Received: March 14, 2023 | Revised: July 17, 2023 | Accepted: July 18, 2023

JOURNAL OF APPLIED RESEARCH ON SCIENCE AND TECHNOLOGY
JARST
INSTITUTE OF RESEARCH AND DEVELOPMENT
RAJAMANGALA UNIVERSITY OF TECHNOLOGY THANYABURI
ISSN ONLINE: 2773-9473
https://ph01.tci-thaijo.org/index.php/rmutt-journal

Further investigation on fixed point theorems via C-class functions in extended b-metric spaces

Tiwabhorn Khanpanuk* **and Chainarong Khunpanuk**

Mathematics and Computing Science Program, Faculty of Science and Technology, Phetchabun Rajabhat University, Phetchabun 67000, THAILAND

*Corresponding author: tiwabhorn.kha@pcru.ac.th

ABSTRACT

The purpose of this research project is to develop new theories, discuss, and extend some recent common fixed point results established when the underlying ambient space is an extended b-metric space and the contraction condition involves a new class of ψ-φ-C-contraction type mappings where ψ is the altering distance function and φ is the ultra-altering distance function. The unique fixed point theorems for such mappings in the setting of ψ-φ-complete metric spaces are proven. We also prove the fixed point theorem in partially ordered metric spaces. Moreover, some examples supporting the main results are given. Our results extend and generalize corresponding results in the literature. The start of the development of the theory of fixed points is tied to the end of the 19th century. The method of successive approximations is used in order to prove the solution's existence and uniqueness at the beginning of differential and integral equations. This branch of nonlinear analysis has been developed through various classes of spaces, such as metric spaces, topological spaces, probabilistic metric spaces, fuzzy metric spaces, and others. In developing the theory of fixed points, achievements are applied in various sciences, such as optimization, economics, and approximation theory. A very important step in the development of fixed point theory was taken by A.H. Ansari through the introduction of a C-class function. Using C-class functions, we generalize some known fixed point results, and Kamran et al. introduced a new intuitive concept of distance measure to extend the notion of b-metric space by further weakening the triangle inequality.

Keywords: Fixed point, Extended b-metric space, C-class function

INTRODUCTION

It is widely known that Thailand has a strategy to develop the nation using science and technology, particularly in applying scientific understanding to biotechnology, materials science technology, and suitable use of electronic and computer technology. Therefore, acquiring information is crucial for academic success, and it is evident that mathematics is an essential instrument for discovering and growing those above. Using knowledge of mathematics to create a mathematical model to predict the effect of soil temperature on plant growth, for example, In 2020, Boonwan J, et al. [3] created a mathematical model to predict soil temperature for the growth of chrysanthemum sprouts. However, fixed point theory is another tool used to solve many nonlinear problems in mathematical analysis. In 1922, Stefan Banach [17] began modern functional analysis and subsequently studied how to extend this principle for generalized contraction transmission in many different ways. Later, many researchers extended metric spaces to generalize metric spaces such as partial, G-metric, and cone metric spaces. For fixed point theorems in metric spaces, see [2, 4, 6, 10-13, 15, 16] and references therein. The concept of a b-metric space was introduced by Bakhtin [7], and Czerwik [9] generalized the structure of metric space by weakening the triangle inequality called a b-metric space and proved some results of the fixed point theorem in b-metric spaces. Further, many authors use the concepts for trade measures [5] and to measure ice floes [8]. In this context, Kamran and his co-authors [14] introduced the concept of extended b-metric space by further weakening the triangle inequality. Later in 2014, Ansari [1] introduced the concept of C-class functions and proved the unique fixed point theorems for specific contractive mappings concerning the C-class functions.

Following the above results, the motivation of this paper is to introduce the article's idea on some fixed point theorems for C-class functions in b-metric

DOI: 10.60101/jarst.2023.251308

Journal of Applied Research on Science and Technology

spaces [18] to cover more general cases. We then prove the existence of unique fixed points in extended b-metric space. Further, some examples supporting the main results are given.

MATERIALS AND METHODS

This section has compiled definitions and relevant theorems, a tool for further study and research in the main results.

Definition 2.1. [14] Let X be a nonempty set and $\theta: X\times X \to [1,\infty)$. A function $d_\theta: X\times X \to [1,\infty)$ is called an extended b metric space, if for all $x,y,z \in X$, it satisfies

EbM 1. $d_\theta(x,y) = 0$ if and only if $x=y$

EbM 2. $d_\theta(x,y) = d_\theta(y,x)$

EbM 3. $d_\theta(x,z) \le \theta(x,z)[d_\theta(x,y) + d_\theta(y,z)]$

(X, d_θ) is called an extended b-metric space.

To show the concreteness of the idea of extended b-metric space, we give some examples of extended b-metric space in the following:

Example 2.2. Let $X = \{3,4,5\}$, $\theta: X\times X \to R^+$ and $d_\theta: X\times X \to R^+$ as $\theta(x,y) = x + y$

$d_\theta(3,3) = d_\theta(4,4) = d_\theta(5,5) = 0,$

$d_\theta(3,4) = d_\theta(4,3) = 50,$

$d_\theta(3,5) = d_\theta(5,3) = 250,$

$d_\theta(4,5) = d_\theta(5,4) = 400.$

It is obvious from definition 2.1 in EbM 1. and EbM 2. We then prove EbM 3. in the following way:

$50 = d_\theta(3,4) \le \theta(3,4)[d_\theta(3,5) + d_\theta(5,4)] = 4{,}550$

$250 = d_\theta(3,5) \le \theta(3,5)[d_\theta(3,4) + d_\theta(4,5)] = 3{,}600$

$450 = d_\theta(4,5) \le \theta(4,5)[d_\theta(4,3) + d_\theta(3,5)] = 2{,}700$

Therefore, (X, d_θ) is an extended b-metric space.

Example 2.3. [14] Let $X = [0,+\infty)$ and $\theta: X\times X \to [1,+\infty)$, $\theta(x,y) = 1 + x + y$.

Define $d_\theta: X\times X \to [1,+\infty)$, as

$d_\theta(x,y) = x + y$, for $x,y \in X, x \ne y$

$d_\theta(x,y) = 0$, for $x,y \in X, x = y$.

