_0) \in A$, x_1 is negative and y_1 is non-negative.

$$\begin{cases} x_2 = |x_1| - y_1 - 3 = -1 \\ y_2 = x_1 - |y_1| + 1 = -2x_0 - 2y_0 - 3 < 0 \end{cases} \quad (5)$$

Since y_1 is non-negative, $y_2 = -2y_1 - 1 < 0$.

$$\begin{cases} x_3 = |x_2| - y_2 - 3 = 2x_0 + 2y_0 + 1 \\ y_3 = x_2 - |y_2| + 1 = -2x_0 - 2y_0 - 3 < 0 \end{cases} \quad (6)$$

If $x_3 = 2x_0 + 2y_0 + 1 \le 0$ then the next iteration

$$\begin{cases} x_4 = |x_3| - y_3 - 3 = -1 \\ y_4 = x_3 - |y_3| + 1 = -1 \end{cases}.$$

It means that $(x_4, y_4) = (-1, -1)$. We have the following lemma.

Lemma 1. *Let* $(x_0, y_0) \in A$ *be an initial condition where* $2x_0 + 2y_0 + 1 \le 0$. *Then the fourth iteration of the solution of the system* (3) *is an equilibrium point* $(-1, -1)$.

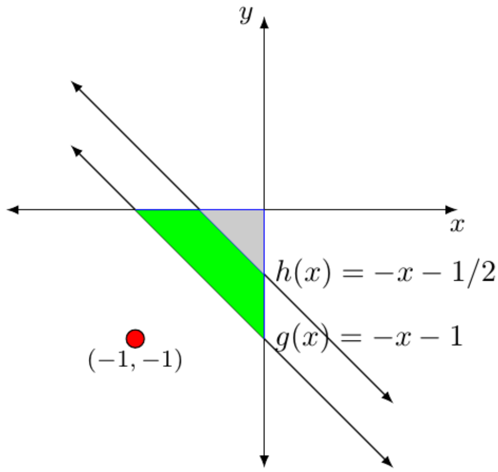

Figure 2 The region A is separated into two subregions.

We now suppose $(x_0, y_0) \in A$ to be an initial condition where $2x_0 + 2y_0 + 1 > 0$, which corresponds to the gray region in Figure 2. In Figure 2, The green region represents the points that satisfy the condition in Lemma 1, while the gray region is the remaining part of set A. We have the closed form of the solution in (4) – (6), except for $x_3 = 2x_0 + 2y_0 + 1 > 0$. Then

$$\begin{cases} x_4 = |x_3| - y_3 - 3 = 4x_0 + 4y_0 + 1 \\ y_4 = x_3 - |y_3| + 1 = -1 \end{cases}, \quad (7)$$

If $x_4 = 4x_0 + 4y_0 + 1 \le 0$ then the next iteration

$$\begin{cases} x_5 = |x_4| - y_4 - 3 = -4x_0 - 4y_0 - 3 < 0 \\ y_5 = x_4 - |y_4| + 1 = 4x_0 + 4y_0 + 1 \le 0 \end{cases},$$
$$\begin{cases} x_6 = |x_5| - y_5 - 3 = -1 \\ y_6 = x_5 - |y_5| + 1 = -1 \end{cases}.$$

This means that $(x_6, y_6) = (-1, -1)$. We have the following lemma.

Lemma 2. *Let* $(x_0, y_0) \in A$ *be an initial condition where* $4x_0 + 4y_0 + 1 \le 0$. *Then the sixth iteration of the solution of the system* (3) *is an equilibrium point* $(-1, -1)$.

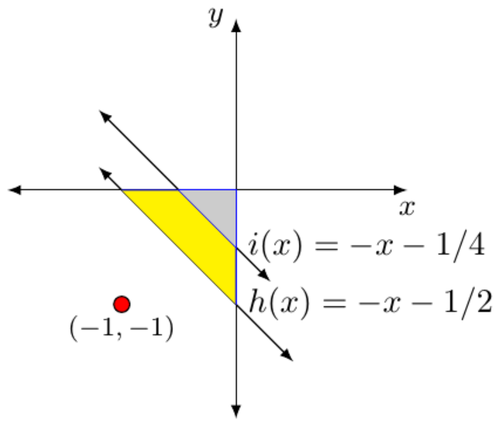

Figure 3 The gray region in Figure 2 is separated into two subregions.

