Generalized Solutions of the Diophantine Equation

$$\frac{1^n}{u_1}+\frac{2^n}{u_2}+\frac{3^n}{u_3}+\cdots+\frac{k^n}{u_k}=1$$

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Abstract

This paper investigates the positive integer solutions of the Diophantine equation $\frac{1^n}{u_1}+\frac{2^n}{u_2}+\frac{3^n}{u_3}+\cdots+\frac{k^n}{u_k}=1$ where n is a positive integer and $u_1<u_2<\cdots<u_k$ are positive integers with the condition $k=u_1$. The study focuses on the existence and structural properties of the solution set. For a fixed smallest denominator, it is shown that the equation admits only finitely many solutions, and explicit upper and lower bounds for all denominators are established. Necessary conditions and polynomial growth constraints for admissible solutions are derived. These results clarify the structural behavior of power-type fractional Diophantine equations and provide a theoretical framework for further developments in number theory.

Keywords: Diophantine Equation; Parametric Solutions; Power-Type Equation; Number Theory

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Introduction

Diophantine equations are algebraic equations whose solutions are restricted to integers. They may be linear or nonlinear, involve one or several variables, or form systems in which the number of variables exceeds the number of equations. The study of Diophantine equations constitutes an important branch of number theory.

In 2019, Thamkaew et al. investigated Diophantine equations of the Equation (1) and (2)

$$\frac{1}{u_1}+\frac{3}{u_2}+\frac{5}{u_3}+\cdots+\frac{2k+1}{u_k}=1, \tag{1}$$

and

$$\frac{2}{u_1}+\frac{4}{u_2}+\frac{6}{u_3}+\cdots+\frac{2k}{u_k}=1, \tag{2}$$

where $2\le u_1<u_2<\cdots<u_k$ are integers and $k=u_1$. Their results described the number of solutions depending on the value of u_1. In 2017, Burshtein Nechemia studied reciprocal-type Diophantine equations and developed structural techniques for constructing parametric solution families, contributing significantly to the theory of fractional Diophantine representations. In 2022, Thamkaew and Thamkaew examined the quadratic power case Equation (3)

$$\frac{1^2}{u_1}+\frac{2^2}{u_2}+\cdots+\frac{k^2}{u_k}=1, \tag{3}$$

establishing existence results and identifying families of solutions depending on the smallest denominator. Subsequently, Saethao et al. (2024) investigated the cubic case Equation (4)

$$\frac{1^3}{u_1}+\frac{2^3}{u_2}+\cdots+\frac{k^3}{u_k}=1, \tag{4}$$

and analyzed the structure of positive integer solutions under ordered denominators. Their results provided further insight into the behavior of higher power reciprocal Diophantine equations and demonstrated how the number and structure of solutions change as the exponent increases.

More recently, Muangkarn et al. (2024) extended this line of research to the fourth power case Equation (5)

$$\frac{1^4}{u_1}+\frac{2^4}{u_2}+\cdots+\frac{k^4}{u_k}=1, \tag{5}$$

establishing structural classifications and identifying parametric families of solutions.
Motivated by these developments for $n=1, 2, 3, 4$ the present study considers the generalized power by Equation (6)

$$\sum_{i=1}^{k}\frac{i^n}{u_i}=1, \tag{6}$$

and aims to investigate the existence and structural properties of its positive integer solutions for arbitrary $n\in\mathbb{N}$.

Preliminaries

Diophantine equations are algebraic equations for which only integer solutions are sought. They may be linear or nonlinear, involve one or several variables, or even form systems in which the number of variables exceeds the number of equations. The study of Diophantine equations constitutes a central area of number theory, focusing on existence criteria, structural properties, and classifications of integer solutions. In particular, equations involving reciprocal sums have attracted considerable attention due to their connections with classical problems in number theory.