It is easy to show EbM1. and EbM 2. Hold. For EbM 3. We split the consideration into four cases:

Case 1. If $x = y$, we have EbM 3. hold.

Case 2. If $x \ne y, x = z$, then

$\theta(x,y)\,[d_\theta(x,z) + d_\theta(z,y)]$
$= (1 + x + y)[0 + (z + y)]$
$= (1 + x + y)(z + y)$
$\ge x + y = d_\theta(x,y).$

Case 3. If $x \ne y, y = z$, then

$\theta(x,y)[d_\theta(x,z) + d_\theta(z,y)]$
$= (1 + x + y)[(x + z) + 0]$
$= (1 + x + y)(x + y)$
$\ge x + y = d_\theta(x,y)$

Case 4. If $x \ne y, y \ne z, x \ne z$, then

$\theta(x,y)[d_\theta(x,z) + d_\theta(z,y)]$
$= (1 + x + y)[(x + z) + (z + y)]$
$\ge x + 2z + y$
$\ge x + y = d_\theta(x,y).$

In conclusion, for any $x,y,z \in X$,

$d_\theta(x,z) \le \theta(x,z)[d_\theta(x,y) + d_\theta(y,z)].$

Hence, (X, d_θ) is an extended b metric space.

In the next section, the concepts of convergence, Cauchy sequence, and completeness are introduced in extended b-metric space.

Definition 2.4. [14] Let (X, d_θ) be an extended b-metric space. Then a sequence (x_n) in X is said to be:

1) convergent if and only if there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$,

2) Cauchy if and only if $\lim_{m,n\to\infty} d_\theta(x_m, x_n) = 0$.

Definition 2.5. An extended b metric space. (X, d_θ) is complete if every Cauchy sequence in X is convergent.

Lemma 2.6. Let (X, d_θ) be a complete extended b-metric space. If d_θ is continuous, then every convergent sequence has a unique limit.

Definition 2.7. [1] A mapping $F: [0,\infty)^2 \to R$ is called a C-class function if it is continuous and for all $a, b \in [0,\infty)$

1) $F(a,b) \le a$;

2) $F(a,b) = a$ implies that either $a = 0$ or $b = 0$.

We denote C as the family set of all C-class functions.

Example 2.8. [1] The following functions $F: [0,\infty)^2 \to R^2$ are elements in C. For all $a, b \in [0,\infty)$

1) $f(a,b) = a - b, f(a,b) = a \to b = 0;$

2) $f(a,b) = xa, f(a,b) = a \to a = 0$ where $0 < x < 1;$

3) $f(a,b) = (a + x)^{\left(\frac{1}{1+b}y\right)} - x, f(a,b) = a \to b = 0$ where $x > 1, y \in (0,\infty);$

4) $f(a,b) = \log\frac{b + xa}{1 + b}, x > 1,$
$(a,b) = a \Rightarrow a = 0$ or $b = 0;$

5) $f(a,b) = \ln(1 + xa)/2, x > e,$
$f(a,b) = a \Rightarrow a = 0;$

6) $f(a,b) = a\,(1 + b)\,x; x \in (0,\infty),$
$f(a,b) = a \Rightarrow a = 0$ or $b = 0;$

7) $f(a,b) = a\log_{b+x} x, x > 1,$
$f(a,b) = a \Rightarrow a = 0$ or $b = 0;$

8) $f(a,b) = a - \left(\frac{1+a}{2+a}\right)\left(\frac{b}{1+b}\right),$
$f(a,b) = a \Rightarrow b = 0$;

9) $f(a,b) = a; \beta(a), \beta: [0,\infty) \to [0,1)$ is continuous
$f(a,b) = a \Rightarrow a = 0;$

10) $f(a,b) = a - \frac{b}{k+b}, f(a,b) = a \Rightarrow b = 0;$

11) $f(a,b) = a - \varphi(a), f(a,b) = a \Rightarrow a = 0,$
$\varphi: [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \to t = 0;$

12) $f(a,b) = ah(a,b), f(a,b) = a \Rightarrow a = 0$
$h: [0,\infty)\times[0,\infty) \to [0,\infty)$ is a continuous function such that $h(b,a) < 1$ for all $b, a > 0;$

13) $f(a,b) = a - \left(\frac{2-b}{1+b}\right)b, f(a,b) = a \Rightarrow a = 0;$

14) $f(a,b) = \sqrt[n]{\ln(1 + a^n)}, f(a,b) = a \Rightarrow a = 0;$

15) $f(a,b) = \varphi(a), f(a,b) = a \Rightarrow a = 0$
$\varphi: [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(0) = 0$ and $\varphi(b) < b$ for $b > 0;$

16) $f(a,b) = \frac{a}{(1+a)^x}, x \in (0,\infty),$
$f(a,b) = a \Rightarrow a = 0.$

DOI: 10.60101/jarst.2023.251308

Definition 2.9. [1] $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:
1) ψ is non-decreasing and continuous;
2) $\psi(t) = 0$ if and only if $t = 0$.
The family of all altering distance functions is denoted by ψ.
Example 2.10. The following functions
$\psi_i : R^+ \to R^+, i \in \{1,2,3,\dots,6\}$ are elements in ψ
1) $\psi_1(x) = kx$ where $k > 0$,
2) $\psi_2(x) = x^\alpha$ where $\alpha > 0$,
3) $\psi_3(x) = \sinh^{-1} x$,
4) $\psi_4(x) = \cosh x - 1$,
5) $\psi_5(x) = a^x - 1$ where $0 < a \neq 1$.
Definition 2.11. [1] A function $\varphi : [0,\infty) \to [0,\infty)$ is called an ultra-altering distance function if the following properties are satisfied in the following
1) φ is continuous;
2) $\varphi(t) > 0$ if and only if $t > 0$.
We denoted Ψu as the family of all ultra-altering distance functions.

