On Certain Properties of the Laplace-type Integral Transform Via Post Quantum Calculus

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Abstract

In this paper, we apply the notion of (p,q)-calculus or post quantum calculus to establish theoretical results of (p,q)-analogues of Laplace-type integral transform of the first and second kind, which is a symmetric relation between (p,q)-analogues of the Laplace-type integral and Laplace transforms. Additionally, we discuss (p,q)-analogues of Laplace-type integral transform on various classes of some (p,q)-special functions, (p,q)-exponential function, (p,q)-trigono-metric types, (p,q)-differential operator, and (p,q)-convolution theorem. Finally, we establish results related to (p,q)-Aleph function.

Keywords: (p,q)-Integral Transform; (p,q)-Calculus; (p,q)-Convolution Theorem; (p,q)-Aleph Function

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Introduction

Quantum calculus, also called q-calculus known as calculus without limits, has been used in the applications of diverse areas such as mathematics, applied mathematics, and physics. Euler was the first mathematician to study quantum calculus in the early eighteenth, which Gauss and Ramanujan later developed. In 1910, Jackson [1], [2] introduced the q-derivative, or Jackson derivative, and the q-integral, or Jackson integral. Many researchers have come up with the generalization and development of the q-calculus as found in [3] - [12] and their respective references. Also, the fundamental explanation of the q-calculus aspects can be found in the book by Kac and Cheung [13].

Around a decade ago, the topic of the q-integral transform piqued many researchers' interest, leading to various investigation forms. Many researchers study the properties of q-integral transform such as q-Laplace transform of two variables [14], q-analogues of Sumudu transform [15], q-analogues of the Laplace transform [16], q-analogues of the natural transform [17], q-analogues of the Laplace-type integral transform [18], and q-theory of the q-Mellin transform [19], see [20] - [22] for more details.

The post quantum calculus, indicated as (p,q)-calculus, is a generalized form of q-calculus that was first considered in 1991 by Chakrabarti and Jagannathan [23]. It is pertinent that the direct substitution of q by q/p in q-calculus cannot provide valid quantum calculus; however, if $p = 1$ in (p,q)-calculus, it will reduce to q-calculus. In 2013, Sadjang [24] studied the concept of the (p,q)-derivative, the (p,q)-integration, the fundamental theorem of (p,q)-calculus, and the (p,q)-Taylor formulas. The studies and developments of the (p,q)-calculus have been conducted many times as found in [25] - [28] and their references. A slew of extensive research about (p,q)-integral transforms can also be seen lately. many researchers study the properties of (p,q)-integral transform and apply some (p,q)-differential equations such as (p,q)-analogues of the Laplace transform [29], (p,q)-analogues of the Sumudu transform [30], and (p,q)-analogues of Laplace-type integral transforms [31].

Many integral transforms are created to solve diverse differential equations. One of the most well-known integral transforms is the Laplace transform, which is wildly and extensively used in several branches of applied mathematics and engineering [32]. In 1991, Yurekli and Sasek [33] established a new integral transform, called the Laplace-type integral transform, which is defined by

$$L_2(F(\varkappa);\xi) = \int_0^\infty \varkappa F(\varkappa)\exp(-\varkappa^2\xi^2)d\varkappa, \quad \operatorname{Re}(\xi) > 0.$$

It is a close relation to the Laplace transform given as

$$L_2(F(\varkappa);\xi) = \frac{1}{2}L(F(\sqrt{\varkappa});\xi^2).$$

In the past century, the generalized H-functions have been established. In 1998, Sudland et al. [34] introduced a generalization of Saxena's I-function [35], which is considered to be another form of the generalization of Fox's H-function. This function is also named as Aleph function. In 2019, Bhat et al. [36] introduced q-Sumudu and q-Laplace transforms of the q-analogue of Aleph function. In 2020, Tassaddiq extended Bhat's and others' study to (p,q)-Sumudu and (p,q)-Laplace transforms based on (p,q)-Aleph function [37].

We are greatly inspired by a series of the above-mentioned literature, and therefore, propose to extend the q-analogues of the Laplace type integral transform to this new (p,q)-analogues of the Laplace type integral transform of the first and second kind while giving some properties such as (p,q)-special functions, (p,q)-exponential function, (p,q)-trigonometric types, (p,q)-differential operator, and (p,q)-Aleph function, which could be practically be utilized to solve some (p,q)-differential equations.

The outline of this paper is as follows: Section 2 consists of some basic knowledge of q and (p,q)-calculus that is used in the following sections; Section 3 is comprised of some (p,q)-special functions of the (p,q)-analogues of Laplace-typed integral transform; Section 4 demonstrates the (p,q)-differential operator; Section 5 shows the (p,q)-convolution theorem; in Section 6, we write (p,q)-analogues of Laplace-typed integral transform for (p,q)-Aleph function; and the last section includes the conclusion.

The q and (p,q)-calculus

In this section, some mathematical symbols used in the (p,q)-calculus are denoted to be of help in further study and can be found in [23] - [25], [29]. In the entirety of this work, let $0 < q < p \leq 1$ be constants. If $p = 1$, then we can reduce (p,q) of any forms to q-classical, see [13].

The (p,q)-number for $n \in \mathbb{N}$ is defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p-q}, & \text{if } p,q \neq 1; \\ [n]_q, & \text{if } p = 1; \\ n, & \text{if } p = 1 \text{ and take } \lim\limits_{q\to 1}. \end{cases} \tag{1}$$

Also, we write the (p,q)-factorial for $n \in \mathbb{N}$ is defined as

$$[n]_{p,q}! = \begin{cases} \prod_{j=1}^{n} [j]_{p,q} = [1]_{p,q}[2]_{p,q}\cdots[n]_{p,q}, & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases} \tag{2}$$

If $p = 1$ in (1) and (2), then we called this the q-number and q-factorial, respectively, see [13].

The (p,q)-derivatives of a function $F: [0,\infty) \to \mathbb{R}$ is defined by

$$\begin{aligned} D_{p,q}F(\varkappa) &= \frac{F(p\varkappa) - F(q\varkappa)}{(p-q)\varkappa}, \quad \varkappa \neq 0, \\ D_{p,q}F(0) &= \lim_{\varkappa \to 0} D_{p,q}F(\varkappa). \end{aligned} \tag{3}$$

If $p = 1$ in (3), then we called this the q-derivative of the function F, see [13].

The (p,q)-derivatives of higher order are given by

$$\left(D_{p,q}^{0}F\right)(\varkappa) = F(\varkappa) \quad \text{and} \quad \left(D_{p,q}^{k}F\right)(\varkappa) = D_{p,q}\left(D_{p,q}^{k-1}F\right)(\varkappa), \quad k \in \mathbb{N}.$$

Example 2.1 Define the function $F: [0,\infty) \to \mathbb{R}$ by $F(\varkappa) = \varkappa^3 + \varkappa^2 + 2\varkappa + r$, where r is a constant, then

$$\begin{aligned} D_{p,q}(\varkappa^3 + \varkappa^2 + 2\varkappa + r) &= \frac{(p^3\varkappa^3 + p^2\varkappa^2 + 2p\varkappa + r) - (q^3\varkappa^3 + q^2\varkappa^2 + 2q\varkappa + r)}{(p-q)\varkappa} \\ &= \frac{(p^3 - q^3)\varkappa^3 + (p^2 - q^2)\varkappa^2 + 2\varkappa(p-q)}{(p-q)\varkappa} \\ &= (p^2 + pq + q^2)\varkappa^2 + (p+q)\varkappa + 2 \\ &= [3]_{p,q}\varkappa^2 + [2]_{p,q}\varkappa + 2. \end{aligned}$$

The (p,q)-derivatives of the product and quotient rules of two functions are as follows:

$$D_{p,q}(F(\varkappa)g(\varkappa)) = F(p\varkappa)D_{p,q}g(\varkappa) + g(q\varkappa)D_{p,q}F(\varkappa), \tag{4}$$

and

$$D_{p,q}\left(\frac{F(\varkappa)}{g(\varkappa)}\right) = \frac{g(q\varkappa)D_{p,q}F(\varkappa) - F(q\varkappa)D_{p,q}g(\varkappa)}{g(p\varkappa)g(q\varkappa)}, \quad g(\varkappa) \neq 0. \tag{5}$$

If $p = 1$ in (4) and (5), then we called these the q-derivative of the product and quotient rules of two functions, respectively, see [13].

