Received: September 5, 2024 | Revised: January 23, 2025 | Accepted: January 27, 2025

JOURNAL OF APPLIED RESEARCH ON SCIENCE AND TECHNOLOGY
JARST
INSTITUTE OF RESEARCH AND DEVELOPMENT
RAJAMANGALA UNIVERSITY OF TECHNOLOGY THANYABURI
ISSN ONLINE: 2773-9473
https://ph01.tci-thaijo.org/index.php/rmutt-journal

On the generalization neo balancing sequence and some applications

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ABSTRACT

We investigate the generalization of sequence neo balancing numbers and their recurrence relations by extending the catalyst of certain sequences in balancing numbers to any integer k in the sequence neo balancing numbers. We derive the Diophantine equation for sequence neo balancing numbers in terms of k, which corresponds to the Diophantine equation for neo balancing numbers. We derive the Diophantine equation for the sequence neo balancing numbers and solve it via Pell's equation and Brahmagupta's identity. We examine the square root term in the derived Diophantine equation for the sequence neo balancing numbers by treating it as the generalized Pell's equation. Simultaneously, we consider the well-known Pell's equation. We integrate the generalized Pell's equation for the sequence neo balancing numbers to the well-known Pell's equation by using the Brahmagupta's identity. We obtain two solutions for both the generalized Pell's equation for the sequence neo balancing numbers and the well-known Pell's equation. The obtained solutions are sometimes analogous for some values of k. Then we investigate more precisely each case and substitute the solutions into the derived Diophantine equation for the sequence neo balancing numbers. Therefore, there are values of k that make both solutions analogous implying the recurrence relation can give all terms in the sequence neo balancing numbers. Simultaneously, there are values of k that make both solutions different implying that we need two recurrence relations generated by the two solutions to complete the sequence neo balancing numbers. Moreover, we establish a few theorems to explain why some values of k generate similar sequence of neo balancing numbers.

Keywords: Balancing numbers, Neo balancing numbers, Sequence neo balancing numbers, Pell's equation, Brahmagupta's identity

INTRODUCTION

In the field of number theory, sequences play a fascinating role in unraveling hidden patterns and relationships among integers. The study of balancing numbers has emerged as a significant area of interest within number theory, offering valuable insights and applications to researchers. Over the past two decades, extensive investigations into balancing numbers have enriched the field, highlighting their mathematical elegance and intricate properties.

Balancing numbers, characterized by their unique relationship to sums of consecutive integers, have been explored in various contexts, including their connections to Pell's equations, recurrence relations, and integer sequences. This body of work has not only deepened the theoretical understanding of these numbers but has also uncovered potential applications in areas such as cryptography and algebraic structures [1-8].

The sustained scholarly focus on balancing numbers underscores their importance as a rich and fruitful subject of mathematical inquiry. In 1999, Panda and Behera [9] laid the foundation for an intriguing exploration with the introduction of the founding important results on the square roots of triangular numbers as balancing numbers. A positive integer n is called a balancing number if it satisfies the Diophantine equation [10]

$$1+2+3+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \quad (1.1)$$

for some positive integer r. Here r is called the balancer corresponding to the balancing number n. Then, Panda and Ray [4] discovered generalization balancing numbers to cobalancing numbers. A positive integer n is defined as a cobalancing number if n is a solution of Diophantine equation

$$1+2+3+\cdots+n=(n+1)+(n+2)+(n+3)+\cdots+(n+r) \quad (1.2)$$

for some positive integer r. Here r is called the cobalancer corresponding to the cobalancing number n. Later, Panda [11] found the many results on the sequence of balancing numbers, the sequence of the cobalancing numbers and also connected to Fibonacci numbers. These pioneering works have served as a catalyst for further investigation and have paved the way for new inquiries to neo balancing numbers.