RESULTS AND DISCUSSION

In this section, we are now ready to prove our main results.
Theorem 3.1. Let a function $d_\theta: X \times X \to [0,\infty)$ is an extended b metric space, and $f : X \to X$ is a self-mapping. Suppose

$$\psi\left(\theta(x,y)d_\theta(f_x, f_y)\right) \leq F(\psi(M(x,y)), \phi(M(x,y))) + LN(x,y) \quad (3.1)$$

for all $x, y \in X$, where $L \geq 0, F : [0,\infty)^2 \to R$ is an element in C, $\psi : [0,\infty) \to [0,\infty)$ is an element in Ψ, $\varphi : [0,\infty) \to [0,\infty)$ is an element in Ψu and

$$M(x,y) = \max\left\{d_\theta(x,y), \frac{d_\theta(x,f_x)d_\theta(y,f_y)}{1+d_\theta(f_x,f_y)}\right\}$$ and

$$N(x,y) = \min\{d_\theta(x,f_x), d_\theta(x,f_y), d_\theta(y,f_x), d_\theta(y,f_y)\}.$$

Then f has a unique fixed point.
Proof. Let $x_0 \in X$. Define a sequence $(x_n) \subset X$ by $x_n = f^n(x_0) = fx_{n-1}$ for $n \in N \cup \{0\}$. We now prove that (x_n) is a Cauchy sequence. First, we show $\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0$.
From 3.1 we have,

$$d_\theta(x_n,x_{n+1}) \leq \theta(x_n,x_{n+1})d_\theta(x_n,x_{n+1}) = \theta(x_n,x_{n+1})d_\theta(f(x_{n-1}), f(x_n)).$$

Consequently $\psi d_\theta(x_n,x_{n+1})$

$$\leq \psi(\theta(x_n,x_{n+1})d_\theta(f(x_{n-1}),f(x_n)) \leq F(\psi(M(x_{n-1},x_n)), \varphi(M(x_{n-1},x_n))) + LN(x_{n-1},x_n) \quad (3.2)$$

where $M(x_{n-1},x_n)$

$$= \max\left\{d_\theta(x_{n-1},x_n), \frac{d_\theta(x_{n-1},fx_{n-1})d_\theta(x_n,fx_n)}{1+d_\theta(fx_{n-1},fx_n)}\right\}$$

$$= \max\left\{d_\theta(x_{n-1},x_n), \frac{d_\theta(x_{n-1},x_n)d_\theta(x_n,x_{n-1})}{1+d_\theta(x_n,x_{n+1})}\right\}$$

$$= d_\theta(x_{n-1},x_n)$$

and $N(x_{n-1},x_n)$

$$= \min\{d_\theta(x_{n-1},fx_{n-1}), d_\theta(x_{n-1},fx_n), d_\theta(x_n,fx_{n-1}), d_\theta(x_n,fx_n)\}$$

$$= \min\{d_\theta(x_{n-1},x_n), d_\theta(x_{n-1},x_{n+1}), d_\theta(x_n,x_n), d_\theta(x_n,x_{n+1})\}$$

$$= \min\{d_\theta(x_{n-1},x_n), d_\theta(x_{n-1},x_{n+1}), 0, d_\theta(x_n,x_{n+1})\} = 0.$$

Therefore, it follows from 3.2 that we have

$$\psi(\theta(x_n,x_{n+1})d_\theta(f(x_{n-1}),f(x_n)) \leq F(\psi(d_\theta(x_{n-1},x_n)), \varphi(d_\theta(x_{n-1},x_n)) + LN(x_{n-1},x_n)$$

$$= F(\psi(d_\theta(x_{n-1},x_n)), \varphi(d_\theta(x_{n-1},x_n)) + L(0)$$

$$= F(\psi(d_\theta(x_{n-1},x_n)), \varphi(d_\theta(x_{n-1},x_n))$$

thus

$$\psi(d_\theta(x_n,x_{n+1}) \leq F(\psi\big(d_\theta(x_{n-1},x_n)\big), \varphi\big(d_\theta(x_{n-1},x_n)\big). \quad (3.3)$$

Since F is a function in C, we have

$$\psi(d_\theta(x_n,x_{n+1}) \leq F(\psi(d_\theta(x_{n-1},x_n)), \varphi(d_\theta(x_{n-1},x_n)) \leq \psi\big(d_\theta(x_{n-1},x_n)\big) \quad (3.4)$$

$$= \psi\big(d_\theta(x_n,x_{n-1})\big)$$

And ψ is non-decreasing, thus
$d_\theta(x_n,x_{n+1}) \geq 0 \ \forall n \in N$.
And $\{d_\theta(x_n,x_{n+1})\}$ is a descending sequence.
Then it converges, and there exists $r \geq 0$ such that $\lim_{n\to\infty} d_\theta(x_n,x_{n+1}) = r$.
Let $n \to \infty$, then from 3.4 it implies that

$$\psi(r) = \psi \lim_{n\to\infty} d_\theta(x_n,x_{n+1}) \leq \psi(x_n,x_{n-1})$$

$$= \lim_{n\to\infty} \psi(d_\theta(x_n,x_{n+1})) \leq \psi \lim_{n\to\infty} d_\theta(x_{n-1},x_n)$$

$$\leq \lim_{n\to\infty} F(\psi(d_\theta(x_{n-1},x_n)), \varphi(d_\theta(x_{n-1},x_n)) \leq \psi(r)$$

$$= F \lim_{n\to\infty}(\psi(d_\theta(x_{n-1},x_n)), \varphi(d_\theta(x_{n-1},x_n)) \leq \psi(r)$$

$$= F(\psi(\lim_{n\to\infty}((d_\theta(x_{n-1},x_n))), (\varphi(\lim_{n\to\infty}((d_\theta(x_{n-1},x_n))) \leq \psi(r)$$

$$= F(\psi(r), \varphi(r)) \leq \psi(r)$$

therefore $r = 0$ and $\lim_{n\to\infty} d_\theta(x_{n-1},x_n) = 0. \quad (3.5)$
Next, it is proved that the sequence (x_n) is a Cauchy sequence. Suppose that (x_n) is not a Cauchy sequence. By definition 2.4, we have $\varepsilon > 0$, for which we can find two sequences of positive integers $(m(k))$ and $(n(k))$ such that for all positive integers $k, n(k) > m(k) > k$ and $d\big(x_{m(k)},x_{n(k)}\big) \geq \varepsilon$. Let $n(k)$ be the smallest such positive integer $n(k) > m(k) > k$ such that $\forall k \in I^+$