The (p,q)-integral from 0 to a and from 0 to ∞ of F are defined as

$$\int_0^a F(\varkappa)d_{p,q}\varkappa = (p-q)a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}F\left(\frac{q^k}{p^{k+1}}a\right) \tag{6}$$

and

$$\int_0^\infty F(\varkappa)d_{p,q}\varkappa = (p-q)\sum_{k=-\infty}^{\infty}\frac{q^k}{p^{k+1}}F\left(\frac{q^k}{p^{k+1}}\right), \tag{7}$$

respectively. If $p = 1$ in (6) and (7), then we called these the q-integral from 0 to a and from 0 to ∞, respectively, see [13].

The (p,q)-integral in an interval $[a,b]$ of $D_{p,q}F$ is given by

$$\int_a^b D_{p,q}F(\varkappa)d_{p,q}\varkappa = F(b) - F(a).$$

The (p,q)-integration by parts representation as

$$\int_a^b g(q\varkappa)D_{p,q}F(\varkappa)d_{p,q}\varkappa = F(b)g(b) - F(a)g(a) - \int_a^b F(p\varkappa)\left(D_{p,q}g(\varkappa)\right)d_{p,q}\varkappa, \tag{8}$$

also note that $b = \infty$ is allowed. If $p = 1$ in (8), then we called this the q-integration by parts, see [13].

The two types of (p,q)-exponential functions are defined in [29] as follows:

$$e_{p,q}(\varkappa) = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}}{[n]_{p,q}!}\varkappa^n, \tag{9}$$

$$E_{p,q}(\varkappa) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{[n]_{p,q}!}\varkappa^n. \tag{10}$$

If $p = 1$ in (9) and (10), then we called these the q-exponential function, see [13].

In addition, the derivative of the (p,q)-exponential functions are given as

$$D_{p,q}e_{p,q}(n\varkappa) = ne_{p,q}(np\varkappa),$$
$$D_{p,q}E_{p,q}(n\varkappa) = nE_{p,q}(nq\varkappa).$$

The (p,q)-trigonometric functions cosine and sine are as follows:

$$\cos_{p,q}(\varkappa) = \frac{e_{p,q}(i\varkappa)+e_{p,q}(-i\varkappa)}{2} = \sum_{n=0}^{\infty}\frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!}\varkappa^{2n}, \tag{11}$$

$$\mathrm{Cos}_{p,q}(\varkappa) = \frac{E_{p,q}(i\varkappa)+E_{p,q}(-i\varkappa)}{2} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_{p,q}!}\varkappa^{2n}, \tag{12}$$

$$\sin_{p,q}(\varkappa) = \frac{e_{p,q}(i\varkappa)-e_{p,q}(-i\varkappa)}{2i} = \sum_{n=0}^{\infty}\frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!}\varkappa^{2n+1}, \tag{13}$$

$$\mathrm{Sin}_{p,q}(\varkappa) = \frac{E_{p,q}(i\varkappa) - E_{p,q}(-i\varkappa)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!}\varkappa^{2n+1}. \tag{14}$$

If $p = 1$ in (11)-(14), then we called this the q-analogues of the trigonometric functions cosine and sine, see [13].

The function of the hyperbolic (p, q)-cosine and the hyperbolic (p, q)-sine of $\varkappa$ are as follows:

$$\cosh_{p,q}\varkappa = \frac{e_{p,q}(\varkappa) + e_{p,q}(-\varkappa)}{2} \quad \text{and} \quad \sinh_{p,q}\varkappa = \frac{e_{p,q}(\varkappa) - e_{p,q}(-\varkappa)}{2}. \tag{15}$$

If $p = 1$ in (15), then we called this the hyperbolic (p, q)-cosine and the hyperbolic (p, q)-sine functions, see [16].

The first kind of the (p, q)-gamma function is defined in [29] as follows:

$$\Gamma_{p,q}(n) = p^{\frac{n(n-1)}{2}} \int_0^{\infty} \varkappa^{n-1} E_{p,q}(-q\varkappa) d_{p,q}\varkappa. \tag{16}$$

It satisfies the following properties:

$$\Gamma_{p,q}(n+1) = [n]_{p,q}\Gamma_{p,q}(n) \quad \text{and} \quad \Gamma_{p,q}(n+1) = [n]_{p,q}!. \tag{17}$$

If $p = 1$ in (16) and (17), then we called these the q-gamma function of the first kind and properties of q-gamma function, respectively, see [13].

The second kind of the (p, q)-gamma function is defined in [29] as follows:

$$\gamma_{p,q}(n) = q^{\frac{n(n-1)}{2}} \int_0^{\infty} \varkappa^{n-1} e_{p,q}(-p\varkappa) d_{p,q}\varkappa.$$

The above equation satisfies the following properties:

$$\gamma_{p,q}(n+1) = [n]_{p,q}\gamma_{p,q}(n) \quad \text{and} \quad \gamma_{p,q}(n+1) = [n]_{p,q}!. \tag{18}$$

The (p, q)-integral on $(0, \infty)$ for $\beta \in \mathbb{R}\backslash\{0\}$ is given by

$$\int_0^{\infty} F(\beta\varkappa) d_{p,q}\varkappa = \frac{1}{\beta}\int_0^{\infty} F(\varkappa) d_{p,q}\varkappa. \tag{19}$$

The (p, q)-beta function is defined as

$$B_{p,q}(s, t) = \int_0^1 \varkappa^{s-1}(1 - q\varkappa)_{p,q}^{t-1} d_{p,q}\varkappa. \tag{20}$$

The relation between the (p, q)-gamma function and the (p, q)-beta function is

$$B_{p,q}(s, t) = p^{\frac{(t-1)(2s+t-2)}{2}} \frac{\Gamma_{p,q}(s)\Gamma_{p,q}(t)}{\Gamma_{p,q}(s+t)}. \tag{21}$$

If $p = 1$ in (20) and (21), then we called these the q-beta function and the relation between the q-gamma function and the q-beta function , see [13].

The (p,q)-Aleph function is defined by Jain el al. [27] as follows:

$$\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left(z;(p,q)\Big|^{(a_j,A_j)_{1,n};[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}_{(b_j,B_j)_{1,m};[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}}\right)= \int_L \frac{\frac{1}{2\pi\psi}\prod_{j=1}^{m}\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^{n}\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}d_{p,q}s}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)\sin\pi s\right]}, \quad (22)$$

where $z\neq 0$, $\psi=\sqrt{-1}$, and L is contour of integration running from $-i\infty$ to $+i\infty$ in such a manner so that all poles of $\Gamma_{p,q}(b_j+B_js); 1\leq j\leq m$ are to right of the path and those of $\Gamma_{p,q}(1-a_j-A_js); 1\leq j\leq n$, are to left. The integral converges if $Re[s\log(z)-\log(\sin(\pi s))]<0$, for large values of $|s|$ on the contour L. If $p=1$ in (22), then we called this the q-Aleph function, see [11].

The (p,q)-analogues of Laplace-type integral transform to some (p,q)-special functions

In this section, we evaluate ${}_{p,q}L_2$ (first kind) and ${}_{p,q}\mathbb{L}_2$ (second kind) of some special functions, which are defined in Definition 3.1.

