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All solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+2}$

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ABSTRACT

In the history of mathematics, many mathematical researchers have investigated the Diophantine equation in the form $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{v}$, where x, y, z, u and v are positive integers. Without loss of generality, we may assume that $x \le y \le z$. This Diophantine equation, also known as the Egyptian fraction equation of length 3, is to write the fraction as a sum of three fractions with the numerator being one and the denominators being different positive integers. Examples of research such as, in 2021, Sandor and Atanassov studied and found that the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1}$ has forty-four positive integer solutions. In this paper, we will study and find the complete positive integer solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+2}$, by using elementary methods of number theory and computer calculations. In the process, we can see that $1 \le x \le 9$. Then, we will consider separately the value of a positive integer x in nine cases. The first case is impossible. For the second and third cases, we will separate to consider the value of y. For the remaining cases, we will separate to consider the value of u. The research results showed that all positive integer solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+2}$ are eighty-seven positive integer solutions. Moreover, from the steps to find the above positive integer solutions, we expect that it can be used to find the complete positive integer solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+k}$, where k is a positive integer with $k \ge 3$.

Keywords: Diophantine equation, Positive integer solution, Egyptian fraction equation

INTRODUCTION

A Diophantine equation is an equation, for which only integer solutions are of interest. One of the Diophantine equations that has caught the attention of many researchers is the equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{v}$, where x, y, z, u and v are positive integers with $x \le y \le z$. For example, in 2013 Rabago and Tagle [1] found that if $u = 1$ and $v = 2$, then the equation has only ten positive integer solutions. Tadee and Poopra [2] proved that if $u = 1$ and $v = 3$, then the equation has exactly twenty-one positive integer solutions. Meanwhile, Delang [3] showed some conditions of the non-existence of the solutions for the equation, where $u = 4$ and $v > 1$. Later, Kishan, Rani and Agarwal [4] investigated all positive integer solutions of the equation, where $u = 3$ and $v \equiv 1 \pmod 2$. In 2021, Zhao, Lu and Wang [5] studied the equation for some prime number v. In the same year, Banderier et al. [6] gave the bounds to the number of integer solutions to the equation, where u and v are relatively prime. Recently, Sandor and Atanassov [7] proved that if $v = u + 1$, then the equation has forty-four solutions.

In this article, we will show that if $v = u + 2$, then the equation has eighty-seven solutions.

MATERIALS AND METHODS

Let x, y, z and u be positive integers with $x \le y \le z$ such that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+2}. \quad (1)$$

Since $x \le y \le z$, we get

$$\frac{1}{z} \le \frac{1}{y} \le \frac{1}{x}. \quad (2)$$

From (1) and (2), it implies that $\frac{u}{u+2} \le \frac{3}{x}$. Then

$$u(x-3) \le 6. \quad (3)$$

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Since $u \geq 1$, we have $x \leq 9$. We consider the following cases:

Case 1. $\boldsymbol{x = 1}$. Then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1$. From (1), we obtain $\frac{u}{u+2} > 1$. This is impossible.

Case 2. $\boldsymbol{x = 2}$. From (1), we have

$$\frac{1}{y} + \frac{1}{z} = \frac{u-2}{2u+4}. \tag{4}$$

Thus $u \geq 3$. From (2), it implies that $\frac{u-2}{2u+4} \leq \frac{2}{y}$. Therefore,

$$u(y-4) \leq 2y + 8. \tag{5}$$

Subcase 2.1. $y = 1$. Then $x > y$, a contradiction.

Subcase 2.2. $y = 2$. Then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 + \frac{1}{z} > 1$. From (1), we have $\frac{u}{u+2} > 1$, which is impossible.