$$d_\theta\big(x_{m(k)},x_{n(k)}\big) \geq \varepsilon, \quad d\big(x_{m(k)},x_{n(k)-1}\big) > \varepsilon \quad (3.6)$$

by 3.6 and $\theta : X \times X \to [1,\infty)$
we have

$$\liminf_{n\to\infty}(\varepsilon) \leq \liminf_{n\to\infty}(d_\theta\big(x_{m(k)},x_{n(k)-1}\big))$$

$$\varepsilon \leq \liminf_{n\to\infty}(d_\theta\big(x_{m(k)},x_{n(k)-1}\big)) \quad (3.7)$$

$$\theta(x_{m(k)},x_{n(k)-1}) \geq 1. \quad (3.8)$$

Since 3.7 and 3.8, we have

$$0 < \frac{\varepsilon}{\theta(x_{m(k)},x_{n(k)-1})} \leq \varepsilon \leq \liminf_{n\to\infty}(d_\theta\big(x_{m(k)},x_{n(k)-1}\big))$$

where $M\big(x_{m(k)},x_{n(k)-1}\big)$

$$= \max\{d_\theta\big(x_{m(k)},x_{n(k)-1}\big),$$

DOI: 10.60101/jarst.2023.251308

$$\frac{d_\theta\left(x_{n(k)-1}, fx_{n(k)-1}\right)d_\theta\left(x_{m(k)}, fx_{m(k)}\right)}{1 + d_\theta\left(fx_{n(k)-1}, fx_{m(k)}\right)}\}$$

$= \max\{d_\theta\left(x_{m(k)}, x_{n(k)-1}\right),$
$\frac{d_\theta(x_{n(k)-1}, x_{n(k)})d_\theta(x_{m(k)}, x_{m(k)+1})}{1+d_\theta(x_{n(k)}, x_{m(k)+1})}\}.$

Let $k \to \infty$ and apply 3.4, 3.5, and 3.6. We get

$$\frac{\varepsilon}{\theta^2 x_{m(k)}, x_{n(k)-1}} \le \lim_{k\to\infty} \inf f\left(M\left(x_{m(k)}, x_{n(k)-1}\right)\right). \tag{3.9}$$

Also $\lim_{k\to\infty} N\left(x_{m(k)}, x_{n(k)-1}\right)$

$= \lim_{k\to\infty} \min\{d_\theta\left(x_{n(k)-1}, fx_{n(k)-1}\right), d_\theta\left(x_{m(k)}, fx_{m(k)-1}\right),$
$d_\theta\left(x_{n(k)}, fx_{m(k)}\right)\}$

$= \min\{\lim_{k\to\infty} d_\theta\left(x_{n(k)-1}, x_{n(k)}\right), \lim_{k\to\infty} d_\theta\left(x_{m(k)}, x_{m(k)+1}\right),$
$\lim_{k\to\infty} d_\theta\left(x_{n(k)}, x_{m(k)+1}\right), \lim_{k\to\infty} d_\theta\left(x_{m(k)}, x_{m(k)}\right)\}$

$= \min\{\lim_{k\to\infty} d_\theta\left(x_{n(k)-1}, x_{n(k)}\right), \lim_{k\to\infty} d_\theta\left(x_{m(k)}, x_{m(k)+1}\right),$
$\lim_{k\to\infty} d_\theta\left(x_{m(k)}, x_{m(k)}\right)\}$

$= 0.$

Then

$d_\theta\left(x_{m(k)}, x_{n(k)}\right) - \theta\left(x_{m(k)}, x_{n(k)}\right)d_\theta\left(x_{m(k)}, x_{m(k)+1}\right)$

$$\le \theta\left(x_{m(k)}, x_{n(k)}\right)d_\theta\left(x_{m(k)+1}, x_{n(k)}\right). \tag{3.10}$$

from 3.7 and 3.9 we get

$\psi(\varepsilon) \le \psi(\lim_{k\to\infty} \sup d_\theta(x_{m(k)}, x_{n(k)}))$

$\le \psi(\lim_{k\to\infty} \sup d_\theta(x_{m(k)}, x_{n(k)}))d_\theta(x_{m(k)+1}, x_{n(k)})$

$\le F(\psi\left(\lim_{k\to\infty} \sup M\left(x_{m(k)}, x_{n(k)-1}\right)\right),$
$\varphi(\left(\lim_{k\to\infty} \inf M\left(x_{m(k)}, x_{n(k)-1}\right)\right))$

$\le \psi(\varepsilon)$

and $F\left(\psi(\varepsilon), \varphi\left(\lim_{k\to\infty} \inf M\left(x_{m(k)}, x_{n(k)-1}\right)\right)\right) = \psi(\varepsilon)$. by definition 2.7 2) we get $\psi(\varepsilon) = 0$ or

$\varphi\left(\lim_{k\to\infty} \inf M\left(x_{m(k)}, x_{n(k)-1}\right)\right) = 0$

and by definition 2.9 2) we get $\varepsilon = 0$ or

$\lim_{k\to\infty} \inf M\left(x_{m(k)}, x_{n(k)-1}\right) = 0.$

it is a contradiction with $\varepsilon > 0$ and

$\lim_{k\to\infty} \inf \left(M\left(x_{m(k)}, x_{n(k)-1}\right)\right) \ge \frac{\varepsilon}{d_\theta\left(x_{m(k)}, x_{n(k)}\right)^2}$

thus (x_n) is an extended b-Cauchy sequence in X. Since (X, d) is a completely extended b metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$ now, we show u is a fixed point of f since

$1 \le \theta(u, fu)$ and $d_\theta(u, fu) \ge 0$

we get $d_\theta(u, fu) \le \theta(u, fu)d_\theta(u, fu)$.