Definition 3.1 The (p,q)-analogues of Laplace-type integral transform of function $F(\varkappa)$, $Re(\xi)>0$ is defined by

$${}_{p,q}L_2(F(\varkappa);\xi)=\int_0^\infty \varkappa F(\varkappa)E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa, \quad (23)$$

and

$${}_{p,q}\mathbb{L}_2(F(\varkappa);\xi)=\int_0^\infty \varkappa F(\varkappa)e_{p,q}(-p\varkappa^2\xi^2)d_{p,q}\varkappa. \quad (24)$$

If $p=1$ in (23) and (24), then (23) and (24) reduce to the q-analogues of Laplace-type integral transform of function $F(\varkappa)$, which appeared in [17].

Theorem 3.1 (Linearity): Let $F(\varkappa)$ and $g(\varkappa)$ be two functions, then the following formula holds:

(i) ${}_{p,q}L_2(\mu F(\varkappa)+\phi g(\varkappa);\xi)=\mu\, {}_{p,q}L_2(F(\varkappa);\xi)+\phi\, {}_{p,q}L_2(g(\varkappa);\xi)$;
(ii) ${}_{p,q}\mathbb{L}_2(\mu F(\varkappa)+\phi g(\varkappa);\xi)=\mu\, {}_{p,q}\mathbb{L}_2(F(\varkappa);\xi)+\phi\, {}_{p,q}\mathbb{L}_2(g(\varkappa);\xi)$;

where μ and ϕ are constants.
Proof. Theorem 3.1 follows immediately from Definition 3.1. □

Theorem 3.2 (Scaling): Let $f(\omega\varkappa)$ be a function, then the following formula holds:
(i) ${}_{p,q}L_2(F(\omega\varkappa);\xi)=\frac{1}{\omega^2}{}_{p,q}L_2(F(\varkappa);\frac{\xi}{\sqrt{\omega}})$;
(ii) ${}_{p,q}\mathbb{L}_2(F(\omega\varkappa);\xi)=\frac{1}{\omega^2}{}_{p,q}\mathbb{L}_2(F(\varkappa);\frac{\xi}{\sqrt{\omega}})$;

where ω are constants.
Proof. The proof of part (i) and (ii) using (19), (23), and (24), and we get

$$
\begin{aligned}
{}_{p,q}L_2(F(\omega\varkappa);\xi) &= \int_0^\infty \varkappa F(\omega\varkappa)E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa \\
&= \frac{1}{\omega^2}\int_0^\infty \varkappa F(\varkappa)E_{p,q}\left(-q\varkappa^2\left(\frac{y}{\sqrt{\omega}}\right)^2\right)d_{p,q}\varkappa \\
&= \frac{1}{\omega^2}{}_{p,q}L_2(F(\varkappa);\frac{\xi}{\sqrt{\omega}}).
\end{aligned}
$$

${}_{p,q}\mathbb{L}_2(F(\omega\varkappa);\xi)$ can be proven similarly. □

Theorem 3.3 For $f(\varkappa)=\varkappa^{2n}, n\in\mathbb{N}$, the following properties hold:

(i) ${}_{p,q}L_2(\varkappa^{2n};\xi) = \frac{[n]_{p,q}!}{[2]_{p,q}p^{\frac{n(n+1)}{2}}\xi^{2n+2}}$;

(ii) ${}_{p,q}\mathbb{L}_2(\varkappa^{2n};\xi) = \frac{[n]_{p,q}!}{[2]_{p,q}q^{\frac{n(n+1)}{2}}\xi^{2n+2}}$.

Proof. The proof of part (i) requires the use of (16) and (23). We get

$$
{}_{p,q}L_2(\varkappa^{2n};\xi) = \int_0^\infty \varkappa^{2n+1}E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa.
$$

By the change of variables $\varkappa^2\xi^2 = z$, we obtain $d_{p,q}\varkappa = \frac{d_{p,q}z}{[2]_{p,q}\varkappa\xi^2}$ and get that

$$
\begin{aligned}
{}_{p,q}L_2(\varkappa^{2n};\xi) &= \frac{1}{[2]_{p,q}\xi^{2n+2}}\int_0^\infty z^nE_{p,q}(-qz)d_{p,q}z \\
&= \frac{[n]_{p,q}!}{[2]_{p,q}p^{\frac{n(n+1)}{2}}\xi^{2n+2}}.
\end{aligned}
$$

The proof of part (ii) of this theorem follows a similar process from the proof of part (i). Therefore, the proof is completed. □

Theorem 3.4 For $n\in\mathbb{N}$, then the following properties hold:

(i) ${}_{p,q}L_2(e_{p,q}(n\varkappa^2);\xi) = \frac{p}{[2]_{p,q}(p\xi^2-n)}$ $(n<p\xi^2)$;

(ii) ${}_{p,q}\mathbb{L}_2(e_{p,q}(n\varkappa^2);\xi) = \frac{1}{[2]_{p,q}\xi^2}\sum_{k=0}^\infty \frac{p^{\binom{k}{2}}n^k}{q^{\frac{k(k+1)}{2}}\xi^{2k}}$;

(iii) ${}_{p,q}L_2(E_{p,q}(n\varkappa^2);\xi) = \frac{1}{[2]_{p,q}\xi^2}\sum_{k=0}^\infty \frac{q^{\binom{k}{2}}n^k}{p^{\frac{k(k+1)}{2}}\xi^{2k}}$;

(iv) ${}_{p,q}\mathbb{L}_2(E_{p,q}(n\varkappa^2);\xi) = \frac{q}{[2]_{p,q}(q\xi^2-n)}$ $(n<q\xi^2)$.

Proof. The proof of part (i) utilizes (9), (23), and Theorem 3.3(i), and we have

$$
\begin{aligned}
{}_{p,q}L_2(e_{p,q}(n\varkappa^2);\xi) &= \int_0^\infty \varkappa e_{p,q}(n\varkappa^2)E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa \\
&= \int_0^\infty \varkappa\sum_{k=0}^\infty \frac{p^{\binom{k}{2}}(n\varkappa^2)^k}{[k]_{p,q}!}E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa \\
&= \sum_{k=0}^\infty \frac{p^{\binom{k}{2}}n^k}{[k]_{p,q}!}\int_0^\infty \varkappa^{2k+1}E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa.
\end{aligned}
$$

By the change of variables $\varkappa^2\xi^2 = z$, the equation above can be written as follows:

$$\begin{aligned}{}_{p,q}L_2(e_{p,q}(n\varkappa^2);\xi) &= \sum_{k=0}^{\infty}\frac{p^{\binom{k}{2}}n^k}{[k]_{p,q}!}\frac{\Gamma_{p,q}(k+1)}{[2]_{p,q}p^{\frac{k(k+1)}{2}}y^{2k+2}}\\ &= \sum_{k=0}^{\infty}\frac{pn^k}{[2]_{p,q}p^{\frac{k(k+1)}{2}}\xi^{2k+2}}\\ &= \frac{1}{[2]_{p,q}\xi^2}\sum_{k=0}^{\infty}\frac{n^k}{p^k\xi^{2k}}\\ &= \frac{p}{[2]_{p,q}(p\xi^2-n)} \quad (n < p\xi^2).\end{aligned}$$

The proof of part (ii) share a similar process from the proof of part (i), which can be written as follows:

$${}_{p,q}\mathbb{L}_2(e_{p,q}(n\varkappa^2);\xi) = \sum_{k=0}^{\infty}\frac{p^{\binom{k}{2}}n^k}{[k]_{p,q}!}\int_0^{\infty}\varkappa^{2k+1}e_{p,q}(-p\varkappa^2\xi^2)d_{p,q}\varkappa.$$

The proof of part (iii) and (iv) procceds in the similar manners from the proof of part (i). Hence, the proof is completed. □

Theorem 3.5 For $n \in \mathbb{N}$, then the following properties hold:

(i) ${}_{p,q}L_2(\cos_{p,q}(n\varkappa^2);\xi) = \frac{p^2\xi^2}{[2]_{p,q}(p^2\xi^4+n^2)}$ $(n^2 < p^2\xi^4)$;

(ii) ${}_{p,q}\mathbb{L}_2(\cos_{p,q}(n\varkappa^2);\xi) = \frac{1}{[2]_{p,q}\xi^2}\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{2k}{2}}n^{2k}}{q^{k(2k+1)}\xi^{4k}}$;

(iii) ${}_{p,q}L_2(\mathrm{Cos}_{p,q}(n\varkappa^2);\xi) = \frac{1}{[2]_{p,q}\xi^2}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{2k}{2}}n^{2k}}{p^{k(2k+1)}\xi^{4k}}$;

(iv) ${}_{p,q}\mathbb{L}_2(\mathrm{Cos}_{p,q}(n\varkappa^2);\xi) = \frac{q^2\xi^2}{[2]_{p,q}(q^2\xi^4+n^2)}$ $(n^2 < q^2\xi^4)$.