DOI: 10.60101/jarst.2024.257946

Chailangka and Pakapongpun [12] defined neo balancing numbers by the study of Diophantine equation

$$1+2+3+\cdots+(n-1) = (n-1)+n+(n+1)+\cdots+(n+r), \qquad (1.3)$$

for each positive integer n which is called a neo balancing number and for some integer r which is called a neo balancer corresponding to the neo balancing number n.

MATERIALS AND METHODS

Pell's equation

The Diophantine equation of the form

$$x^2 - ny^2 = 1, \qquad (1.4)$$

where n is a given positive nonsquare integer, seeks integer solutions sought for x and y [13].

Generalized Pell's equation

The Diophantine equation of the form

$$x^2 - ny^2 = k, \qquad (1.5)$$

where n is a given positive nonsquare integer, k is a non zero integer and integer solutions are sought for x and y [13].

Additional solutions from the fundamental solution

If (x_1, y_1) is a fundamental solution, then the algebraic form is

$$x_k + y_k\sqrt{n} = (x_1 + y_1\sqrt{n})^k \qquad (1.6)$$

and also yields the recurrence relations

$$x_{k+1} = x_1x_k + ny_1y_k \qquad (1.7)$$

$$y_{k+1} = x_1y_k + y_1x_k, \qquad (1.8)$$

where n is a given positive non-square integer and k is a positive integer [14].

Brahmagupta's identity

If the triples (x_1, y_1, k_1) and (x_2, y_2, k_2) are solutions of Diophantine equation

$$x^2 - ny^2 = k, \qquad (1.9)$$

then we can compose the triples to generate new triples $(x_1x_2 + ny_1y_2, x_1y_2 + x_2y_1, k_1k_2)$ and $(x_1x_2 - ny_1y_2, x_1y_2 - x_2y_1, k_1k_2)$,

where n is a given positive non-square integer, k is a non zero integer and integer solutions are sought for x and y [15].

RESULTS AND DISCUSSION

1. Sequence neo balancing numbers

In this section, we will introduce neo balancing numbers in the sequence a_m. Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of real numbers. Then a_m is called a sequence neo balancing number if a_m satisfies the Diophantine equation

$$a_1 + a_2 + a_3 + \cdots + a_{m-1} = a_{m-1} + a_m + a_{m+1} + \cdots + a_{m+r} \qquad (2.1)$$

for some integer r, simultaneous m_n is called a sequence neo balancing number's index and r_n is called the sequence neo balancer's index which corresponding to m_n. Then we will get sequences for neo balancing numbers by equation (2.1) in the following sections.

2. Generalized neo balancing numbers

In this section, we will generalize the idea of sequence for neo balancing numbers by equation (2.1). For any integer k in the sequence $a_m = 2m+k$, we derive the sequence neo balancing number $2m+k$ as a Diophantine equation (2.1). Currently, we substitute $a_m = 2m+k$ and obtain the equation

$$(2+k)+(4+k)+(6+k)+\cdots+(2m-2+k) = (2m-2+k)+(2m+k)+(2m+2+k) +\cdots+(2m+2r+k). \qquad (2.2)$$

From equation (2.2), we derive to

$$2m = 2r+5-k+\sqrt{8r^2+24r+k^2+2k+17}, \qquad (2.3)$$

for any integer k. Henceforth we will consider k. If k is odd, then $k^2+2k+17$ can be divided by 4 and we can derive equation (2.3) as the following.