And since ψ is a nondecreasing function implies

$$\psi(d_\theta(u, fu)) \le \psi(\theta(u, fu)d_\theta(u, fu)) \tag{3.11}$$

$\psi(\theta(u, fu)d_\theta(f\left(\lim_{n\to\infty} x_n, f(\lim_{n\to\infty} x_n\right))$

$= \psi(\theta(u, fu)(d_\theta fu, f(fu)))$

$= \psi(\theta(u, fu)d_\theta(u, fu))\psi(\theta(u, fu)\theta(u, fu)(u, fu))$

$$\le F(\psi(M(u, fu)), \varphi(M(u, fu))) + LN(u, fu) \tag{3.12}$$

but $M(u, fu) = d_\theta(u, fu)$ and $N(u, fu) = 0$ thus

$\psi(d(u, f(u))) \le F\,\psi(\theta(u, f(u))), \varphi d(u, f(u)$
$\le \psi(d(u, f(u)))$

So $\psi(d(u, f(u))) = F\,\psi(\theta(u, f(u))), \varphi d(u, f(u))$.

$\therefore \psi(\theta(u, f(u))) = 0$ or $\varphi d(u, f(u)) = 0$.

By definition 2.1 we have $d(u, f(u)) = 0$ so $u = f(u)$

Now, we will show that u is a unique fixed point of f. Suppose $v \neq u$ is another fixed point of f from 3.1. We have

$\psi(d(u, f(u))) \le \psi(\theta(u, v), d(u, v))$

$= \psi(\theta(u, v), d(f(u), f(v))$

$\le F(\psi(M(u, v)), \varphi M(u, v)) + LN(u, v)$

$\le F(\psi(M(u, v)), \varphi M(u, v))$

$\le F(\psi(d(u, v)), \varphi d(u, v))$

$\le \psi(d(u, v))$

so $\psi d(u, v) = F(\psi(d(u, v), \varphi\, d(u, v))$

thus $\psi(d(u, v) = 0$ or $\varphi d(u, v) = 0$.

By definition 2.9, we have $d(u, v) = 0$ so that $u = v$ It means that f has a unique fixed point.

Example 3.2. Let $d_\theta: X \times X \to R^+$ and (X, d_θ) is an extended b metric space. $f: X \to X$ be such that $f(x) = \frac{x}{2}$, $\theta: X \times X \to R^+$ satisfy

$$\theta(x, y) = \begin{cases} |x - y|^3 & ; x \neq y \\ 1 & ; x = y \end{cases}$$

And define $F: [0, \infty)^2 \to R^2$ by $F(x, y) = x - y$ and define $\psi: [0, \infty) \to [0, \infty)$, $\varphi: [0, \infty) \to [0, \infty)$ by $\psi(x) = 2x$ and $\varphi(x) = x$ respectively.

$M(x, y) = \max\{d_\theta(x, y), \frac{d_\theta(x, fx)d_\theta(y, fy)}{1+d_\theta(fx, fy)}\}$ and

$N(x, y) = \min\{d_\theta(x, fx), d_\theta(x, fy), d_\theta(y, fx), d_\theta(y, fy)\}$.

From, example 2.8, definition 2.9, and definition 2.11 invoke that F is in C,ψ is in Ψ, and φ is in Ψu. Next, it will be considered that

$\psi(\theta(x, y)d_\theta(fx, fy))$

$\le F(\psi(M(x, y)), \varphi(M(x, y))) + LN(x, y)$.

It will be considered in 3 cases: $x = y$, $x > y$ and $x < y$ as follows.

Case 1: if $x = y$

Since $x = y$ therefore that $\theta(x, y) = 1$ and

$d_\theta(x, y) = d_\theta(fx, fy) = 0$ and

$d_\theta(x, fx) = d_\theta(y, fy) = d_\theta(x, fy) = d_\theta(y, fx)$

$= (x - \frac{x}{2})^4 = \frac{x^4}{16}$.

So that $\frac{d_\theta(x, fx)d_\theta(y, fy)}{1+d_\theta(fx, fy)} = \frac{\left(\frac{x^4}{16}\right)^2}{1+0} = \frac{x^8}{256}$.

And consider that

$M(x, y) = \max\left\{d_\theta(x, y), \frac{d_\theta(x, fx)d_\theta(y, fy)}{1+d_\theta(fx, fy)}\right\}$

$= \max\left\{0, \frac{x^8}{256}\right\} = \frac{x^8}{256}$ and

$N(x, y) = \min\{d_\theta(x, fx), d_\theta(x, fx), d_\theta(y, fx), d_\theta(y, fy)\}$

$= \frac{x^4}{16}$.

Next, it will be considered that

$\psi(\theta(x, y)d_\theta(fx, fy))$ by $\theta(x, y) = 1$ and

$d_\theta(x, y) = d_\theta(fx, fy) = 0$. We have

$\psi(\theta(x, y)d_\theta(fx, fy)) = \psi(1(0)) = \psi(0) = 0$.

And $F(\psi(M(x, y)), \varphi(M(x, y))) + LN(x, y)$

$\psi(M(x, y)) = \psi(\frac{x^8}{256}) = 2(\frac{x^8}{256}) = \frac{x^8}{128}$

$\varphi(M(x, y)) = \varphi(\frac{x^8}{256}) = \frac{x^8}{256}$ then we have

$F\left(\psi(M(x, y)), \varphi(M(x, y))\right) = F\left(\frac{x^8}{128}, \frac{x^8}{256}\right) = \frac{x^8}{256}$.

Hence $F(\psi(M(x, y)), \varphi(M(x, y))) + N(x, y)$

DOI: 10.60101/jarst.2023.251308

$= \frac{x^8}{256} + \frac{x^4}{16}$ such that $0 \leq \frac{x^8}{256} + \frac{x^4}{16}$ so if $x = y$ then we have

$\psi(\theta(x,y) d_\theta(fx, fy))$
$\leq F(\psi(M(x,y)), \varphi(M(x,y))) + N(x,y)$.