Subcase 2.3. $y = 3$. From (4), we get $\frac{1}{z} = \frac{u-10}{6u+12}$. Then $z = 6 + \frac{72}{u-10}$. Since z is a positive integer, it implies that $u - 10 = 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36$ or 72. Therefore, $u = 11, 12, 13, 14, 16, 18, 19, 22, 28, 34, 46$ or 82.
If $u = 11$, then $z = 78$ and so $(x, y, z, u) = (2, 3, 78, 11)$.
If $u = 12$, then $z = 42$ and so $(x, y, z, u) = (2, 3, 42, 12)$.
If $u = 13$, then $z = 30$ and so $(x, y, z, u) = (2, 3, 30, 13)$.
If $u = 14$, then $z = 24$ and so $(x, y, z, u) = (2, 3, 24, 14)$.
If $u = 16$, then $z = 18$ and so $(x, y, z, u) = (2, 3, 18, 16)$.
If $u = 18$, then $z = 15$ and so $(x, y, z, u) = (2, 3, 15, 18)$.
If $u = 19$, then $z = 14$ and so $(x, y, z, u) = (2, 3, 14, 19)$.
If $u = 22$, then $z = 12$ and so $(x, y, z, u) = (2, 3, 12, 22)$.
If $u = 28$, then $z = 10$ and so $(x, y, z, u) = (2, 3, 10, 28)$.
If $u = 34$, then $z = 9$ and so $(x, y, z, u) = (2, 3, 9, 34)$.
If $u = 46$, then $z = 8$ and so $(x, y, z, u) = (2, 3, 8, 46)$.
If $u = 82$, then $z = 7$ and so $(x, y, z, u) = (2, 3, 7, 82)$.

Subcase 2.4. $y = 4$. From (4), we have $\frac{1}{z} = \frac{u-6}{4u+8}$. Then $z = 4 + \frac{32}{u-6}$. Since z is a positive integer, it implies that $u - 6 = 1, 2, 4, 8, 16$ or 32. Therefore, $u = 7, 8, 10, 14, 22$ or 38.
If $u = 7$, then $z = 36$ and so $(x, y, z, u) = (2, 4, 36, 7)$.
If $u = 8$, then $z = 20$ and so $(x, y, z, u) = (2, 4, 20, 8)$.
If $u = 10$, then $z = 12$ and so $(x, y, z, u) = (2, 4, 12, 10)$.
If $u = 14$, then $z = 8$ and so $(x, y, z, u) = (2, 4, 8, 14)$.
If $u = 22$, then $z = 6$ and so $(x, y, z, u) = (2, 4, 6, 22)$.
If $u = 38$, then $z = 5$ and so $(x, y, z, u) = (2, 4, 5, 38)$.

Subcase 2.5. $y = 5$. From (5), we obtain $u \leq 18$. From (4), it follows that $z = \frac{10u+20}{3u-14}$. Since z is a positive integer, we get $u = 5, 6, 8, 13$ or 18.
If $u = 5$, then $z = 70$ and so $(x, y, z, u) = (2, 5, 70, 5)$.
If $u = 6$, then $z = 20$ and so $(x, y, z, u) = (2, 5, 20, 6)$.
If $u = 8$, then $z = 10$ and so $(x, y, z, u) = (2, 5, 10, 8)$.
If $u = 13$, then $z = 6$ and so $(x, y, z, u) = (2, 5, 6, 13)$.
If $u = 18$, then $z = 5$ and so $(x, y, z, u) = (2, 5, 5, 18)$.

Subcase 2.6. $y = 6$. From (5), we obtain $u \leq 10$. From (4), it follows that $z = \frac{3u+6}{u-4}$. Since z is a positive integer, we get $u = 5, 6, 7$ or 10.
If $u = 5$, then $z = 21$ and so $(x, y, z, u) = (2, 6, 21, 5)$.
If $u = 6$, then $z = 12$ and so $(x, y, z, u) = (2, 6, 12, 6)$.
If $u = 7$, then $z = 9$ and so $(x, y, z, u) = (2, 6, 9, 7)$.
If $u = 10$, then $z = 6$ and so $(x, y, z, u) = (2, 6, 6, 10)$.

Subcase 2.7. $y = 7$. From (5), we have $u \leq 7$. From (4), it follows that $z = \frac{14u+28}{5u-18}$. Since z is a positive integer, we get $u = 4$ or 5.
If $u = 4$, then $z = 42$ and so $(x, y, z, u) = (2, 7, 42, 4)$.
If $u = 5$, then $z = 14$ and so $(x, y, z, u) = (2, 7, 14, 5)$.