We further suppose $(x_0, y_0) \in A$ to be an initial condition where $4x_0 + 4y_0 + 1 > 0$, which corresponds to the gray region in Figure 3. In Figure 3, The yellow region represents the points that satisfy the condition in Lemma 2, while the gray region is the remaining part of the set A. We will show that the solution will become an equilibrium point, proving by induction the following lemma.

Lemma 3. *Let* $(x_0, y_0) \in A$ *be an initial condition where* $4x_0 + 4y_0 + 1 > 0$. *Then, the solution of the system* (3) *is eventually the equilibrium point.*

Proof. We have the closed form of the solution in (4) – (7), except for $x_4 = 4x_0 + 4y_0 + 1 > 0$. Let $A_n = \{(x, y) \in Q_3 | 2^{2n}x + 2^{2n}y + 1 > 0\}$ and $P(n)$ be the following statement "for $(x_0, y_0) \in A_n$,

$$\begin{cases} x_{4n+1} = 2^{2n}x_0 + 2^{2n}y_0 - 1 < 0 \\ y_{4n+1} = 2^{2n}x_0 + 2^{2n}y_0 + 1 > 0 \end{cases},$$

$$\begin{cases} x_{4n+2} = -2^{2n+1}x_0 - 2^{2n+1}y_0 - 3 < 0 \\ y_{4n+2} = -1 \end{cases},$$

$$\begin{cases} x_{4n+3} = 2^{2n+1}x_0 + 2^{2n+1}y_0 + 1 \\ y_{4n+3} = -2^{2n+1}x_0 - 2^{2n+1}y_0 - 3 < 0 \end{cases}.$$

If $(x_0, y_0) \in B_n = \{(x, y) \in A_n | 2^{2n+1}x + 2^{2n+1}y + 1 \le 0\}$

then $x_{4n+3} = 2^{2n+1}x_0 + 2^{2n+1}y_0 + 1 \le 0$. Thus

$$\begin{cases} x_{4n+4} = -1 \\ y_{4n+4} = -1 \end{cases}.$$

If $(x_0, y_0) \in A_n - B_n = \{(x, y) \in A_n | 2^{2n+1}x + 2^{2n+1}y + 1 > 0\}$

then $x_{4n+3} = 2^{2n+1}x_0 + 2^{2n+1}y_0 + 1 > 0$. So

$$\begin{cases} x_{4n+4} = 2^{2n+2}x_0 + 2^{2n+2}y_0 + 1 \\ y_{4n+4} = -1 \end{cases},$$

If $(x_0, y_0) \in (A_n - B_n) - A_{n+1} =$

$$\{(x, y) \in A_n - B_n | 2^{2n+2}x + 2^{2n+2}y + 1 \le 0\},$$

then $x_{4n+4} = 2^{2n+2}x_0 + 2^{2n+2}y_0 + 1 \le 0$. So

$$\begin{cases} x_{4n+5} = -2^{2n+2}x_0 - 2^{2n+2}y_0 - 3 < 0 \\ y_{4n+5} = 2^{2n+2}x_0 + 2^{2n+2}y_0 + 1 \le 0 \end{cases},$$

$$\begin{cases} x_{4n+6} = -1 \\ y_{4n+6} = -1 \end{cases}.$$

DOI: 10.60101/jarst.2024.258602

If $(x_0, y_0) \in A_{n+1} =$

$$\{(x,y) \in Q_3 | 2^{2n+2}x + 2^{2n+2}y + 1 > 0\}$$

then $x_{4n+4} = 2^{2n+2}x_0 + 2^{2n+2}y_0 + 1 > 0.$"

We first show that $P(1)$ is true. Since $x_4 = 2^2x_0 + 2^2y_0 + 1 > 0$ and $y_4 = -1$ for $n = 1$ with $(x_0, y_0) \in A_1 = \{(x,y) \in Q_3 | 2^2x + 2^2y + 1 > 0\}$, we have

$$\begin{cases} x_{4(1)+1} = x_5 = 2^2x_0 + 2^2y_0 - 1 < 0 \\ y_{4(1)+1} = y_5 = 2^2x_0 + 2^2y_0 + 1 > 0 \end{cases}.$$

