การลู่เข้าของขั้นตอนการทำซ้ำของมานน์ด้วยเงื่อนไขการหดตัวอย่างอ่อน
THE CONVERGENCE OF MODIFIED MANN ITERATIVE SCHEME WITH WEAKLY CONTRACTIVE CONDITION

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บทคัดย่อ

ในงานวิจัยนี้ เราได้ศึกษาการลู่เข้าของลำดับของขั้นตอนการทำซ้ำของมานน์ด้วยการส่งแบบหดตัวอย่างอ่อน สองลำดับซึ่งขยายมาจากขั้นตอนวิธีการทำซ้ำของมานน์จะถูกพิจารณาการลู่เข้า ยิ่งไปกว่านั้น เรายังแสดงด้วยว่าลำดับดังที่กล่าวข้างต้นนั้นลู่เข้าไปยังจุดตรึงร่วม โดยสิ่งที่ได้รับในงานวิจัยนี้ได้ขยายกว่าสิ่งที่ได้เคยศึกษามาแล้วในอดีต

คำสำคัญ: ขั้นตอนการทำซ้ำของมานน์ การส่งแบบหดตัวอย่างอ่อน จุดตรึงร่วม

Abstract

In this paper, we study the convergence of a sequence, which is modified from Mann iterative scheme, with a weakly contractive mapping condition. The convergence of two sequences which are generalized from the modified Mann iterative scheme are also considered. Furthermore, we show that these sequences converge to a common fixed point. Our results generalize and extend various known results in the literature.

Keywords: modified Mann iterative scheme, weakly contractive mapping, common fixed point

Introduction

The well known iteration which is used in many fields is Picard iteration, that is,

$$x_{n+1} = T(x_n)$$

where $x_0 \in X, n \geq 0$ with X is a nonempty closed convex subset of real Banach space and $T: X \to X$ is a mapping. The development of Picard iteration was presented in many papers and a iteration which was introduced is Mann iteration method, that is, a sequence $\{x_n\}$ which as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T(x_n)$$

when $n \geq 0, 0 \leq \alpha_n \leq 1$and $x_0 \in X$ is chosen arbitrarily. It is easy to see that Picard iteration is a special case of the Mann iteration. So, the concept of Mann iteration is generalized in many ways, see references therein.

In the other hand, the Banach contraction principle is well known and the important theorem to study in fixed point theorem. The Banach contraction principle states that: if (X, d) is a complete metric space and $T: X \to X$ is a contraction mapping such that

$$d(Tx, Ty) \leq \alpha d(x, y) \quad (1)$$

for all $x, y \in X$ where $0 \leq \alpha \leq 1$ is a constant, then T has a unique fixed point, (see Banach, 1922). This principle has become a power tool for solving the existence problems in mathematical analysis and applying in many fields. By this reasons, Banach contraction principle was developed in the long time. In 2006, I. Beg and M. Abbas (Beg & Abbas, 2006) introduced the weakly contractive mapping with respect to f as follows:

Definition 1 Let X be a metric space. A mapping $T: X \to X$ is called *weakly contractive with respect to* f if for each $x, y \in X$ satisfy

$$d(Tx, Ty) \leq d(fx, fy) - \phi(d(fx, fy)) \quad (2)$$

where $\phi: [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that ϕ is positive on $(0, \infty), \phi(0) = 0$ with $\lim_{t\to\infty} \phi(t) = \infty$ and $f: X \to X$.

I. Beg and M. Abbas showed that if the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have coincidence point in X. In addition, they introduced the modified Mann iterative scheme as the following definition.

Definition 2 Let X be a Banach space and let T be a weakly contractive mapping with respect to f on X. Assume that $T(X) \subseteq f(X)$ and $f(X)$ is a convex subset of X. Define a sequence $\{y_n\}$ in $f(X)$ as

$$y_n = f(x_{n+1}) = (1 - \alpha_n)f(x_n) + \alpha_n T(x_n) \quad (3)$$

where $x_n \in X$, $0 \leq \alpha_n \leq 1$ for each $n \geq 0$. The sequence (3) is called *modified Mann iterative scheme*.