Subcase 2.8. $y = 8$. From (5), we have $u \leq 6$. From (4), it follows that $z = \frac{8u+16}{3u-10}$. Since z is a positive integer, we get $u = 4$ or 6.
If $u = 4$, then $z = 24$ and so $(x, y, z, u) = (2, 8, 24, 4)$.
If $u = 6$, then $z = 8$ and so $(x, y, z, u) = (2, 8, 8, 6)$.

Subcase 2.9. $y = 9$. From (5), we have $u \leq 5$. From (4), it follows that $z = \frac{18u+36}{7u-22}$. Since z is a positive integer, we get $u = 4$. Then $z = 18$ and $(x, y, z, u) = (2, 9, 18, 4)$.

Subcase 2.10. $y = 10$. From (5), we have $u \leq 4$. From (4), it follows that $z = \frac{5u+10}{2u-6}$. Since z is a positive integer, we get $u = 4$. Then $z = 15$ and $(x, y, z, u) = (2, 10, 15, 4)$.

Subcase 2.11. $y = 11$. From (5), we have $u \leq 4$. From (4), it follows that $z = \frac{22u+44}{9u-26}$. Since z is a positive integer, we get $u = 3$. Then $z = 110$ and $(x, y, z, u) = (2, 11, 110, 3)$.

Subcase 2.12. $y = 12$. From (5), we have $u \leq 4$. From (4), it follows that $z = \frac{12u+24}{5u-14}$. Since z is a positive integer, we get $u = 3$ or 4.
If $u = 3$, then $z = 60$ and so $(x, y, z, u) = (2, 12, 60, 3)$.
If $u = 4$, then $z = 12$ and so $(x, y, z, u) = (2, 12, 12, 4)$.

Subcase 2.13. $y \geq 13$. Then $\frac{2y+8}{y-4} < 4$. From (5) and (4), we have $u = 3$ and $\frac{1}{y} + \frac{1}{z} = \frac{1}{10}$. Then $z = \frac{10y}{y-10}$. From (2), it follows that $\frac{2}{y} \geq \frac{1}{10}$ or $y \leq 20$. Since z is a positive integer, we get $y = 14, 15$ or 20.
If $y = 14$, then $z = 35$ and so $(x, y, z, u) = (2, 14, 35, 3)$.
If $y = 15$, then $z = 30$ and so $(x, y, z, u) = (2, 15, 30, 3)$.

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If $y = 20$, then $z = 20$ and so $(x, y, z, u) = (2, 20, 20, 3)$.

Case 3. $\boldsymbol{x = 3}$. From (1), we have

$$\frac{1}{y} + \frac{1}{z} = \frac{2u-2}{3u+6}. \quad (6)$$

From (2), it implies that $\frac{2u-2}{3u+6} \leq \frac{2}{y}$. Therefore,

$$u(y-3) \leq y + 6. \quad (7)$$

Since $x \leq y$, we have $3 \leq y$.

<u>Subcase 3.1.</u> $y = 3$. From (6), we have $\frac{1}{z} = \frac{u-4}{3u+6}$. Then $z = 3 + \frac{18}{u-4}$. Since z is a positive integer, we get $u - 4 = 1, 2, 3, 6, 9$ or 18. Therefore, $u = 5, 6, 7, 10, 13$ or 22.
If $u = 5$, then $z = 21$ and so $(x, y, z, u) = (3, 3, 21, 5)$.
If $u = 6$, then $z = 12$ and so $(x, y, z, u) = (3, 3, 12, 6)$.
If $u = 7$, then $z = 9$ and so $(x, y, z, u) = (3, 3, 9, 7)$.
If $u = 10$, then $z = 6$ and so $(x, y, z, u) = (3, 3, 6, 10)$.
If $u = 13$, then $z = 5$ and so $(x, y, z, u) = (3, 3, 5, 13)$.
If $u = 22$, then $z = 4$ and so $(x, y, z, u) = (3, 3, 4, 22)$.

<u>Subcase 3.2.</u> $y = 4$. From (7) and (6), we have $u \leq 10$ and $z = \frac{12u+24}{5u-14}$, consequently. Since z is a positive integer, we have $u = 3, 4, 6$ or 10.
If $u = 3$, then $z = 60$ and so $(x, y, z, u) = (3, 4, 60, 3)$.
If $u = 4$, then $z = 12$ and so $(x, y, z, u) = (3, 4, 12, 4)$.
If $u = 6$, then $z = 6$ and so $(x, y, z, u) = (3, 4, 6, 6)$.
If $u = 10$, then $z = 4$ and so $(x, y, z, u) = (3, 4, 4, 10)$.