$$m = r + \frac{5-k}{2} + \frac{1}{2}\sqrt{8r^2+24r+k^2+2k+17}.$$

Similarly, if k is even, then we can derive equation (2.3) as

$$m = \frac{2r-k}{2} + \frac{1}{2}(5+\sqrt{8r^2+24r+k^2+2k+17}).$$

In the present, we will consider $x^2-2y^2 = D$ which we can compose the solution to the fundamental of Pell's equation by Brahmagupta's identity, see [14-17]. Then we will obtain the expanded solutions to $x^2-2y^2 = D$. Since m is a neo balancing number,

$$y = \sqrt{8r^2+24r+k^2+2k+17}$$

must be an integer. Hence, we set

$$y = \sqrt{8r^2+24r+k^2+2k+17}$$

Then there exists $x = 4r+6$ such that

$$x^2-2y^2 = -2k^2-4k+2. \qquad (2.4)$$

Currently, we let $D = -2k^2-4k+2$ and rewrite D as $D = 4-2(k+1)^2$, which implies that $(2, |k+1|)$ is a solution for the equation (2.4). Then the next process will be helped by the quite well-known Pell's equation and the fascinating Brahmagupta's identity. We consider Pell's equation

DOI: 10.60101/jarst.2024.257946

$$x^2 - 2y^2 = 1, \quad (2.5)$$

with a fundamental solution $(3,2)$. Since the Pell's equation (2.5) is investigated for many years, it is easy to find the other solution by relations

$$\bar{x}_n = \frac{\alpha^n + \beta^n}{2}$$

$$\bar{y}_n = \frac{\alpha^n + \beta^n}{2\sqrt{2}},$$

where $\alpha = 3+2\sqrt{2}$ and $\beta = 3-2\sqrt{2}$. Therefore, $(2,|k+1|)$ is a solution for the equation (2.4) and $(\bar{x}_n, \bar{y}_n)$ is also a solution for the equation (2.5). Then we use Brahmagupta's identity to combine equation (2.4) and equation (2.5) and their solutions. Thus, we obtain the combined solution as relations

$$X_n = 2\bar{x}_n + 2|k+1|\bar{y}_n$$

$$Y_n = |k+1|\bar{x}_n + 2\bar{y}_n$$

and

$$X_n^* = 2\bar{x}_n - 2|k+1|\bar{y}_n$$

$$Y_n^* = |k+1|\bar{x}_n - 2\bar{y}_n.$$

Therefore, (X_n, Y_n) and (X_n^*, Y_n^*) are solutions to equation (2.4). Moreover, we find that the recurrence relation for both two sequences of X_n, Y_n and X_n^*, Y_n^* as given satisfy the recurrence

$$X_n = 6X_{n-1} - X_{n-2} \quad (2.6)$$

$$Y_n = 6Y_{n-1} - Y_{n-2} \quad (2.7)$$

and

$$X_n^* = 6X_{n-1}^* - X_{n-2}^* \quad (2.8)$$

$$Y_n^* = 6Y_{n-1}^* - Y_{n-2}^* \quad (2.9)$$

Since equation (2.6), (2.7), (2.8) and equation (2.9), we get

$$2X_n = \alpha^n(2+|k+1|\sqrt{2}) + \beta^n(2-|k+1|\sqrt{2})$$

$$2\sqrt{2}Y_n = \alpha^n(2+|k+1|\sqrt{2}) - \beta^n(2-|k+1|\sqrt{2})$$

and

$$2X_n^* = \alpha^n(|k+1|\sqrt{2}-2) - \beta^n(|k+1|\sqrt{2}+2)$$

$$2\sqrt{2}Y_n^* = \alpha^n(|k+1|\sqrt{2}-2) + \beta^n(|k+1|\sqrt{2}+2).$$

Therefore, we obtain two sequences both X_n, Y_n and X_n^*, Y_n^* which make the sets of solution of the equation (2.4) are sometimes analogous. Then we have to consider this critical property $X_n = |X_{n+n_0}^*|$ and $Y_n = |Y_{n+n_0}^*|$. Since $X_n = |X_{n+n_0}^*|$, we will obtain

$$\left(\beta^{n_0}(2+|k+1|\sqrt{2})\right)^2 = (2-|k+1|\sqrt{2})^2.$$

Then we have

$$\left(\beta^{n_0}(2+|k+1|\sqrt{2})\right) - (2-|k+1|\sqrt{2}) = 0$$

and

$$\left(\beta^{n_0}(2+|k+1|\sqrt{2})\right) + (2-|k+1|\sqrt{2}) = 0.$$

Since $\beta^{n_0} = (3-2\sqrt{2})^{n_0}$ can be calculated algebraically form $x_{n_0} + y_{n_0}\sqrt{2} = (3+2\sqrt{2})^{n_0}$ [9] when $x_0 = 1$, $y_0 = 0$, $x_1 = 3$, $y_1 = 2$ and the above discussion, we have important 4 cases as the following.