Proof. The proof of part (i) can be done using (11), (16), and (23), and we obtain the following:

$$\begin{aligned}{}_{p,q}L_2(\cos_{p,q}(n\varkappa^2);\xi) &= \int_0^{\infty}\varkappa\cos_{p,q}(n\varkappa^2)E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{2k}{2}}n^{2k}}{[2k]_{p,q}!}\int_0^{\infty}\varkappa^{4k+1}E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa.\end{aligned}$$

By applying the change of variables $\varkappa^2\xi^2 = z$, the equation above can be put into the form

$$\begin{aligned}{}_{p,q}L_2(\cos_{p,q}(n\varkappa^2);\xi) &= \sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{2k}{2}}n^{2k}}{[2k]_{p,q}!}\frac{1}{[2]_{p,q}\xi^{4k+2}}\int_0^{\infty}z^{2k}E_{p,q}(-qz)d_{p,q}z\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{2k}{2}}n^{2k}\Gamma_{p,q}(2k+1)}{[2k]_{p,q}![2]_{p,q}p^{k(2k+1)}\xi^{4k+2}}\\ &= \frac{1}{[2]_{p,q}\xi^2}\sum_{k=0}^{\infty}\left(-\frac{n^2}{p^2\xi^4}\right)^k\\ &= \frac{p^2\xi^2}{[2]_{p,q}(p^2\xi^4+n^2)} \quad (n^2 < p^2\xi^4).\end{aligned}$$

The proof of part (ii) uses a similarly process from part (i). We write

$$ {}_{p,q}\mathbb{L}_2(\cos_{p,q}(n\varkappa^2);\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{2k}{2}} n^{2k}}{[2k]_{p,q}!} \int_0^{\infty} \varkappa^{4k+1} e_{p,q}(-q\varkappa^2\xi^2) d_{p,q}\varkappa. $$

The proof of part (iii) and (iv) follows the same procedure from the proof of part (i). Hence, the proof is completed. □

Theorem 3.6 For $n \in \mathbb{N}$, then the following properties hold:

(i) ${}_{p,q}L_2(\sin_{p,q}(n\varkappa^2);\xi) = \frac{np}{[2]_{p,q}(p^2\xi^4+n^2)}\ \ (n^2 < p^2\xi^4)$;
(ii) ${}_{p,q}\mathbb{L}_2(\sin_{p,q}(n\varkappa^2);\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{2k+1}{2}} n^{2k+1}}{[2]_{p,q} q^{(k+1)(2k+1)} \xi^{4k+4}}$;
(iii) ${}_{p,q}L_2\left(\mathrm{Sin}_{p,q}(n\varkappa^2);\xi\right) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{2k+1}{2}} n^{2k+1}}{[2]_{p,q} p^{(k+1)(2k+1)} \xi^{4k+4}}$;
(iv) ${}_{p,q}\mathbb{L}_2\left(\mathrm{Sin}_{p,q}(n\varkappa^2);\xi\right) = \frac{nq}{[2]_{p,q}(q^2\xi^4+n^2)}\ \ (n^2 < q^2\xi^4)$.

Proof. The proof of this theorem follows the definitions and proceeds according to Theorem 3.5. The details are, therefore, omitted. □

Theorem 3.7 For $n \in \mathbb{N}$, then the following properties hold:

(i) ${}_{p,q}L_2(\cosh_{p,q}(n\varkappa^2);\xi) = \frac{p^2\xi^2}{[2]_{p,q}(p^2\xi^4-n^2)}\ \ (n^2 < p^2\xi^4)$;
(ii) ${}_{p,q}L_2(\sinh_{p,q}(n\varkappa^2);\xi) = \frac{np}{[2]_{p,q}(p^2\xi^4-n^2)}\ \ (n^2 < p^2\xi^4)$.

Proof. The proof of Theorem 3.7 directly follows from (15), Theorem 3.5 and 3.6; therefore, details are omitted. □

Theorem 3.8 (Transforms of the Heaviside function): For $b \geq 0$, let

$$ H(\varkappa - b) = \begin{cases} 1, & \text{for } \varkappa \geq b; \\ 0, & \text{for } 0 \leq \varkappa \leq b. \end{cases} \tag{25} $$

Then, we have

(i) ${}_{p,q}L_2(H(\varkappa-b);\xi) = \frac{E_{p,q}(-b)}{[2]_{p,q}\xi^2}$;
(ii) ${}_{p,q}\mathbb{L}_2(H(\varkappa-b);\xi) = \frac{e_{p,q}(-b)}{[2]_{p,q}\xi^2}$.

Proof. The proof of part (i) using (23) and (25). We obtain

$$ \begin{aligned} {}_{p,q}L_2(H(\varkappa-b);\xi) &= \int_0^{\infty} \varkappa H(\varkappa-b) E_{p,q}(-q\varkappa^2\xi^2) d_{p,q}\varkappa \\ &= \int_b^{\infty} \varkappa E_{p,q}(-q\varkappa^2\xi^2) d_{p,q}\varkappa. \end{aligned} $$

By the change of variables $z = \varkappa^2\xi^2$, we get $d_{p,q}\varkappa = \frac{1}{[2]_{p,q}\varkappa\xi^2} d_{p,q}z$, and we obtain

$$ {}_{p,q}L_2(H(\varkappa-b);\xi) = \frac{1}{[2]_{p,q}\xi^2} \int_b^{\infty} E_{p,q}(-qz) d_{p,q}z $$

$$= -\frac{1}{[2]_{p,q}\xi^2}\int_b^\infty D_{p,q}E_{p,q}(-z)d_{p,q}z$$
$$= \frac{E_{p,q}(-b)}{[2]_{p,q}\xi^2}.$$

The proof of part (ii) follows the same procedure from the proof of part (i). Hence, the proof is completed.

Theorem 3.9 (Transforms of the Dirac delta function): For $b \geq 0$, let

$$f_k(\varkappa - b) = \begin{cases} \frac{1}{k}, & \text{for } b \leq \varkappa \leq b+k; \\ 0, & \text{otherwise}. \end{cases} \tag{26}$$

If $\delta(\varkappa - b)$ denotes the limit of f_k as $k \to 0$, then we have

(i) ${}_{p,q}L_2(\delta(\varkappa - b); \xi) = \frac{E_{p,q}(-qb)}{[2]_{p,q}\xi^2}$,

(ii) ${}_{p,q}\mathbb{L}_2(\delta(\varkappa - b); \xi) = \frac{e_{p,q}(-pb)}{[2]_{p,q}\xi^2}$,

where delta is the Dirac delta function.