Since $(x_0, y_0) \in Q_3$, we have $x_0 < 0$ and $y_0 < 0$, thus $x_5 = 2^2x_0 + 2^2y_0 - 1 < 0$. Additionally, since $(x_0, y_0) \in A_1$, we have $y_5 = 2^2x_0 + 2^2y_0 + 1 > 0$. Then

$$\begin{cases} x_{4(1)+2} = x_6 = -2^3x_0 - 2^3y_0 - 3 < 0 \\ y_{4(1)+2} = y_6 = -1 \end{cases}.$$

Since $x_6 = -2y_5 - 1$, we have $x_6 < 0$.

$$\begin{cases} x_{4(1)+3} = x_7 = 2^3x_0 + 2^3y_0 + 1 \\ y_{4(1)+3} = y_7 = -2^3x_0 - 2^3y_0 - 3 < 0 \end{cases}.$$

Since $y_7 = x_6$, we have $y_7 < 0$.

If $(x_0, y_0) \in B_1 = \{(x,y) \in A_1 | 2^3x + 2^3y + 1 \le 0\}$

then $x_7 = 2^3x_0 + 2^3y_0 + 1 \le 0$.

Then $\begin{cases} x_{4(1)+4} = x_8 = -1 \\ y_{4(1)+4} = y_8 = -1 \end{cases}$.

If $(x_0, y_0) \in A_1 - B_1 =$

$$\{(x,y) \in A_1 |\ 2^3x + 2^3y + 1 > 0\}$$

then $x_7 = 2^3x_0 + 2^3y_0 + 1 > 0$.

$$\begin{cases} x_{4(1)+4} = x_8 = 2^4x_0 + 2^4y_0 + 1 \\ y_{4(1)+4} = y_8 = -1 \end{cases}.$$

If $(x_0, y_0) \in (A_1 - B_1) - A_2 =$

$$\{(x,y) \in A_1 - B_1 |\ 2^4x + 2^4y + 1 \le 0\}$$

then $x_8 = 2^4x_0 + 2^4y_0 + 1 \le 0$.

$$\begin{cases} x_{4(1)+5} = x_9 = -2^4x_0 - 2^4y_0 - 3 < 0 \\ y_{4(1)+5} = y_9 = 2^4x_0 + 2^4y_0 + 1 < 0 \end{cases},$$

$$\begin{cases} x_{4(1)+6} = x_{10} = -1 \\ y_{4(1)+6} = y_{10} = -1 \end{cases}.$$

If $(x_0, y_0) \in A_2 = \{(x,y) \in Q_3 | 2^4x + 2^4y + 1 > 0\}$ then $x_8 = 2^4x_0 + 2^4y_0 + 1 > 0$. Hence, $P(1)$ is true.

The base case of the induction, A_1 is the triangular region of the initial condition (x_0, y_0) that is bounded by the graph of the function $(x) = -x - 1/4$, the x-axis, and the y-axis, which corresponds to the blue dotted region in Figure 4. In Figure 4, the region of the initial condition A_1 for the base case of induction is indicated by the dotted blue line. The green region represents set B_1. The orange and white regions represent $A_1 - B_1$, and the white region represents A_2. B_1 is the region inside A_1 below the graph of $j(x) = -x - 1/8$, which is the green region in Figure 4. $A_1 - B_1$ is the triangular region above $j(x)$ and A_2 is the triangular region above $k(x)$ in Figure 4.

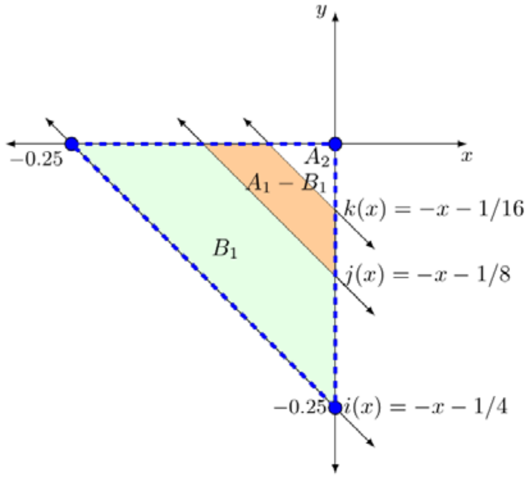

Figure 4 The region of the initial condition A_1 for the base case of induction.