1. $|k+1| = \dfrac{x_{n_0} - 1}{y_{n_0}}$
2. $|k+1| = \dfrac{2y_{n_0}}{x_{n_0} + 1}$
3. $|k+1| = \dfrac{x_{n_0} + 1}{y_{n_0}}$
4. $|k+1| = \dfrac{2y_{n_0}}{x_{n_0} - 1}$.

Theorem 2.1 Let $A = \{-3,-2,-1,0,1\}$. For any integer $k \in A$, the recurrence relations for the sequence neo balancing number $a_m = 2m + k$ is

$$a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8$$

and the recurrence relations for the sequence neo balancing number's index is $m_n = 6m_{n-1} - m_{n-2} + 2k - 4$, where $n \geq 3$.

Proof.

Consider 4 given cases. If we do not shift any index, then $n_0 = 0$. We obtain that

$$|k+1| = 0$$

by case 1 and 2. Hence, $k = -1$. Therefore, $X_n = |X_n^*|$ meanwhile X_n and X_n^* have same set of solution of equation (2.4) when $a_m = 2m-1$. Similarly, $Y_n = |Y_n^*|$ meanwhile Y_n and Y_n^* have same set of solution of equation (2.4) when $a_m = 2m-1$. Subsequently, if we shift an index 1 step for $X_n = |X_{n+n_0}^*|$, then $n_0 = 1$. Hence, we obtain that $|k+1| = 1$ by case 1 and case 2 meanwhile we obtain that $|k+1| = 2$ by case 3 and case 4. Then we have $k = 0,-2,-3$ and 1, respectively. Therefore, $X_n = |X_{n+1}^*|$ meanwhile X_n and X_{n+1}^* have same set of solution of equation (2.4) when $a_m = 2m, a_m = 2m-2, a_m = 2m-3$ and $a_m = 2m+1$. Since we have already discovered $2m = 2r + 5 - k + \sqrt{8r^2 + 24r + k^2 + 2k + 17}$, $y = \sqrt{8r^2 + 24r + k^2 + 2k + 17}$ and $x = 4r + 6$, we

DOI: 10.60101/jarst.2024.257946

will combine them. Hence, we get the recurrence relation for sequence neo balancing number's index $m_n = 6m_{n-1} - m_{n-2} + 2k - 4$. Since $a_m = 2m + k$, we will get fascinating results on the generalized recurrence relation for the sequence neo balancing number as $a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8$.

Theorem 2.2 Let $A = \{-3, -2, -1, 0, 1\}$. For any integer $k \in A^c$, the recurrence relations for the sequence neo balancing number $a_m = 2m + k$ are

$$a_{m_{2n-1}} = 6a_{m_{2n-3}} - a_{m_{2n-5}} - 8$$

and

$$a_{m_{2n}} = 6a_{m_{2n-2}} - a_{m_{2n-4}} - 8.$$

Moreover, the recurrence relations for the sequence neo balancing number's index are

$$m_{2n-1} = 6m_{2n-3} - m_{2n-5} + 2k - 4$$

and

$$m_{2n} = 6m_{2n-2} - m_{2n-4} + 2k - 4,$$

respectively, where $n \geq 3$.

Proof.