Proof. The proof of part (i) using (23) and (26). We get

$${}_{p,q}L_2(f_k(\varkappa - b); \xi) = \int_0^\infty \varkappa f_k(\varkappa - b)E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa$$
$$= \frac{1}{k}\int_b^{b+k} \varkappa E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa.$$

By the change of variables $z = \varkappa^2\xi^2$, we get $d_{p,q}\varkappa = \frac{1}{[2]_{p,q}\varkappa\xi^2}d_{p,q}z$, and we obtain

$${}_{p,q}L_2(f_k(\varkappa - b); \xi) = \frac{1}{[2]_{p,q}\xi^2 k}\int_b^{b+k} E_{p,q}(-qz)d_{p,q}z$$
$$= -\frac{1}{[2]_{p,q}\xi^2 k}\int_b^{b+k} D_{p,q}E_{p,q}(-z)d_{p,q}z$$
$$= -\frac{1}{[2]_{p,q}\xi^2 k}\left[E_{p,q}(-(b+k)) - E_{p,q}(-b)\right].$$

If we take the limit of f_k as $k \to 0$, then

$${}_{p,q}L_2(\delta(\varkappa - b); \xi) = \lim_{k\to 0} {}_{p,q}L_2(f_k(\varkappa - b); \xi) = \frac{E_{p,q}(-qb)}{[2]_{p,q}\xi^2}.$$

The proof of part (ii) follows the same procedure from the proof of part (i). Hence, the proof is completed. □

The (p, q)-analogues of Laplace-type integral transform to the first order (p, q)-differential operator

We offer this section to computations related to the ${}_{p,q}L_2$ and ${}_{p,q}\mathbb{L}_2$, and some differential operators. First and foremost, we derive the following theorem:

Theorem 4.1 Let $\mathrm{Re}(\xi) > 0$. Then, we have

$$D_{p,q}E_{p,q}(-\varkappa^2\xi^2) = -\varkappa\xi^2 \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!}(p^{k+1} + q^{k+1})\xi^{2k}\varkappa^{2k}. \quad (27)$$

Proof. Taking (p,q)-derivative on $E_{p,q}(-\varkappa^2\xi^2)$ with respect to $\varkappa$ and shifting the lower bound of the summation, we get

$$\begin{aligned} D_{p,q}E_{p,q}(-\varkappa^2\xi^2) &= D_{p,q} \sum_{k=1}^{\infty} \frac{q^{\binom{k}{2}}(-\varkappa^2\xi^2)^k}{[k]_{p,q}!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{[k]_{p,q}!}[2k]_{p,q}\varkappa^{2k-1}\xi^{2k} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{[k]_{p,q}!}(p^k + q^k)[k]_{p,q}\varkappa^{2k-1}\xi^{2k} \\ &= -\varkappa\xi^2 \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!}(p^{k+1} + q^{k+1})\xi^{2k}\varkappa^{2k}. \end{aligned}$$

Hence, the proof of this theorem is completed. □

Theorem 4.2 Let $\mathrm{Re}(\xi) > 0$. Then, we have

$$D_{p,q}e_{p,q}(-\varkappa^2\xi^2) = -\varkappa\xi^2 \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!}(p^{k+1} + q^{k+1})\xi^{2k}\varkappa^{2k}. \quad (28)$$

Proof. The proof of Theorem 4.2 is similar to the one of Theorem 4.1, which can be expressed by

$$\begin{aligned} D_{p,q}e_{p,q}(-\varkappa^2\xi^2) &= D_{p,q} \sum_{k=1}^{\infty} \frac{p^{\binom{k}{2}}}{[k]_{p,q}!}(-\varkappa^2\xi^2)^k \\ &= -\varkappa\xi^2 \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!}(p^{k+1} + q^{k+1})\xi^{2k}\varkappa^{2k}. \end{aligned}$$

Therefore, the proof is completed. □

Theorem 4.3 Let $\nabla_{p,q} F(\varkappa) = \frac{1}{\varkappa} D_{p,q}F(\varkappa)$ and $\mathrm{Re}(\xi) > 0$. Then, we have

$$_{p,q}L_2(\nabla_{p,q} F(\varkappa);\xi) = -F(0) + \frac{\xi^2}{p}\,_{p,q}L_2\left(F(\varkappa);\frac{\xi}{\sqrt{p}}\right) + \frac{q\xi^2}{p^2}\,_{p,q}L_2\left(F(\varkappa);\frac{\sqrt{q}\xi}{p}\right). \quad (29)$$

Proof. According to the (p,q)-integration by parts (8) and Theorem 4.1, we can write

$$\begin{aligned} {}_{p,q}L_2(\nabla_{p,q} F(\varkappa);\xi) &= \int_0^{\infty} D_{p,q}F(\varkappa)E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa \\ &= \lim_{a\to\infty}\left[F(\varkappa)E_{p,q}(-\varkappa^2\xi^2)\right]_{\varkappa=0}^{a} - \int_0^{\infty} F(p\varkappa)D_{p,q}E_{p,q}(-\varkappa^2\xi^2)d_{p,q}\varkappa \\ &= -F(0) - \int_0^{\infty} F(p\varkappa)D_{p,q}E_{p,q}(-\varkappa^2\xi^2)d_{p,q}\varkappa \\ &= -F(0) + \xi^2 \int_0^{\infty} \varkappa F(p\varkappa) \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!}(p^{k+1} + q^{k+1})\xi^{2k}\varkappa^{2k}d_{p,q}\varkappa \end{aligned}$$

$$= -F(0) + \xi^2 \int_0^\infty \varkappa F(p\varkappa) \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!} p^{k+1} \xi^{2k} \varkappa^{2k} d_{p,q}\varkappa$$
$$+\xi^2 \int_0^\infty \varkappa F(p\varkappa) \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!} q^{k+1} \xi^{2k} \varkappa^{2k} d_{p,q}\varkappa.$$

By the change of variables $p\varkappa = z$, we get $d_{p,q}\varkappa = \frac{1}{p} d_{p,q}z$, and we obtain

$${}_{p,q}L_2(\nabla_{p,q} F(\varkappa);\xi) = -f(0) + p^{-1}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!} p^{-k} \xi^{2k} z^{2k} d_{p,q}z$$
$$+qp^{-2}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!} q^k \xi^{2k} z^{2k} p^{-2k} d_{p,q}z.$$

Next, multiplying $q^{-k}q^k$, we have

$${}_{p,q}L_2(\nabla_{p,q} F(\varkappa);\xi) = -F(0) + p^{-1}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!} q^{-k} q^k p^{-k} \xi^{2k} z^{2k} d_{p,q}z$$
$$+qp^{-2}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k+1}{2}}}{[k]_{p,q}!} q^{-k} q^k q^k \xi^{2k} z^{2k} p^{-2k} d_{p,q}z$$
$$= -F(0) + p^{-1}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}}{[k]_{p,q}!} q^k p^{-k} \xi^{2k} z^{2k} d_{p,q}z$$
$$+qp^{-2}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}}{[k]_{p,q}!} q^{2k} \xi^{2k} z^{2k} p^{-2k} d_{p,q}z$$
$$= -F(0) + p^{-1}\xi^2 \int_0^\infty zF(z) E_{p,q}\left(-\frac{q\xi^2 z^2}{p}\right) d_{p,q}z$$
$$+qp^{-2}\xi^2 \int_0^\infty zF(z) E_{p,q}\left(-\frac{q^2\xi^2 z^2}{p^2}\right) d_{p,q}z$$
$$= -F(0) + \frac{\xi^2}{p}\,{}_{p,q}L_2\left(F(\varkappa);\frac{\xi}{\sqrt{p}}\right) + \frac{q\xi^2}{p^2}\,{}_{p,q}L_2\left(F(\varkappa);\frac{\sqrt{q}\xi}{p}\right).$$

This completes the proof of the theorem. □

Theorem 4.4 Let $\nabla_{p,q} F(\varkappa) = \frac{1}{\varkappa} D_{p,q}F(\varkappa)$ and $\mathrm{Re}(\xi) > 0$. Then, we have

$${}_{p,q}\mathbb{L}_2(D_{p,q}F(\varkappa);\xi) = -F(0) + \frac{\xi^2}{q^2}\,{}_{p,q}\mathbb{L}_2\left(F(\varkappa);\frac{\sqrt{p}\xi}{q}\right) + \xi^2\,{}_{p,q}\mathbb{L}_2\left(F(\varkappa);\frac{\xi}{\sqrt{q}}\right). \tag{30}$$