Case 2: $x > y$

Since $x > y$ then we have

$\theta(x,y) = |x-y|^3 = (x-y)^3$

and $d_\theta(x,y) = (x-y)^4$

$$d_\theta(x, fx) = \left(x - \frac{x}{2}\right)^4 = \frac{x^4}{16}$$

$$d_\theta(y, fy) = \left(y - \frac{y}{2}\right)^4 = \frac{y^4}{16}$$

$$d_\theta(x, fy) = \left(x - \frac{y}{2}\right)^4 = \frac{(2x-y)^4}{16}$$

$$d_\theta(y, fx) = \left(y - \frac{x}{2}\right)^4 = \frac{(2y-x)^4}{16}$$

$$d_\theta(fx, fy) = \left(\frac{x}{2} - \frac{y}{2}\right)^4 = \frac{(x-y)^4}{16}.$$

Therefore $\frac{d_\theta(x,fx) d_\theta(y,fy)}{1+d_\theta(fx,fy)} = \frac{\left(\frac{x^4}{16}\right)\left(\frac{y^4}{16}\right)}{1+\left(\frac{(x-y)^4}{16}\right)} = \left(\frac{\frac{x^4y^4}{16}}{16+(x-y)^4}\right)$.

Next, it will be considered that

$$M(x,y) = \max\left\{d_\theta(x,y), \frac{d_\theta(x, fx)\, d_\theta(y, fy)}{1 + d_\theta(fx, fy)}\right\}$$

$$= \max\left\{(x-y)^4, \left(\frac{\frac{x^4y^4}{16}}{16+(x-y)^4}\right)\right\} = (x-y)^4 \text{ and}$$

$$N(x,y) = \min\{d_\theta(x, fx), d_\theta(x, fy), d_\theta(y, fx), d_\theta(y, fy)\}$$

$$= \min\left\{\frac{x^4}{16}, \frac{(2x-y)^4}{16}, \frac{(2y-x)^4}{16}, \frac{y^4}{16}\right\} = \frac{(2y-x)^4}{16}.$$

And $\psi(\theta(x,y) d_\theta(fx, fy))$ by $\theta(x,y) = |x-y|^3$

and $d_\theta(fx, fy) = \frac{(x-y)^4}{16}$

$$\psi(\theta(x,y) d_\theta(fx, fy)) = \psi(|x-y|^3 \frac{(x-y)^4}{16})$$

$$= \psi\left(\frac{(x-y)^7}{16}\right)$$

$$= \frac{(x-y)^7}{8}.$$

And next regard

$F(\psi(M(x,y)), \varphi(M(x,y))) + N(x,y)$

$$\psi(M(x,y)) = \psi((x-y)^4) = 2((x-y)^4)$$

$$\varphi(M(x,y)) = \varphi((x-y)^4) = (x-y)^4.$$

We have

$$F\left(\psi(M(x,y)), \varphi(M(x,y))\right)$$

$$= F(2(x-y)^4, (x-y)^4)$$

$$= 2(x-y)^4 - (x-y)^4 = (x-y)^4$$

So $F(\psi(M(x,y)), \varphi(M(x,y))) + N(x,y)$

$$= (x-y)^4 + \frac{(2y-x)^4}{16}$$

such that $\frac{(x-y)^7}{8} \leq (x-y)^4 + \frac{(2y-x)^4}{16}$. Consequently, if $x > y$ we have

$\psi(\theta(x,y) d_\theta(fx, fy))$
$\leq F(\psi(M(x,y)), \varphi(M(x,y))) + N(x,y)$.

Case 3: if $x < y$

Since $x < y$, we have

$\theta(x,y) = |x-y|^3 = -(x-y)^3$ and

$d_\theta(x,y) = (x-y)^4$

$$d_\theta(x, fx) = \left(x - \frac{x}{2}\right)^4 = \frac{x^4}{16}$$

$$d_\theta(y, fy) = \left(y - \frac{y}{2}\right)^4 = \frac{y^4}{16}$$

$$d_\theta(x, fy) = \left(x - \frac{y}{2}\right)^4 = \frac{(2x-y)^4}{16}$$

$$d_\theta(y, fx) = \left(y - \frac{x}{2}\right)^4 = \frac{(2y-x)^4}{16}$$

$$d_\theta(fx, fy) = \left(\frac{x}{2} - \frac{y}{2}\right)^4 = \frac{(x-y)^4}{16}.$$

Then $\frac{d_\theta(x,fx) d_\theta(y,fy)}{1+d_\theta(fx,fy)} = \frac{\left(\frac{x^4}{16}\right)\left(\frac{y^4}{16}\right)}{1+\left(\frac{(x-y)^4}{16}\right)} = \left(\frac{\frac{x^4y^4}{16}}{16+(x-y)^4}\right)$. Consider that

$$M(x,y) = \max\left\{d_\theta(x,y), \frac{d_\theta(x, fx)\, d_\theta(y, fy)}{1 + d_\theta(fx, fy)}\right\}$$

$$N(x,y) = \min\{d_\theta(x, fx), d_\theta(x, fy), d_\theta(y, fx), d_\theta(y, fy)\}$$

$$\max\left\{(x-y)^4, \left(\frac{\frac{x^4y^4}{16}}{16+(x-y)^4}\right)\right\} = (x-y)^4 \text{ and}$$

$$N(x,y) = \min\{d_\theta(x, fx), d_\theta(x, fy), d_\theta(y, fx), d_\theta(y, fy)\}$$

$$= \min\left\{\frac{x^4}{16}, \frac{(2x-y)^4}{16}, \frac{(2y-x)^4}{16}, \frac{y^4}{16}\right\} = \frac{(2y-x)^4}{16}$$

$\psi(\theta(x,y) d_\theta(fx, fy))$ by $\theta(x,y) = -(x-y)^3$

and $d_\theta(fx, fy) = \frac{(x-y)^4}{16}$

$$\psi(\theta(x,y) d_\theta(fx, fy)) = \psi(|x-y|^3 \frac{(x-y)^4}{16})$$

$$= \psi\left(\frac{-(x-y)^7}{16}\right)$$

$$= \frac{-(x-y)^7}{8}.$$

Next, it will be considered that

$F(\psi(M(x,y)), \varphi(M(x,y))) + N(x,y)$

$\psi(M(x,y)) = \psi((x-y)^4) = 2((x-y)^4)$

$\varphi(M(x,y)) = \varphi((x-y)^4) = (x-y)^4$.