Since (x_{n_0}, y_{n_0}) satisfies the algebraic form recurrence relation

$$x_{k+1} = 3x_k + 4y_k$$
$$y_{k+1} = 3y_k + 2x_k,$$

we rewrite the 4 given cases as

1. $|k+1| = 1 + \dfrac{x_{n_0-1} + y_{n_0-1} - 1}{2x_{n_0-1} + 3y_{n_0-1}}$
2. $|k+1| = 1 + \dfrac{x_{n_0-1} + 2y_{n_0-1} - 1}{3x_{n_0-1} + 4y_{n_0-1} + 1}$
3. $|k+1| = 1 + \dfrac{x_{n_0-1} + y_{n_0-1} + 1}{2x_{n_0-1} + 3y_{n_0-1}}$
4. $|k+1| = 1 + \dfrac{x_{n_0-1} + 2y_{n_0-1} + 1}{3x_{n_0-1} + 4y_{n_0-1} - 1}$.

Obviously, $|k+1|$ of the 4 above cases are not integer for $n_0 > 1$. Therefore, X_n and $X^*_{n+n_0}$ do not have same set of solution to equation (2.4) where $n_0 > 1$. Similarly, we can obtain the important property as the above for $Y_n = |Y^*_{n+n_0}|$. Then we will obtain 2 sets of solutions to equation (2.4) which generated by (X_n, Y_n) and (X^*_n, Y^*_n).

Since we have already discovered $2m = 2r + 5 - k + \sqrt{8r^2 + 24r + k^2 + 2k + 17}$, $y = \sqrt{8r^2 + 24r + k^2 + 2k + 17}$ and $x = 4r + 6$, we will combine them. Hence, we get the recurrence relation for sequence neo balancing number's index $m_{2n-1} = 6m_{2n-3} - m_{2n-5} + 2k - 4$ and $m_{2n} = 6m_{2n-2} - m_{2n-4} + 2k - 4$.

Since $a_m = 2m + k$, we will get fascinating results on the generalized recurrence relation for the sequence neo balancing number as

$a_{m_{2n-1}} = 6a_{m_{2n-3}} - a_{m_{2n-5}} - 8$ and

$a_{m_{2n}} = 6a_{m_{2n-2}} - a_{m_{2n-4}} - 8$.

Theorem 2.3 Let $f : \square \to \square$ be a function that $f(k) = 4 - 2(k+1)^2$. If there are k_1 and k_2 such that $k_1 \neq k_2$, then the sequences $a_m = 2m + k_1$ and $a_m = 2m + k_2$ have the same sequence neo balancing number if and only if $f(k_1) = f(k_2)$.

Proof.

Since we have defined $D = 2 - (2k^2 + 4k)$, we can rewrite it to a parabola equation

$$D - 4 = -2(k+1)^2.$$

Then we will obtain the solution (k_1, D) and (k_2, D) where $k_1 \neq k_2$ which imply the same equation as equation (2.4). Simultaneously, we will get the same sequence neo balancing number by k_1 and k_2 if $f(k_1) = f(k_2)$.

3. Sequence neo balancing numbers in some sequences

In this section, we will demonstrate sequence neo balancing numbers in some sequences.

Example 3.1 The sequence neo balancing numbers in sequence $a_m = 2m - 1$.

By theorem 2.1, we let $k = -1$. Then the recurrence relations for the sequence neo balancing number $a_m = 2m - 1$ are $m_n = 6m_{n-1} - m_{n-2} - 6$ and $a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8$. Then 2, 5, 22, 121, 698 and 4061 are first six examples of the sequence neo balancing number's indices meanwhile $-1, 0, 7, 48, 287$ and 1680 are first six examples of the sequence neo balancer's indices, where $a_m = 2m - 1$. We can demonstrate by putting a_m to the equation (2.1) as the following. If $m = 22$ and $r = 7$, then we obtain

$$
\begin{aligned}
a_1 + a_2 + a_3 + \cdots + a_{22-1} &= a_{22-1} + a_{22} + a_{23} + \cdots + a_{22+7} \\
1 + 3 + 5 + \cdots + 41 &= 41 + 43 + 45 + \cdots + 57 \\
441 &= 441.
\end{aligned}
$$

DOI: 10.60101/jarst.2024.257946

Hence, 3, 9, 43, 241, 1396 and 8121 are the first six example of sequence neo balancing numbers, where $a_m = 2m-1$.