Proof. Using (8) and Theorem 4.2, we have

$${}_{p,q}\mathbb{L}_2(\nabla_{p,q} F(\varkappa);\xi) = \int_0^\infty D_{p,q}F(\varkappa) e_{p,q}(-p\varkappa^2\xi^2) d_{p,q}\varkappa$$
$$= \lim_{a\to\infty} \left[F(\varkappa) e_{p,q}(-\varkappa^2\xi^2)\right]_{\varkappa=0}^{a} - \int_0^\infty F(q\varkappa) D_{p,q} e_{p,q}(-\varkappa^2\xi^2) d_{p,q}\varkappa$$
$$= -F(0) + \xi^2 \int_0^\infty \varkappa F(q\varkappa) \sum_{k=0}^\infty \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!} (p^{k+1} + q^{k+1}) \xi^{2k} \varkappa^{2k} d_{p,q}\varkappa$$
$$= -F(0) + \xi^2 \int_0^\infty \varkappa F(q\varkappa) \sum_{k=0}^\infty \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!} p^{k+1} \xi^{2k} \varkappa^{2k} d_{p,q}\varkappa$$
$$+\xi^2 \int_0^\infty \varkappa F(q\varkappa) \sum_{k=0}^\infty \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!} q^{k+1} \xi^{2k} \varkappa^{2k} d_{p,q}\varkappa.$$

Therefore, by the change of variables $q\varkappa = z$, we get $d_{p,q}\varkappa = \frac{1}{q} d_{p,q}z$, and the equation above can be rearranged to the following form:

$$
{}_{p,q}\mathbb{L}_2(\nabla_{p,q} F(\varkappa);\xi) = -F(0) + q^{-2}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!} p^{k+1}\xi^{2k} z^{2k} q^{-2k} d_{p,q}z
$$

$$
+q^{-1}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!} q^{k+1}\xi^{2k} z^{2k} q^{-2k} d_{p,q}z.
$$

Next, multiplying the previous equation by $p^{-k}p^k$, we get

$$
\begin{aligned}
{}_{p,q}\mathbb{L}_2(\nabla_{p,q} F(\varkappa);\xi) &= -F(0) + q^{-2}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!} p^{-k}p^k p^{k+1}\xi^{2k} z^{2k} q^{-2k} d_{p,q}z \\
&\quad +q^{-1}\xi^2 \int_0^\infty zF(z) \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k+1}{2}}}{[k]_{p,q}!} p^{-k}p^k q^{k+1}\xi^{2k} z^{2k} q^{-2k} d_{p,q}z \\
&= -F(0) + q^{-2}p\xi^2 \int_0^\infty zF(z) \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k}{2}}}{[k]_{p,q}!} p^{2k}\xi^{2k} z^{2k} q^{-2k} d_{p,q}z \\
&\quad +\xi^2 \int_0^\infty zF(z) \sum_{k=0}^{\infty} \frac{(-1)^k p^{\binom{k}{2}}}{[k]_{p,q}!} p^{k}\xi^{2k} z^{2k} q^{-k} d_{p,q}z \\
&= -F(0) + q^{-2}p\xi^2 \int_0^\infty zF(z) e_{p,q}\left(-\frac{p^2\xi^2 z^2}{q^2}\right) d_{p,q}z \\
&\quad +\xi^2 \int_0^\infty zF(z) e_{p,q}\left(-\frac{p\xi^2 z^2}{q}\right) d_{p,q}z \\
&= -F(0) + \frac{\xi^2}{q^2}\, {}_{p,q}\mathbb{L}_2\left(F(\varkappa);\frac{\sqrt{p}\xi}{q}\right) + \xi^2\, {}_{p,q}\mathbb{L}_2\left(F(\varkappa);\frac{\xi}{\sqrt{q}}\right).
\end{aligned}
$$

Hence, the proof is completed. □

The (p,q)-analogues of Laplace-type integral transform to the (p,q)-convolution theorem

In this section, we put an emphasis on giving a (p,q)-convolution theorem for the ${}_{p,q}L_2$ (${}_{p,q}\mathbb{L}_2$ is similar). Let us assume $F(\varkappa) = \varkappa^{2\gamma}$ and $g(\varkappa) = \varkappa^{2\beta-1}, \gamma, \beta > 0$, then the (p,q)-convolution product is defined for F and g as

$$
(F * g)_{p,q}(\varkappa) = \int_0^t F(\eta) g(\varkappa - q\eta) d_{p,q}\eta. \tag{31}
$$

If $p = 1$ in (31), then (31) reduces to the q-convolution theorem, which appeared in [16].

Theorem 5.1 If $F(\varkappa) = \varkappa^{2\gamma}$ and $g(\varkappa) = \varkappa^{2\beta-1}, \gamma, \beta > 0$, then

$$
{}_{p,q}L_2[(F * g)_{p,q}(\varkappa);\xi] = \frac{p^{(7\beta^2-\gamma^2+2\beta\gamma-9\gamma-5\beta)/2}\Gamma_{p,q}(2\gamma+1)\Gamma_{p,q}(2\beta)\Gamma_{p,q}(\gamma+\beta+1)}{[2]_{p,q}\Gamma_{p,q}(2\gamma+2\beta+1)\xi^{2\gamma+2\beta+2}}.
$$

Proof. Using (20) and (31), we get

$$
(F * g)_{p,q}(\varkappa) = \int_0^t \eta^{2\gamma} (\varkappa - q\eta)^{2\beta-1} d_{p,q}\eta
$$

$$= \varkappa \int_0^1 r^{2\gamma}\varkappa^{2\gamma}(\varkappa - qrt)_{p,q}^{2\beta-1} d_{p,q}r$$
$$= \varkappa^{2\gamma+2\beta} \int_0^1 r^{2\gamma}(1 - qr)_{p,q}^{2\beta-1} d_{p,q}r$$
$$= \varkappa^{2\gamma+2\beta} B_{p,q}(2\gamma + 1, 2\beta).$$

Taking ${}_{p,q}L_2$ in the equation above and using (21), we can write

$$\begin{aligned}{}_{p,q}L_2[(F * g)_{p,q}(\varkappa); \xi] &= B_{p,q}(2\gamma + 1, 2\beta) \int_0^\infty \varkappa^{2\gamma+2\beta} E_{p,q}(-q\varkappa^2\xi^2) d_{p,q} \\ &= p^{(2\beta-1)(4\gamma+2\beta)} \frac{\Gamma_{p,q}(2\gamma+1)\Gamma_{p,q}(2\beta)}{\Gamma_{p,q}(2\gamma+2\beta+1)} \, {}_{p,q}L_2(\varkappa^{2(\gamma+\beta)}; \xi) \\ &= p^{(2\beta-1)(4\gamma+2\beta)} \frac{\Gamma_{p,q}(2\gamma+1)\Gamma_{p,q}(2\beta)}{\Gamma_{p,q}(2\gamma+2\beta+1)} \frac{[\gamma+\beta]_{p,q}!}{[2]_{p,q} p^{\frac{(\gamma+\beta)(\gamma+\beta+1)}{2}} \xi^{2\gamma+2\beta+2}} \\ &= \frac{p^{(7\beta^2-\gamma^2+2\beta\gamma-9\gamma-5\beta)/2}\Gamma_{p,q}(2\gamma+1)\Gamma_{p,q}(2\beta)\Gamma_{p,q}(\gamma+\beta+1)}{[2]_{p,q}\Gamma_{p,q}(2\gamma+2\beta+1)\xi^{2\gamma+2\beta+2}}.\end{aligned}$$

This completes the prood of the theorem. □

The (p, q)-analogues of Laplace-type integral transform to the (p, q)-Aleph function