<u>Subcase 3.3.</u> $y = 5$. From (7) and (6), we have $u \leq 5$ and $z = \frac{15u+30}{7u-16}$, consequently. Since z is a positive integer, we have $u = 3$. Then $z = 15$ and so $(x, y, z, u) = (3, 5, 15, 3)$.

<u>Subcase 3.4.</u> $y = 6$. From (7) and (6), we have $u \leq 4$ and $z = \frac{2u+4}{u-2}$, consequently. Since z is a positive integer, we have $u = 3$ or 4.
If $u = 3$, then $z = 10$ and so $(x, y, z, u) = (3, 6, 10, 3)$.
If $u = 4$, then $z = 6$ and so $(x, y, z, u) = (3, 6, 6, 4)$.

<u>Subcase 3.5.</u> $y = 7$. From (7) and (6), we have $u \leq 3$ and $z = \frac{21u+42}{11u-20}$, consequently. Since z is a positive integer, we have $u = 2$. Then $z = 42$ and so $(x, y, z, u) = (3, 7, 42, 2)$.

<u>Subcase 3.6.</u> $y \geq 8$. Then $\frac{y+6}{y-3} < 3$. From (7) and (6), we have $u = 2$ and $\frac{1}{y} + \frac{1}{z} = \frac{1}{6}$. Then $z = \frac{6y}{y-6}$. From (2), it follows that $\frac{2}{y} \geq \frac{1}{6}$ or $y \leq 12$. Since z is a positive integer, we get $y = 8, 9, 10$ or 12.
If $y = 8$, then $z = 24$ and so $(x, y, z, u) = (3, 8, 24, 2)$.
If $y = 9$, then $z = 18$ and so $(x, y, z, u) = (3, 9, 18, 2)$.
If $y = 10$, then $z = 15$ and so $(x, y, z, u) = (3, 10, 15, 2)$.
If $y = 12$, then $z = 12$ and so $(x, y, z, u) = (3, 12, 12, 2)$.

Case 4. $\boldsymbol{x = 4}$. From (3) and (1), we have $u \leq 6$ and

$$\frac{1}{y} + \frac{1}{z} = \frac{3u-2}{4u+8}. \quad (8)$$

<u>Subcase 4.1.</u> $u = 1$. From (2) and (8), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{1}{12}.$$

Then $y \leq 24$ and $z = \frac{12y}{y-12}$. Since z is a positive integer, we have $y = 13, 14, 15, 16, 18, 20, 21$ or 24.
If $y = 13$, then $z = 156$ and so $(x, y, z, u) = (4, 13, 156, 1)$.
If $y = 14$, then $z = 84$ and so $(x, y, z, u) = (4, 14, 84, 1)$.
If $y = 15$, then $z = 60$ and so $(x, y, z, u) = (4, 15, 60, 1)$.
If $y = 16$, then $z = 48$ and so $(x, y, z, u) = (4, 16, 48, 1)$.
If $y = 18$, then $z = 36$ and so $(x, y, z, u) = (4, 18, 36, 1)$.
If $y = 20$, then $z = 30$ and so $(x, y, z, u) = (4, 20, 30, 1)$.
If $y = 21$, then $z = 28$ and so $(x, y, z, u) = (4, 21, 28, 1)$.
If $y = 24$, then $z = 24$ and so $(x, y, z, u) = (4, 24, 24, 1)$.

<u>Subcase 4.2.</u> $u = 2$. From (2) and (8), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{1}{4}.$$

Then $y \leq 8$ and $z = \frac{4y}{y-4}$. Since z is a positive integer, we have $y = 5, 6$ or 8.
If $y = 5$, then $z = 20$ and so $(x, y, z, u) = (4, 5, 20, 2)$.
If $y = 6$, then $z = 12$ and so $(x, y, z, u) = (4, 6, 12, 2)$.
If $y = 8$, then $z = 8$ and so $(x, y, z, u) = (4, 8, 8, 2)$.