Now we suppose further that $P(k)$ is true for a positive integer. We have

$x_{4k+4} = 2^{2k+2}x_0 + 2^{2k+2}y_0 + 1 > 0$ and $y_{4k+4} = -1$

for $(x_0, y_0) \in A_{k+1} =$

$$\{(x_0, y_0) \in Q_3 | 2^{2k+2}x_0 + 2^{2k+2}y_0 + 1 > 0\}.$$

Then

$$\begin{cases} x_{4(k+1)+1} = x_{4k+5} = 2^{2(k+1)}x_0 + 2^{2(k+1)}y_0 - 1 < 0 \\ y_{4(k+1)+1} = y_{4k+5} = 2^{2(k+1)}x_0 + 2^{2(k+1)}y_0 + 1 > 0 \end{cases}.$$

Since $(x_0, y_0) \in Q_3$, we have $x_0, y_0 < 0$, and thus

$x_{4(k+1)+1} = 2^{2(k+1)}x_0 + 2^{2(k+1)}y_0 - 1 < 0.$

Additionally, since $(x_0, y_0) \in A_{k+1}$, we have

$y_{4(k+1)+1} = 2^{2(k+1)}x_0 + 2^{2(k+1)}y_0 + 1 > 0$. Thus

$$\begin{cases} x_{4(k+1)+2} = x_{4k+6} = -2^{2(k+1)+1}x_0 - 2^{2(k+1)+1}y_0 - 3 < 0 \\ y_{4(k+1)+2} = y_{4k+6} = -1 \end{cases}.$$

Since $x_{4(k+1)+2} = -2y_{4(k+1)+1} - 1$,

we have $x_{4(k+1)+2} < 0$.

$$\begin{cases} x_{4(k+1)+3} = x_{4k+7} = 2^{2(k+1)+1}x_0 + 2^{2(k+1)+1}y_0 + 1 \\ y_{4(k+1)+3} = y_{4k+7} = -2^{2(k+1)+1}x_0 - 2^{2(k+1)+1}y_0 - 3 < 0 \end{cases}.$$

Since $y_{4(k+1)+3} = x_{4(k+1)+2}$, we have $y_{4(k+1)+3} < 0$.

If $(x_0, y_0) \in B_{k+1} =$

$$\{(x,y) \in A_{k+1} | 2^{2k+3}x + 2^{2k+3}y + 1 \le 0\}$$

then $x_{4(k+1)+3} = 2^{2(k+1)+1}x_0 + 2^{2(k+1)+1}y_0 + 1 \le 0$.

Thus $\begin{cases} x_{4(k+1)+4} = -1 \\ y_{4(k+1)+4} = -1 \end{cases}$.

If $(x_0, y_0) \in A_{k+1} - B_{k+1} =$

$$\{(x,y) \in A_{k+1} | 2^{2k+3}x + 2^{2k+3}y + 1 > 0\}$$

then $x_{4(k+1)+3} = 2^{2k+3}x_0 + 2^{2k+3}y_0 + 1 > 0$. Thus

DOI: 10.60101/jarst.2024.258602

$$\begin{cases} x_{4(k+1)+4} = x_{4k+8} = 2^{2(k+1)+2}x_0 + 2^{2(k+1)+2}y_0 + 1 \\ y_{4(k+1)+4} = y_{4k+8} = -1 \end{cases}.$$

If $(x_0, y_0) \in (A_{k+1} - B_{k+1}) - A_{k+2} =$

$\{(x,y) \in (x,y) \in A_{k+1} - B_{k+1} | 2^{2k+4}x + 2^{2k+4}y + 1 \le 0\}$

Then $x_{4(k+1)+4} = x_{4k+8} = 2^{2k+4}x_0 + 2^{2k+4}y_0 + 1 \le 0$. Thus

$$\begin{cases} x_{4(k+1)+5} = x_{4k+9} = -2^{2(k+1)+2}x_0 - 2^{2(k+1)+2}y_0 - 3 < 0 \\ y_{4(k+1)+5} = y_{4k+9} = 2^{2(k+1)+2}x_0 + 2^{2(k+1)+2}y_0 + 1 \le 0 \end{cases}.$$

Since $x_{4(k+1)+5} = -2x_{4(k+1)+3} - 1$,

we have $x_{4(k+1)+5} < 0$ and $y_{4(k+1)+5} = x_{4(k+1)+4} \le 0$.