The equation considered in this paper belongs to the class of reciprocal sum Diophantine equations and is given by Equation (7)

$$\sum_{i=1}^{k} \frac{i^n}{u_i} = 1, \tag{7}$$

where n is a positive integer and $u_1, u_2, \ldots, u_k$ are positive integers satisfying the ordering condition $u_1 < u_2 < \cdots < u_k$. This form may be viewed as a power-weighted generalization of Egyptian fraction representations, in which the numerators are no longer constant but instead grow polynomially according to the exponent n. The presence of the power term significantly affects the behavior of the equation, especially with respect to growth and dominance among the summands. The strictly increasing condition imposed on the denominators ensures uniqueness in representation and prevents trivial rearrangements of solutions. Moreover, it facilitates structural analysis by allowing solutions to be classified into distinct parametric families. A parametric solution refers to a family of integer solutions that can be expressed in terms of one or more integer parameters. Such representations are particularly useful in studying infinite families of solutions and understanding their underlying structure. Since the numerator i^n grows polynomially with respect to i, the magnitude of each term in the sum depends critically on both the exponent n and the size of the corresponding denominator v_i. As n increases, the growth rate of the power terms becomes more pronounced, and stronger constraints on the denominators are required in order for the sum to equal one. Consequently, inequality techniques and bounding arguments play an essential role in analyzing the existence and structural properties of solutions. These foundational concepts provide the necessary framework for investigating the generalized Diophantine equation and for developing structural and parametric descriptions of its positive integer solutions.

Results

In this section, we establish structural properties and existence results for the Diophantine Equation (8)

$$\sum_{i=1}^{k} \frac{i^n}{u_i} = 1, \tag{8}$$

where n is a positive integer and $u_1 < u_2 < \cdots < u_k$ are positive integers.

Definition 3.1 (Big-O Class).

Let $g : \mathrm{N} \to R$ be a function. The Big-O class of g, denoted by $O(g)$, is defined as

$$O(g) = \{f : \mathrm{N} \to R : \exists C > 0, \exists k_0 \in \mathrm{N}, |f(k)| \le C|g(k)| \text{ for all } k \ge k_0\}.$$

Thus, $O(g)$ is the set of all functions whose growth is asymptotically bounded above by a constant multiple of g.

Theorem 3.2 Let Equation (9)

$$\sum_{i=1}^{k} \frac{i^n}{u_i} = 1, \text{ where } 2 \le u_1 < u_2 < \cdots < u_k, k = u_1, n \in \mathrm{N}. \tag{9}$$

Assume that $A(u_1, n) < 1$. Then the Diophantine equation admits only finitely many positive integer solutions.

Proof Fix u_1. Since $k = u_1$ where k is any positive integer, the number of variables is fixed.

Because the sequence is strictly increasing,

$$u_1 \ge u_1 + (i-1) \ldots\ldots\ldots\ldots\ldots\ldots (1) \text{ for all } i = 1, \ldots, k.$$

Step 1: Lower bounds are fixed.

From (1),

$$u_k \ge u_1 + (k-1) = u_1 + (u_1 - 1) = 2u_1 - 1.$$

Thus each denominator has a fixed positive lower bound depending only on u_1.

Step 2: Obtain an upper bound for u_k.

We have Equation (10),

$$\frac{k^n}{u_k} = 1 - \sum_{i=1}^{k-1} \frac{i^n}{u_i}. \tag{10}$$

Using (1) by Equation (11),

$$\sum_{i=1}^{k-1} \frac{i^n}{u_i} \le \sum_{i=1}^{k-1} \frac{i^n}{u_1 + i - 1} =: A(u_1, n). \tag{11}$$

The equinity $A(u_1, n)$ is a fixed real number because u_1 and n are fixed.

Since the left-hand side of the original equation equals (8) and all terms are positive, we must have Equation (12) - (13),

$$A(u_1, n) < 1. \tag{12}$$

Therefore,

$$\frac{k^n}{u_k} \ge 1 - A(u_1, n). \tag{13}$$

Because $1 - A(u_1, n) > 0$, we obtain

$$u_k \le \frac{k^n}{1 - A(u_1, n)}. \tag{14}$$

This gives an explicit upper bound for u_k Equation (14).

Step 3: All denominators are bounded.

Since

$$u_1 < u_2 < \cdots < u_k,$$

each u_1 satisfies

$$u_i \le u_k.$$

Thus every denominator is bounded both below and above by constants depending only on u_1 and n.

Step 4: Conclude finiteness

Each u_i lies in a finite interval of integers. Since k is fixed, the number of possible integer tuples $(u_1,\ldots,u_k)$ is finite. Hence the equation admits only finitely many solutions for fixed u_1.

Let $f(x)$ and $g(x)$ be real-valued functions defined for sufficiently large integers k. We write

$$f(k) = O(g(k)),$$

if there exist constants $C > O$ and $k_0 \in \mathrm{N}$ such that $|f(k)| \le C|g(k)|$ for all $k \ge k_0$.

This notation describes the asymptotic upper growth rate of $f(k)$ relative to $g(k)$.