<u>Subcase 4.3.</u> $u = 3$. From (2) and (8), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{7}{20}.$$

Then $y \leq 5$ and $z = \frac{20y}{7y-20}$. Since z is a positive integer, we get $y = 4$. Then $z = 10$ and $(x, y, z, u) = (4, 4, 10, 3)$.

<u>Subcase 4.4.</u> $u = 4$. From (2) and (8), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{5}{12}.$$

Then $y \leq 4$ and $z = \frac{12y}{5y-12}$. Since z is a positive integer, we get $y = 4$. Then $z = 6$ and $(x, y, z, u) = (4, 4, 6, 4)$.

<u>Subcase 4.5.</u> $u = 5$. From (2) and (8), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{13}{28}.$$

Then $y \leq 4$ and $z = \frac{28y}{13y-28}$. This is impossible since z is a positive integer.

<u>Subcase 4.6.</u> $u = 6$. From (2) and (8), it follows that

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$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{1}{2}.$$

Then $y \leq 4$ and $z = \frac{2y}{y-2}$. Since z is a positive integer, we have $y = 4$. Then $z = 4$ and so $(x, y, z, u) = (4, 4, 4, 6)$.

Case 5. $\boldsymbol{x} = \mathbf{5}$. From (3) and (1), we have $u \leq 3$ and

$$\frac{1}{y} + \frac{1}{z} = \frac{4u-2}{5u+10}. \quad (9)$$

Subcase 5.1. $u = 1$. From (2) and (9), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{2}{15}.$$

Then $y \leq 15$ and $z = \frac{15y}{2y-15}$. Since z is a positive integer, we have $y = 8, 9, 10, 12$ or 15.
If $y = 8$, then $z = 120$ and so$(x, y, z, u) = (5, 8, 120, 1)$.
If $y = 9$, then $z = 45$ and so $(x, y, z, u) = (5, 9, 45, 1)$.
If $y = 10$, then $z = 30$ and so $(x, y, z, u) = (5, 10, 30, 1)$.
If $y = 12$, then $z = 20$ and so $(x, y, z, u) = (5, 12, 20, 1)$.
If $y = 15$, then $z = 15$ and so $(x, y, z, u) = (5, 15, 15, 1)$.

Subcase 5.2. $u = 2$. From (2) and (9), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{3}{10}.$$

Then $y \leq 6$ and $z = \frac{10y}{3y-10}$. Since z is a positive integer, we get $y = 5$. Thus $z = 10$ and $(x, y, z, u) = (5, 5, 10, 2)$.

Subcase 5.3. $u = 3$. From (2) and (9), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{2}{5}.$$

Then $y \leq 5$ and $z = \frac{5y}{2y-5}$. Since z is a positive integer, we get $y = 5$. Thus $z = 5$ and $(x, y, z, u) = (5, 5, 5, 3)$.

Case 6. $\boldsymbol{x} = \mathbf{6}$. From (3) and (1), we have $u \leq 2$ and

$$\frac{1}{y} + \frac{1}{z} = \frac{5u-2}{6u+12}. \quad (10)$$

Subcase 6.1. $u = 1$. From (2) and (10), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{1}{6}.$$

Then $y \leq 12$ and $z = \frac{6y}{y-6}$. Since z is a positive integer, we have $y = 7, 8, 9, 10$ or 12.
If $y = 7$, then $z = 42$ and so $(x, y, z, u) = (6, 7, 42, 1)$.
If $y = 8$, then $z = 24$ and so $(x, y, z, u) = (6, 8, 24, 1)$.
If $y = 9$, then $z = 18$ and so $(x, y, z, u) = (6, 9, 18, 1)$.
If $y = 10$, then $z = 15$ and so $(x, y, z, u) = (6, 10, 15, 1)$.
If $y = 12$, then $z = 12$ and so $(x, y, z, u) = (6, 12, 12, 1)$.

Subcase 6.2. $u = 2$. From (2) and (10), it follows that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{1}{3}.$$

Then $y \leq 6$ and $z = \frac{3y}{y-3}$. Since z is a positive integer, we have $y = 6$. Then $z = 6$ and so $(x, y, z, u) = (6, 6, 6, 2)$.