In this section, we derive ${}_{p,q}L_2$ and ${}_{p,q}\mathbb{L}_2$ of the (p, q)-Aleph function with the help of Mellin–Barnes-type integral and (p, q)-gamma function [38]. Now, we use notation

$$\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left(z; (p,q)\middle|\begin{matrix}(a_j, A_j)_{1,n}; [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i} \\ (b_j, B_j)_{1,m}; [\tau_i(b_{ji}, B_{ji})]_{m+1,q_i}\end{matrix}\right) := \frac{1}{2\pi\psi}\int_L \Lambda^{m,n}_{p_i,q_i,\tau_i;r}(A_j; B_j; s)\pi z^{-s} d_{p,q}s$$

where

$$\Lambda^{m,n}_{p_i,q_i,\tau_i;r}(A_j; B_j; s) = \frac{\prod_{j=1}^m \Gamma_{p,q}(b_j + B_j s)\prod_{j=1}^n \Gamma_{p,q}(1 - a_j - A_j s)}{\sum_{i=1}^r \tau_i\left[\prod_{j=m+1}^{q_i} \Gamma_{p,q}(1 - b_{ji} - B_{ji}s)\prod_{j=n+1}^{p_i} \Gamma_{p,q}(a_{ji} + A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1 - s)\sin\pi s\right]}.$$

Furthermore, taking $\tau_i = 1$ in (22), we get the (p, q)-analogue of I-function defined by Ahmad et al. [28] as follows:

$$I^{m,n}_{p_i,q_i;r}\left(z; (p,q)\middle|\begin{matrix}(a_j, A_j)_{1,n}; [(a_{ji}, A_{ji})]_{n+1,p_i} \\ (b_j, B_j)_{1,m}; [(b_{ji}, B_{ji})]_{m+1,q_i}\end{matrix}\right) = \frac{1}{2\pi\psi}\int_L \Lambda^{m,n}_{p_i,q_i,1;r}(A_j; B_j; s)\pi z^{-s} d_{p,q}s. \quad (32)$$

Taking $r = 1$ in (32), we will get the (p, q)-analogue of Fox's H-function defined by Ahmad et al. [28] as follows:

$$H^{m,n}_{P,Q}\left(z; (p,q)\middle|\begin{matrix}(a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q}\end{matrix}\right) = \frac{1}{2\pi\psi}\int_L \Lambda^{m,n}_{p_i,q_i,1;1}(A_j; B_j; s)\pi z^{-s} d_{p,q}s. \quad (33)$$

Taking $A_j = B_j = 1$ in (33), we will get the (p, q)-analogue of Meijer's G-function defined by Pathak et al. [39] as follows:

$$G_{P,Q}^{m,n}\left(z;(p,q)\middle|\begin{matrix}(a_1,a_2,\dots,a_P)\\(b_1,b_2,\dots,b_Q)\end{matrix}\right)=\frac{1}{2\pi\psi}\int_L \Lambda_{p_i,q_i,1;1}^{m,n}(1;1;s)\pi z^{-s}d_{p,q}s.$$

If $p=1$, then the above results of (p,q)-analogue change to well-known results of I-function, H-function, and G-function [11].

Theorem 6.1 Let $z,\rho\in\mathbb{C}$ and $Re(s)>0$. Further, let $A_j,B_j,A_{ji},B_{ji}\in\mathbb{R}^+$, $a_j,b_j,a_{ji},b_{ji}\in\mathbb{C}$ $(i=1,2,\dots,p_i;j=1,2,\dots,q_i),\tau_i>0$ for $i=1,2,\dots,r$. The conditions as given in above for (p,q)-Aleph function also holds. Then, the ${}_{p,q}L_2$ of the $\aleph$-function exists and the following relation holds,

$$ {}_{p,q}L_2\left(\varkappa^{2\rho+1}\aleph_{p_I,q_I,\tau_I;r}^{m,n}\left(z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right)\right)$$
$$=\begin{cases}\dfrac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}\aleph_{p_i,q_i+1,\tau_i;r}^{m+1,n}\\ \times\left(z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0)(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right),\\ \text{if }\Gamma_{p,q}(b_j+B_js)\text{is compared with }\Gamma_{p,q}(\rho);\\ \dfrac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}\aleph_{p_i,q_i+1,\tau_i;r}^{m+1,n}\\ \times\left(z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0)(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right),\\ \text{if }\Gamma_{p,q}(1-a_j+a_js)\text{is compared with }\Gamma_{p,q}(\rho).\end{cases}\tag{34}$$

Proof. In the proof of (34), we first express the (p,q)-analogue of $\aleph$-function occurring on the left hand side of the (34) in terms of Mellin–Barnes contour integral and apply ${}_{p,q}L_2$, then we obtain (say Δ).

$$\Delta=\int_0^\infty \varkappa^{2\rho+1}\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left(z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right)E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa$$
$$=\int_0^\infty\frac{\varkappa^{2\rho+1}}{2\pi\psi}\int_L\Lambda_{p_i,q_i,\tau_i;r}^{m,n}(A_j;B_j;s)\pi z^{-s}E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa d_{p,q}s$$
$$=\frac{1}{2\pi\psi}\int_L\Lambda_{p_i,q_i,\tau_i;r}^{m,n}(A_j;B_j;s)\pi z^{-s}\int_0^\infty\varkappa^{2\rho+1}E_{p,q}(-q\varkappa^2\xi^2)d_{p,q}\varkappa d_{p,q}s.$$

By changing variable with $\varkappa^2\xi^2$ and using (17), we arrive at

$$\Delta=\frac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}$$
$$\times\frac{1}{2\pi\psi}\int_L\frac{\prod_{j=1}^m\Gamma_{p,q}(b_j+B_js)\prod_{j=1}^n\Gamma_{p,q}(1-a_j-A_js)\pi z^{-s}\Gamma_{p,q}(\rho)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_{p,q}(a_{ji}+A_{ji}s)\Gamma_{p,q}(s)\Gamma_{p,q}(1-s)\sin\pi s\right]}d_{p,q}s.$$

Case 1: we consider $\Gamma_{p,q}(b_j+B_js)$ and compare it with $\Gamma_{p,q}(\rho)$. Then, we get

$$\Delta=\frac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}\aleph_{p_i,q_i+1,\tau_i;r}^{m+1,n}\left(z;(p,q)\middle|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0)(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right).$$

Case 2: we consider $\Gamma_{p,q}(1-a_j+a_j s)$ and compare it with $\Gamma_{p,q}(\rho)$. Then, we get

$$\Delta=\frac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left(z;(p,q)\left|\begin{matrix}(1-\rho,0)(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right).$$

By interpreting the Mellin–Barnes counter integral obtained in terms of the (p,q)-analogue of $\aleph$-function, we acquire the result (34). Therefore, the proof is completed. □

Theorem 6.2 Let $z,\rho\in\mathbb{C}$ and $Re(s)>0$. Further, let $A_j,B_j,A_{ji},B_{ji}\in\mathbb{R}^+$, $a_j,b_j,a_{ji},b_{ji}\in\mathbb{C}$ $(i=1,2,\dots,p_i;j=1,2,\dots,q_i),\tau_i>0$ for $i=1,2,\dots,r$. The conditions as given in above for (p,q)-Aleph function also holds. Then, the ${}_{p,q}\mathbb{L}_2$ of the $\aleph$-function exists and the following relation holds,

$$ {}_{p,q}\mathbb{L}_2\left(\varkappa^{2\rho+1}\aleph^{m,n}_{p_I,q_I,\tau_I;r}\left(z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)\right)$$
$$=\begin{cases}\frac{1}{[1+\rho]_{p,q}[2]_{p,q}q^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}\aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r}\\ \times\left(z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0)(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right),\\ \text{if } \gamma_{p,q}(b_j+B_j s) \text{ is compared with } \gamma_{p,q}(\rho);\\ \frac{1}{[1+\rho]_{p,q}[2]_{p,q}q^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\\ \times\left(z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0)(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right),\\ \text{if } \gamma_{p,q}(1-a_j+a_j s) \text{ is compared with } \gamma_{p,q}(\rho).\end{cases}\tag{35}$$