Therefore $F(\psi(M(x,y)), \varphi(M(x,y)))$

$= F(2(x-y)^4, (x-y)^4)$

$= 2(x-y)^4 - (x-y)^4 = (x-y)^4$.

So $F(\psi(M(x,y)), \varphi(M(x,y))) + N(x,y)$

$$= (x-y)^4 + \frac{(2x-y)^4}{16}$$

such that $\frac{-(x-y)^7}{8} \leq (x-y)^4 + \frac{(2x-y)^4}{16}$.

Consequently, if $x < y$ we have

$\psi(\theta(x,y) d_\theta(fx, fy)) \leq F(\psi(M(x,y)), \varphi(M(x,y))) + N(x,y)$.

By the fact that $f(x) = \left(\frac{x}{2}\right)$, function f has a unique fixed point that is 0.

Corollary 3.3, let (X, d) be a complete extended b-metric space on X and $f: X \to X$ be self-mapping. If $\psi(\theta(x,y) d_\theta(fx, fy)) \leq F(\psi(M(x,y)), \varphi(M(x,y))) + N(x,y)$. for all $x, y \in X$, where $\psi: [0,\infty) \to [0,\infty)$ is an ultra altering distance function and

$M(x,y) = \max\left\{d_\theta(x,y), \frac{d_\theta(x,fx)\, d_\theta(y,fy)}{1+d_\theta(fx,fy)}\right\}$ and

$N(x,y) = \min\{d_\theta(x, fx), d_\theta(x, fy), d_\theta(y, fx), d_\theta(y, fy)\}$

DOI: 10.60101/jarst.2023.251308

Journal of Applied Research on Science and Technology

then f has a unique fixed point.
Proof. Let $F(a,b) = a - b$. By example 2.8, we have F as a C-class function.
so $\psi(\theta(x,y)d_\theta(fx,fy))$
$\leq \psi(M(x,y)) - \varphi(M(x,y)) + LN(x,y)$
$= F(\psi(M(x,y)), \phi(M(x,y))) + LN(x,y)$.
Thus all the conditions of theorem 3.1 are satisfied. Hence f has a unique fixed point.
Corollary 3.4, let (X,d) be a complete extended b-metric space on X and $f: X \to X$ be self-mapping. $\theta: X \times X \to [1,\infty)$. Suppose $\psi(\theta(x,y)d_\theta(fx,fy))$
$\leq \psi(M(x,y))\beta\psi(M(x,y) + LN(x,y))$.
for all $x,y \in X$, where $\beta: [0,\infty) \to [0,1)$ is continuous and
$M(x,y) = \max\left\{d_\theta(x,y), \frac{d_\theta(x,fx)\, d_\theta(y,fy)}{1+d_\theta(fx,fy)}\right\}$ and
$N(x,y) = \min\{d_\theta(x,fx), d_\theta(x,fy), d_\theta(y,fx), d_\theta(y,fy)\}$
then f has a unique fixed point.
Proof. Let $F(a,b) = \theta(x,y)\beta(a)$
where $\beta: [0,\infty) \to [0,1)$ is continuous.
And let $\psi(b) = b$ by corollary 3.3 we have
$\psi(\theta(x,y)d_\theta(fx,fy)$
$\leq \psi(M(x,y))\beta(\psi(M(x,y))) + LN(x,y)$
$= F(\psi(M(x,y)), \beta(\psi(M(x,y)))) + LN(x,y)$
Thus, all the conditions of theorem 3.1 are satisfied. Hence, f has a unique fixed point.
Corollary 3.5, let (X,d) be a complete extended b-metric space on X and $f: X \to X$ be self-mapping. $\theta: X \times X \to [1,\infty)$.
Suppose
$d(f(x),f(y)) \leq \left[\frac{\beta(M(x,y))}{s}\right] M(x,y) + LN(x,y)$
for all $x,y \in X$, where $L \geq 0$,
$\beta: [0,\infty) \to [0,1)$ is continuous. And
$M(x,y) = \max\left\{d_\theta(x,y), \frac{d_\theta(x,fx)\, d_\theta(y,fy)}{1+d_\theta(fx,fy)}\right\}$ and
$N(x,y) = \min\{d_\theta(x,fx), d_\theta(x,fy), d_\theta(y,fx), d_\theta(y,fy)\}$
Then f has a unique fixed point.
Proof. Let $F(a,b) = \phi(a)$ by theorem 3.1 where $\phi: [0,\infty) \to [0,\infty)$ is continuous and
$\phi(0) = 0, \phi(b) < b$ where $b > 0$ and $\psi(b) = b$.
Thus, all the conditions of corollary 3.3 are satisfied. Hence, f has a unique fixed point.
Corollary 3.6, let (X,d_θ) be a complete b-metric space, and $f: X \to X$ be a self
Mapping. Suppose
$\theta(x,y)d_\theta(fx,fy) \leq \theta(M(x,y)) + LN(x,y)$
$\forall x,y \in X$ where $L \geq 0$, $\theta: [0,\infty) \to [0,\infty)$
is a continuous function such that
$\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$ and
$M(x,y) = \max\left\{d_\theta(x,y), \frac{d_\theta(x,fx)\, d_\theta(y,fy)}{1+d_\theta(fx,fy)}\right\}$ and
$N(x,y) = \min\{d_\theta(x,fx), d_\theta(x,fy), d_\theta(y,fx), d_\theta(y,fy)\}$
Then f has a unique fixed point.
Proof. With choice $\theta(b) = lb, 0 < l < 1$.
Thus, all the conditions of corollary 3.5 are satisfied. Hence, f has a unique fixed point.
Example 3.7. Let $X = C([a,b]; R)$ be a real value function defined on $[a,b]$, $f: X \to X$
where $f(x) = \frac{2x+5}{7}$ and define (X,d_θ) is an extended b metric space.
By $d_\theta(x,y) = \sup_{t\in[a,b]}|x(t) - y(t)|^2$ and
$\theta: X \times X \to [1,\infty)$ where
$\theta(x,y) = |x(t)| - |y(t)| + 2$.
So (X,d_θ) is a complete b-metric space. Thus