Remark 3.2 If we substitute $k=-1$, then the another k in theorem 2.3 which generate same sequence do not exist cause $k=-1$ is a vertex of the parabola $D-4=-2(k+1)^2$.

Example 3.3 The sequence neo balancing numbers in sequence $a_m = 2m-2$ and $a_m = 2m$.

By theorem 2.1 and theorem 2.3, we can choose $k=-2$ or $k=0$, so let $k=0$. Then the recurrence relations for the sequence neo balancing number $a_m = 2m$ are $m_n = 6m_{n-1} - m_{n-2} - 4$ and $a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8$. Then 2, 7, 36, 205, 1190 and 6931 are first six examples of the sequence neo balancing number's indices(neo balancing numbers) meanwhile $-1, 1, 13, 83, 491$ and 2869 (neo balancer) are first six examples of the sequence neo balancer's indices, where $a_m = 2m$. We can demonstrate by putting $a_m = 2m$ to the equation (2.1) as the following. If $m=36$ and $r=13$, then we obtain

$$\begin{aligned} a_1 + a_2 + \cdots + a_{36-1} &= a_{36-1} + a_{36} + \cdots + a_{36+13} \\ 2+4+\cdots+70 &= 70+72+\cdots+98 \\ 1260 &= 1260. \end{aligned}$$

Hence, 4, 14, 72, 410, 2380 and 13862 are first six example of sequence neo balancing numbers (The twice of neo balancing numbers), where $a_m = 2m$.

Remark 3.4 The sequences $a_m = 2m-2$ and $a_m = 2m$ on sequence neo balancing numbers are twice of neo balancing numbers.

Example 3.5 The sequence neo balancing numbers in sequences $a_m = 2m+1$ and $a_m = 2m-3$.

By theorem 2.1 and theorem 2.3, we can choose $k=1$ or $k=-3$, so let $k=1$. Then the recurrence relations for the sequence neo balancing number $a_m = 2m+1$ are $m_n = 6m_{n-1} - m_{n-2} - 2$ and $a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8$. Then 2, 9, 50, 289, 1682 and 9801 are first six examples of the sequence neo balancing number's indices and $-1, 2, 19, 118, 695$ and 4058 are first six examples of the sequence neo balancer's indices, where $a_m = 2m+1$. We can demonstrate by putting $a_m = 2m+1$ to the equation (2.1) as the following. If $m=50$ and $r=19$, then we obtain

$$\begin{aligned} a_1 + a_2 + \cdots + a_{50-1} &= a_{50-1} + a_{50} + \cdots + a_{50+19} \\ 3+5+\cdots+99 &= 99+101+\cdots+139 \\ 2499 &= 2499. \end{aligned}$$

Hence, 5, 19, 101, 579, 3365 and 19603 are first six example of sequence neo balancing numbers, where $a_m = 2m+1$.

Example 3.6 The sequence neo balancing numbers in sequences $a_m = 2m+3$ and $a_m = 2m-5$.