Remark 3.3 We use the classical estimate for sums of powers: $\sum_{i=1}^{k} i^n = \frac{k^{n+1}}{n+1} + O(k^n)$, which follows from Faulhaber's formula. In particular, this implies $\sum_{i=1}^{k} i^n = O(k^{n+1})$.

Theorem 3.4 (Polynomial Growth Classification) Let Equation (15)

$$\sum_{i=1}^{k} \frac{i^n}{u_i} = 1, \text{ where } 2 \le u_1 < u_2 < \cdots < u_k, k = u_1, n \in \mathrm{N}. \tag{15}$$

Then every solution satisfies the following growth conditions:

1. $u_i \ge i^n$ for all $i = 1,\ldots,k$.
2. $u_1 \le \sum_{i=1}^{k} i^n$.
3. In particular,

$$u_1 = O(k^{n+1}).$$

Proof Step 1: Proof of $u_1 \ge i^n$.

Since all terms in the sum are positive, we have Equation (16)

$$\frac{i^n}{u_i} \le \sum_{j=1}^{k} \frac{j^n}{u_j} = 1 \text{ for every } j = 1,\ldots,k. \tag{16}$$

Hence,

$$\frac{i^n}{u_i} \le 1.$$

Multiplying both sides by $u_i > 0$ gives

$$i^n \le u_i.$$

Therefore, $u_i \geq i^n$ for all $i = 1,\ldots,k$.

Step 2: Proof of $u_1 \leq \sum_{i=1}^{k} i^n$.

Because the denominators are strictly increasing Equation (17),

$$u_1 < u_2 < \cdots < u_k, \tag{17}$$

we have $u_i \geq u_1$ for all i.

Thus Equation (18),

$$\frac{i^n}{u_i} \leq \frac{i^n}{u_1}. \tag{18}$$

Summing from $i = 1$ to k Equation (19),

$$\sum_{i=1}^{k} \frac{i^n}{u_i} \leq \sum_{j=1}^{k} \frac{i^n}{u_1} = \frac{1}{u_1} \sum_{i=1}^{k} i^n. \tag{19}$$

Since the left-hand side equals 1, we obtain Equation (20)

$$1 \leq \frac{1}{u_1} \sum_{i=1}^{k} i^n. \tag{20}$$

Multiplying by $u_1 > 0$ yields Equation (21)

$$u_1 \leq \sum_{i=1}^{k} i^n. \tag{21}$$

Step 3: Asymptotic growth estimate.

By the classical power-sum estimate (Faulhaber's formula) Equation (22),

$$\sum_{i=1}^{k} i^n = \frac{k^{n+1}}{n+1} + 0\left(k^n\right). \tag{22}$$

In particular Equation (23),

$$\sum_{i=1}^{k} i^n = 0\left(k^{n+1}\right). \tag{23}$$

Combining this with Step 2 gives

$$u_i \leq Ck^{n+1},$$

for some constant $C > 0$ and sufficiently large k. Hence,

$$u_1 = O\left(k^{n+1}\right).$$

This completes the proof.

Conclusion and Discussion

In this paper, we investigated the Diophantine equation

$$\sum_{i=1}^{k} \frac{i^n}{u_i} = 1,$$

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under the ordering condition $2 \le u_1 < u_2 < \cdots < u_k$ with $k = u$ and $n \in \mathbf{N}$. The main objective was to analyze how the smallest denominator influences the structure and size of the solution set. The results establish that, for each fixed smallest denominator v_1, the equation admits only finitely many positive integer solutions. This shows that the solution space is completely determined once the initial denominator is fixed. Furthermore, explicit lower and upper bounds for admissible denominators were derived, leading to polynomial growth constraints governed by the exponent n. In particular, every solution must satisfy $u_i \ge i^n,$, while the smallest denominator satisfies an upper bound of order k^{n+1}. These findings clarify the structural behavior of power-type fractional Diophantine equations and extend previously known results for the cases n = 1, 2, 3, and 4 to arbitrary positive integers n. Rather than focusing on explicit enumeration of solutions, this work provides a general structural framework that explains how growth conditions restrict admissible solution families. Future research may address the explicit construction of parametric solution families, the determination of exact solution counts for small values of u_1, and possible extensions to more general reciprocal-type Diophantine equations without the restriction $k = v_1$. Another promising direction is the asymptotic analysis of solution distributions as the exponent n increases.

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