$$d_\theta(x,fx) = \sup_{t\in[a,b]}\left|x(t) - \frac{2x(t)+5}{7}\right|^2$$
$$= \sup_{t\in[a,b]}\left|\frac{6x(t)+5}{7}\right|^2$$
$$d_\theta(y,fy) = \sup_{t\in[a,b]}\left|y(t) - \frac{2y(t)+5}{7}\right|^2$$
$$= \sup_{t\in[a,b]}\left|\frac{6y(t)+5}{7}\right|^2$$
$$d_\theta(fx,fy) = \sup_{t\in[a,b]}\left|\frac{2x(t)+5}{7} - \frac{2y(t)+5}{7}\right|^2$$
$$= \sup_{t\in[a,b]}\left|\frac{6x(t)-2y(t)}{7}\right|^2$$
$$M(x,y) = \max\left\{d_\theta(x,y), \frac{d_\theta(x,fx)\, d_\theta(y,fy)}{1+d_\theta(fx,fy)}\right\}$$
$$= \max\left\{\sup_{t\in[a,b]}|x(t) - y(t)|^2, \frac{\sup_{t\in[a,b]}\left|x(t) - \frac{2x(t)+5}{7}\right|^2 \sup_{t\in[a,b]}\left|\frac{6y(t)+5}{7}\right|^2}{1+\sup_{t\in[a,b]}\left|\frac{6x(t)-2y(t)}{7}\right|^2}\right\}$$
$$= \sup_{t\in[a,b]}|x(t) - y(t)|^2.$$

Such that $\theta(x,y)d_\theta(fx,fy)$

$$= (|x(t)| - |y(t)| + 2)\sup_{t\in[a,b]}\left|\frac{2x(t)-2y(t)}{7}\right|^2$$
$$= 4(|x(t)| - |y(t)| + 2)\sup_{t\in[a,b]}\left|\frac{x(t)-y(t)}{7}\right|^2$$
$$\leq \frac{4}{49}(|x(t)| - |y(t)| + 2)\sup_{t\in[a,b]}|x(t) - y(t)|^2$$
$$\leq \frac{4}{49}d_\theta(x,y)M(x,y)$$

Thus, all the conditions of corollary 3.6 are satisfied. Hence, 1 is a unique fixed point of f.

CONCLUSIONS

This article presents the concept of C-class functions of the fixed theorem in incomplete extended b-metric spaces. We also prove that a fixed point of C-class functions exists in incomplete extended b-metric spaces. Further, some examples supporting the main results are provided. Our results extend and generalize corresponding results in the literature. The work presented provides a basis for researchers to work on in the future, and the work presented here is likely to provide a ground for the researchers to work in different structures by using these conditions.

ACKNOWLEDGEMENT

The first author was supported by Phetchabun Rajabhat University and Thailand Science Research and Innovation (Grant no. FRB640052/05).

DOI: 10.60101/jarst.2023.251308

REFERENCES

1. Note on ϕ,ψ-contractive type mappings and related fixed point. The 2nd regional conference on mathematics and applications; 2014 Sep; Tehran, Iran; 2014.
2. Choudhury BS, Metiya N, Postolache M. A generalized weak contraction principle with applications to coupled coincidence point problems. Fixed Point Theory and Applications. 2013;152: 1-21.
3. Boonwan C, Yamfang M, Thepa S, Sukchana T, Pruengam P, Pairintra R, et al. Development of a mathematical model to predict soil temperature for the growth of chrysanthemum sprouts. J Appl Res Sci Tech. 2020;19(1):74-85.
4. Bhaskar G, Lakshmikantham V. Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Analysis. 2005;65(7): 1379-93.
5. Cortelazzo G, Mian G, Vezzi G, Zamperoni P. Trademark shapes description by string matching techniques. Pattern Recognition. 1994;27(8): 1005-18.
6. Aydi H, Jellali M, Karapinar E. On fixed point results for alpha-implicit contractions in quasimetric spaces and consequences. Nonlinear Anal Model Control. 2016;21(1):40-56.
7. Bakhtin IA. The contraction mapping principle in almost metric spaces. Journal of Functional Analysis. 1989;30:26-37.
8. McConnell R, Kwok R, Curlander J, Kober W, Pang S. ψ-S correlation and dynamic time warping: Two methods for tracking ice floes. IEEE Transactions on Geoscience and Remote Sensing. 1991;29:1004-12.
9. Czerwik S. Nonlinear set-valued contraction mappings in b-metric spaces. Atti Semin mat fis Univ Modena Reggio Emilia. 1998;46:263-76.
10. Abdeljawad T, Alzabut JO, Mukheimer A, Zaidan Y. Banach contraction principle for cyclical mappings on partial metric spaces. Fixed Point Theory A. 2012;154:1-7.
11. Abdeljawad T, Alzabut JO, Mukheimer A, Zaidan Y. Best proximity points for cyclical contraction mappings with boundedly compact decompositions. J Comput Anal Appl. 2013;15(4):678-85.
12. Kamran T, Ali MU, Postolache M, Ghiura A, Farheen M. Best proximity points for a new class of generalized proximal mappings. Int J Anal Appl. 2017;13(2):198-205.
13. Kamran T, Postolache M, Uddin F, Ali MU. Fixed point theorems on generalized metric space endowed with graph. J Nonlinear Sci Appl. 2016; 9(6):4277-85.
14. Kamran T, Samreen M, Ain QU. A generalization of b-metric space and some fixed point theorems. Mathematics. 2017;5(2):19.
15. Suzuki T. A generalized Banach contraction principle that characterizes metric completeness. Proc Am Math Soc. 2008;136:1861-9.
16. Shatanawi W, Postolache M, Ansari AH, Kassab W. Common fixed points of dominating and weak annihilators in ordered metric spaces via C-class functions. J Math Anal. 2017;8(3):54-68.
17. Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations integrals. Fund Math. 1922;3:133-81.
18. Ansari AH, Razani A. Some fixed point theorems for C-class functions in b-metric spaces. Sahand Commum Math Anal. 2018;10(1):85-96.

DOI: 10.60101/jarst.2023.251308