By theorem 2.2 and theorem 2.3, we can choose $k=3$ or $k=-5$, so let $k=3$. Then the recurrence relations for the sequence neo balancing number $a_m = 2m+3$ are $m_{2n-1} = 6m_{2n-3} - m_{2n-5} + 2$ and $a_{m_{2n-1}} = 6a_{m_{2n-3}} - a_{m_{2n-5}} - 8$ and another one sequence $m_{2n} = 6m_{2n-2} - m_{2n-4} + 2$ and $a_{m_{2n}} = 6a_{m_{2n-2}} - a_{m_{2n-4}} - 8$ where $n \geq 3$. Then 2, 6, 13, 37, 78, 218, 457, 1273, 2666 and 7422 are first ten examples of sequence neo balancing number's indices and $-1, 1, 4, 14, 31, 89, 188, 526, 1103$ and 3073 are first ten examples of sequence neo balancer's indices, where $a_m = 2m+3$. We can demonstrate by putting $a_m = 2m+3$ to the equation (2.1) as the following. If $m=37$, then we obtain $r=14$ and

$$\begin{aligned} a_1 + a_2 + \cdots + a_{37-1} &= a_{37-1} + a_{37} + \cdots + a_{37+14} \\ 5+7+\cdots+75 &= 75+77+\cdots+105 \\ 1440 &= 1440. \end{aligned}$$

Hence, 7, 15, 29, 77, 159, 439, 917, 2549, 5335 and 14847 are first ten example of sequence neo balancing numbers, where $a_m = 2m+3$. Similarly, we let $k=-5$. Then 1, 2, 6, 10, 17, 41, 82, 222, 461 and 1277 are first ten examples of sequence neo balancing number's indices simultaneously -3, -1, 7, 15, 29, 77, 159, 439, 917 and 2549 are first ten example of sequence neo balancing numbers, where $a_m = 2m-5$.

Discussion

In this study, we explored sequence neo balancing numbers and extended the concept from sequence balancing numbers, as published by Panda [4]. The study of sequence balancing numbers involves specific sequences, such as $a_m = 2m-1$ in [4], but we established the sequence $a_m = 2m+k$ for sequence neo balancing numbers where k is any integer. Additionally, we derived their recurrence relations and found that our findings reveal the

DOI: 10.60101/jarst.2024.257946

significant role of combining the generalized Pell's equation and Brahmagupta's identity in deriving solutions. Specifically, we successfully obtained two sequences that demonstrate analogous behavior under specific conditions. An interesting result emerged from analyzing the indices, showing that certain sequences share similar solutions under particular constraints. This similarity highlights the structured nature of sequence neo balancing numbers and opens up possibilities for further algebraic exploration. Furthermore, we established that for any integer k, the sequence of neo balancing numbers is always accompanied by equivalent sequences, affirming the robustness of the generalization. These findings provide a foundation for future investigations into more complex relationships among generalized sequences.

Further work could explore connections between these sequences and other classical number theory problems or identify practical applications in computational mathematics and cryptography.

CONCLUSIONS

We have investigated the generalization of sequence of neo balancing numbers and their recurrence relations by extending the catalyst that the certain sequence $a_m = 2m \pm 1$ in sequence balancing number to any integer k in sequence neo balancing number. We have solved the equation by composing equations (2.4) and Pell's equation via Brahmagupta's identity. Hence, we have obtained two sequence solutions (X_n, Y_n) and (X_n^*, Y_n^*). Subsequently, we have known that the 2 given solutions are sometimes analogous. Then we have delved into each index. We have found that there exist $k = -3, -2, -1, 0, 1$ which can force (X_n, Y_n) similar to (X_n^*, Y_n^*). Then the sequence $a_m = 2m + k$ which is generated by (X_n, Y_n) and (X_n^*, Y_n^*) are analogous. Alternatively, if $k \neq -3, -2, -1, 0, 1$, there are 2 sequences $a_m = 2m + k$ for sequence neo balancing number which is generated by (X_n, Y_n) and (X_n^*, Y_n^*) such that each sequence individually satisfy the recurrence relations. Furthermore, for all integer $k \neq -1$, the sequence $a_m = 2m + k$ for sequence neo balancing number always has another equivalent sequence.

ACKNOWLEDGEMENT

This work is supported by Faculty of Science, Burapha University, Thailand.

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DOI: 10.60101/jarst.2024.257946