Case 7. $\boldsymbol{x} = \mathbf{7}$. From (3), we have $u = 1$. From (1) and (2), it implies that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{4}{21}.$$

Then $z = \frac{21y}{4y-21}$ and $y \leq 10$, consequently. Since z is a positive integer, we have $y = 7$. Therefore, $z = 21$ and so $(x, y, z, u) = (7, 7, 21, 1)$.

Case 8. $\boldsymbol{x} = \mathbf{8}$. From (3), we have $u = 1$. From (1) and (2), it implies that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{5}{24}.$$

Then $z = \frac{24y}{5y-24}$ and $y \leq 9$, consequently. Since z is a positive integer, we have $y = 8$. Therefore, $z = 12$ and so $(x, y, z, u) = (8, 8, 12, 1)$.

Case 9. $\boldsymbol{x} = \mathbf{9}$. From (3), we have $u = 1$. From (1) and (2), it implies that

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} = \frac{2}{9}.$$

Then $z = \frac{9y}{2y-9}$ and $y \leq 9$, consequently. Since z is a positive integer, we have $y = 9$. Therefore, $z = 9$ and so $(x, y, z, u) = (9, 9, 9, 1)$.

RESULTS AND DISCUSSION

In the previous section, we show all positive integer solutions of the Diophantine equation.

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{u}{u+2},$$

where x, y, z and u are positive integers with $x \leq y \leq z$, are exactly eighty-seven solutions including: $(x, y, z, u) =$ $(2,3,78,11)$, $(2,3,42,12)$, $(2,3,30,13)$, $(2,3,24,14)$, $(2,3,18,16)$, $(2,3,15,18)$, $(2,3,14,19)$, $(2,3,12,22)$, $(2,3,10,28)$, $(2,3,9,34)$, $(2,3,8,46)$, $(2,3,7,82)$, $(2,4,36,7)$, $(2,4,20,8)$, $(2,4,12,10)$, $(2,4,8,14)$, $(2,4,6,22)$, $(2,4,5,38)$, $(2,5,70,5)$, $(2,5,20,6)$, $(2,5,10,8)$, $(2,5,6,13)$, $(2,5,5,18)$, $(2,6,21,5)$, $(2,6,12,6)$, $(2,6,9,7)$, $(2,6,6,10)$, $(2,7,42,4)$, $(2,7,14,5)$, $(2,8,24,4)$, $(2,8,8,6)$, $(2,9,18,4)$, $(2,10,15,4)$, $(2,11,110,3)$, $(2,12,60,3)$, $(2,12,12,4)$, $(2,14,35,3)$, $(2,15,30,3)$, $(2,20,20,3)$, $(3,3,21,5)$, $(3,3,12,6)$, $(3,3,9,7)$, $(3,3,6,10)$, $(3,3,5,13)$, $(3,3,4,22)$, $(3,4,60,3)$, $(3,4,12,4)$, $(3,4,6,6)$,

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$(3,4,4,10)$, $(3,5,15,3)$, $(3,6,10,3)$, $(3,6,6,4)$, $(3,7,42,2)$, $(3,8,24,2)$, $(3,9,18,2)$, $(3,10,15,2)$, $(3,12,12,2)$, $(4,13,156,1)$, $(4,14,84,1)$, $(4,15,60,1)$, $(4,16,48,1)$, $(4,18,36,1)$, $(4,20,30,1)$, $(4,21,28,1)$, $(4,24,24,1)$, $(4,5,20,2)$, $(4,6,12,2)$, $(4,8,8,2)$, $(4,4,10,3)$, $(4,4,6,4)$, $(4,4,4,6)$, $(5,8,120,1)$, $(5,9,45,1)$, $(5,10,30,1)$, $(5,12,20,1)$, $(5,15,15,1)$, $(5,5,10,2)$, $(5,5,5,3)$, $(6,7,42,1)$, $(6,8,24,1)$, $(6,9,18,1)$, $(6,10,15,1)$, $(6,12,12,1)$, $(6,6,6,2)$, $(7,7,21,1)$, $(8,8,12,1)$, $(9,9,9,1)$.

CONCLUSIONS

In the process of finding the positive integer solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+2}$, only basic mathematical knowledge is required. It is expected that the above procedure can be applied to find all positive integer solutions of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+k}$, where x, y, z, u and k are positive integers with $x \leq y \leq z$ and $k \geq 3$. Thus, it is interesting to study and research further.

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