Thus $\begin{cases} x_{4(k+1)+6} = -1 \\ y_{4(k+1)+6} = -1 \end{cases}$.

If $(x_0, y_0) \in A_{k+2} =$ $\{(x,y) \in Q_3 | 2^{2k+4}x + 2^{2k+4}y + 1 > 0\}$ then

$x_{4k+8} = 2^{2k+4}x_0 + 2^{2k+4}y_0 + 1 > 0$.

Hence $P(k+1)$ is true. By mathematical induction, we conclude that $P(n)$ is true for every positive integer $n \ge 1$. If n increases, the regions of A_n and B_n will become progressively smaller. By mathematical induction, $P(n)$ shows that if the initial condition is within region A_1, the solution will eventually converge to an equilibrium point. □
By the above lemmas, we immediately have the following theorem.

Theorem *Let* (x_0, y_0) *be an initial condition in the set. Then the solution of the system (3) is eventually the equilibrium point.*

The second region, the set B, which corresponds to the pink region in Figure 1. We obtain the following solutions:

$$\begin{cases} x_1 = |x_0| - y_0 - 3 = -x_0 - y_0 - 3 < 0 \\ y_1 = x_0 - |y_0| + 1 = x_0 + y_0 + 1 < 0 \end{cases},$$

$$\begin{cases} x_2 = |x_1| - y_1 - 3 = -1 \\ y_2 = x_1 - |y_1| + 1 = -1 \end{cases}.$$

We have the following proposition.

Proposition *Let* $(x_0, y_0) \in B$ *be an initial condition where* $2x_0 + 2y_0 + 1 \le 0$. *Then the second iteration of the solution of the system* (3) *is an equilibrium point* $(-1, -1)$.

The remaining region, which is below the line of the function $f(x) = -x - 3$, is in the set C of the third quadrant in Figure 1. We have (x_1, y_1) belonging to the fourth quadrant. The behaviors of the solution are more complicated than those of the other two sub-regions and are interesting to study, which we leave for future work. This result agrees with the findings in article [12]; there are regions where the solution eventually becomes an equilibrium point.

From the Theorem and Proposition, the solution will eventually converge to an equilibrium point when we begin with an initial condition in set A or B. These results expand the understanding of the behavior of system (3), building on [12], which examined initial conditions only in the first quadrant, and further findings of [13] and [14], which focused specifically on initial conditions in the second quadrant. However, according to our results, no initial condition will converge to a 4-cycle. We conjecture that the solution of system (3) with an initial condition in set C will eventually converge to a 4-cycle $P_{4.1}$ or $P_{4.2}$.

CONCLUSIONS

In conclusion, system (3) analysis reveals that its long-term behavior is highly sensitive to the initial conditions, particularly within different sub-regions of the third quadrant. For initial conditions in the set A, the system converges to the equilibrium point (-1, -1) after either four or six iterations, depending on the specific region of A. Similarly, for initial conditions in set B, the solution also converges to the equilibrium point in just two iterations. However, the behavior of the system for initial conditions in set C is notably more complex, with the possibility of convergence to a 4-cycle rather than an equilibrium point. This complexity presents fascinating opportunities for future research. Although our current findings suggest that no initial conditions in sets A or B lead to a 4-cycle, we conjecture that the system's behavior in set C may eventually result in convergence to one of the identified 4-cycles, $P_{4.1}$ or $P_{4.2}$, as highlighted in previous studies. Further investigation into this sub-region is necessary to fully understand the behaviors of system (3).

ACKNOWLEDGEMENT

The authors are supported by the National Research Council of Thailand and Pibulsongkram Rajabhat University.

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DOI: 10.60101/jarst.2024.258602

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DOI: 10.60101/jarst.2024.258602