Proof. In the proof of (35), we first express the (p,q)-analogue of $\aleph$-function occurring on the left hand side of the (35) in terms of Mellin–Barnes contour integral and apply ${}_{p,q}L_2$, then we obtain (say Θ).

$$\begin{aligned}\Theta&=\int_0^\infty\varkappa^{2\rho+1}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z;(p,q)\left|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right)e_{p,q}(-p\varkappa^2\xi^2)d_{p,q}\varkappa\\&=\int_0^\infty\frac{\varkappa^{2\rho+1}}{2\pi\psi}\int_L\Lambda^{m,n}_{p_i,q_i,\tau_i;r}(A_j;B_j;s)\pi z^{-s}e_{p,q}(-p\varkappa^2\xi^2)d_{p,q}\varkappa d_{p,q}s\\&=\frac{1}{2\pi\psi}\int_L\Lambda^{m,n}_{p_i,q_i,\tau_i;r}(A_j;B_j;s)\pi z^{-s}\int_0^\infty\varkappa^{2\rho+1}e_{p,q}(-p\varkappa^2\xi^2)d_{p,q}\varkappa d_{p,q}s.\end{aligned}$$

By changing variable with $\varkappa^2\xi^2$ and using (18), we arrive at

$$\Theta=\frac{1}{[1+\rho]_{p,q}[2]_{p,q}q^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}$$
$$\times\frac{1}{2\pi\psi}\int_L\frac{\prod_{j=1}^m\gamma_{p,q}(b_j+B_js)\prod_{j=1}^n\gamma_{p,q}(1-a_j-A_js)\pi z^{-s}\gamma_{p,q}(\rho)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\gamma_{p,q}(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\gamma_{p,q}(a_{ji}+A_{ji}s)\gamma_{p,q}(s)\gamma_{p,q}(1-s)\sin\pi s\right]}d_{p,q}s.$$

Case 1: we consider $\gamma_{p,q}(b_j+B_js)$ and compare it with $\gamma_{p,q}(\rho)$. Then, we get

$$\Theta=\frac{1}{[1+\rho]_{p,q}[2]_{p,q}q^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}\aleph^{m+1,n}_{p_i,q_i+1,\tau_i;r}\left(z;(p,q)\Big|\begin{matrix}(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0)(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right).$$

Case 2: we consider $\gamma_{p,q}(1-a_j+a_js)$ and compare it with $\gamma_{p,q}(\rho)$. Then, we get

$$\Theta=\frac{1}{[1+\rho]_{p,q}[2]_{p,q}q^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}\aleph^{m,n+1}_{p_i+1,q_i,\tau_i;r}\left(z;(p,q)\Big|\begin{matrix}(1-\rho,0)(a_j,A_j)_{1,n}\cdots[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\cdots[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right).$$

By interpreting the Mellin–Barnes counter integral obtained in terms of the (p,q)-analogue of $\aleph$-function, we acquire the result (35). Therefore, the proof is completed. □

Corollary 6.1 Let $\mathrm{Re}(\xi)>0$. Then, we have
(i) if we choose $\tau_i=1$ in (34), then we called this the (p,q)-analogue of Laplace-type integral transform of I-function as follows:

$$ {}_{p,q}L_2\left(\varkappa^{2\rho+1}I^{m,n}_{p_i,q_i;r}\left(z;(p,q)\Big|\begin{matrix}(a_j,A_j)_{1,n}\cdots[(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\cdots[(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right)\right)$$
$$=\begin{cases}\dfrac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}I^{m+1,n}_{p_i,q_i+1;r}\\ \times\left(z;(p,q)\Big|\begin{matrix}(a_j,A_j)_{1,n}\cdots[(a_{ji},A_{ji})]_{n+1,p_i}\\(\rho,0)(b_j,B_j)_{1,m}\cdots[(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right),\\ \text{if }\ \Gamma_{p,q}(b_j+B_js)\text{ is comparedwith }\Gamma_{p,q}(\rho);\\ \dfrac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}I^{m,n+1}_{p_i+1,q_i;r}\\ \times\left(z;(p,q)\Big|\begin{matrix}(1-\rho,0)(a_j,A_j)_{1,n}\cdots[(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}\cdots[(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right),\\ \text{if }\ \Gamma_{p,q}(1-a_j+a_js)\text{ is comparedwith }\Gamma_{p,q}(\rho).\end{cases}$$

(ii) If we choose $\tau_i=1,r=1,p_i=P$, and $q_i=Q$ in (34), then we called this the (p,q)-analogue of Laplace-type integral transform of H-function as follows:

$$ {}_{p,q}L_2\left(\varkappa^{2\rho+1}H^{m,n}_{P,Q}\left(z;(p,q)\Big|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right)\right)$$
$$=\begin{cases}\dfrac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}H^{m+1,n}_{P,Q+1}\left(z;(p,q)\Big|\begin{matrix}(a_j,A_j)_{1,P}\\(\rho,0)(b_j,B_j)_{1,Q}\end{matrix}\right),\\ \text{if }\ \Gamma_{p,q}(b_j+B_js)\text{ is comparedwith }\Gamma_{p,q}(\rho);\\ \dfrac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}H^{m,n+1}_{P+1,Q}\left(z;(p,q)\Big|\begin{matrix}(1-\rho,0)(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right),\\ \text{if }\ \Gamma_{p,q}(1-a_j+a_js)\text{ is comparedwith }\Gamma_{p,q}(\rho).\end{cases}$$

(iii) If we choose $\tau_i=1,r=1,p_i=P,q_i=Q$, and $A_j=B_j=1$ in (34), then we called this the (p,q)-analogue of Laplace-type integral transform of G-function as follows:

$$
{}_{p,q}L_2\left(\varkappa^{2\rho+1}G_{P,Q}^{m,n}\left(z;(p,q)\Big|\begin{matrix}(a_1,a_2,\dots,a_P)\\(b_1,b_2,\dots,b_Q)\end{matrix}\right)\right)
$$

$$
=\begin{cases}\dfrac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}G_{P,Q+1}^{m+1,n}\left(z;(p,q)\Big|\begin{matrix}(a_1,a_2,\dots,a_P)\\(\rho,0)(b_1,b_2,\dots,b_Q)\end{matrix}\right),\\ \text{if } \Gamma_{p,q}(b_j+B_js) \text{ is comparedwith } \Gamma_{p,q}(\rho);\\ \dfrac{1}{[1+\rho]_{p,q}[2]_{p,q}p^{\frac{\rho(\rho+1)}{2}}\xi^{2\rho+2}}G_{P+1,Q}^{m,n+1}\left(z;(p,q)\Big|\begin{matrix}(1-\rho,0)(a_1,a_2,\dots,a_P)\\(b_1,b_2,\dots,b_Q)\end{matrix}\right),\\ \text{if } \Gamma_{p,q}(1-a_j+a_js) \text{ is comparedwith } \Gamma_{p,q}(\rho).\end{cases}
$$

The ${}_{p,q}\mathbb{L}_2$ is similar to the (i), (ii) and (iii), but only changes the $p^{\frac{\rho(\rho+1)}{2}}$ to $q^{\frac{\rho(\rho+1)}{2}}$.

Conclusion

In this work, we introduced the properties of the (p,q)-analogues of Laplace-type integral transform of the ${}_{p,q}L_2$ and ${}_{p,q}\mathbb{L}_2$ which consisted of the (p,q)-special functions, (p,q)-trigonometric types, (p,q)-differential operator, and (p,q)-convolution theorem. Also, we introduced (p,q)-Aleph function and obtained some interesting results by applying ${}_{p,q}L_2$ and ${}_{p,q}\mathbb{L}_2$. In the future study, we see the potential to unlock a novel way to solve some (p,q) differential equations with all of the results derived through this research.

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