They obtained the results that if T and f are weakly compatible and $T(X) \subseteq f(X)$, $f(X)$ is a complete subspace of X, then a modified Mann iterative scheme satisfying a condition converges to a common fixed point of T and f. Later, in 2009, M. Abbas and M. A. Khan (Abbas & Khan, 2009) generalized the Banach contractive mapping as follows: let $G = \{\psi: [0, \infty] \to [0, \infty] | \psi$ is continuous and nondecreasing mapping with $\psi(t) = 0$ if and only if $t = 0\}$ and let $T: X \to X$and $f: X \to X$ be mappings which satisfy

$$\psi\left(d\big(T(x), T(y)\big)\right) \leq \psi\left(d\big(f(x), f(y)\big)\right) - \phi(d(f(x), f(y))) \quad \text{.....................(4)}$$

for all $x, y \in X$, which $\psi, \phi \in G$. M. Abbas and M. A. Khan shown that if the range of f contains the range of $T, f(X)$ is a complete subspace of X and if T and f are weakly compatible, then T and f have a unique point of coincidence in X and T and f have a unique common fixed point. The interesting of the concepts of the modified Mann iterative scheme (3) and the generalized of Banach contraction (4), many researchers are analyze and extend the concepts for studying the fixed point theorem, see references therein.

The aim of this paper, we will show the convergence of both the modified Mann iterative scheme and generalized modified Mann iterative scheme. Furthermore, we also show that the iterative schemes converge to the common fixed point.

Preliminaries

Definition 3 (Kreyszig, 1989) A *linear operator* T is an operator such that

i) the domain $D(T)$ of T is a vector space and $R(T)$ lies in a vector space over the same field,

ii) for all $x, y \in D(T)$ and scalar α

$$T(x + y) = Tx + Ty$$
$$T(\alpha x) = \alpha Tx$$

Definition 4 (Beg & Abbas, 2006) A point x in X is a *common fixed point* of T and f if

$$f(x) = T(x) = x.$$

Definition 5 (Abbas& Khan, 2009) Two mappings of T and f are said to be *weakly compatible* if they commute at their coincidence points.

Main Results

Throughout this paper, let $T: X \to X$ and $f: X \to X$ be mappings. Define $G = \{\psi: [0, \infty] \to [0, \infty] | \psi$ is continuous, linear and nondecreasing mapping with $\psi(t) = 0$ if and only if $t = 0\}$. In (Abbas & Khan, 2009), M. Abbas and M. A. Khan proved the following theorem:

Theorem 7 Let T and f be two self mappings of a metric space (X, d) satisfying (3). If the range of f contains the range of T and $f(X)$ is complete subspace of X, then T

and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

We are interesting the following generalized weakly contractive mapping in the sense of norm as follows: let $T: X \to X$and $f: X \to X$ be mappings which satisfy

$$\psi(\|T(x) - T(y)\|) \leq \psi(\|f(x) - f(y)\|) - \phi(\|f(x) - f(y)\|) \quad \text{.....................(5)}$$

for all $x, y \in X$, where $\psi, \phi \in G$.

Next, we show that the modified Mann iterative scheme converges to a common fixed point of T and f.

<u>Theorem 8</u> Let X be a normed space and a mapping $T: X \to X$ be a weakly contractive mapping with respect to f, as $f: X \to X$. If T and f are weakly compatible, $T(X) \subseteq f(X)$ and $f(X)$ is complete subspace of X. Then, the modified Mann iterative scheme (3) with $\sum \alpha_n = \infty$ converges to a common fixed point of T and f.

<u>Proof</u> By Theorem 7, we obtain that T and f have a common fixed point. Assume that q is a common fixed point of T and f, that is, $T(q) = f(q) = q$ for some $q \in X$.
Since ψ is a linear mapping and let $\{y_n\}$ be a sequence which satisfies modified Mann iterative scheme, we see that

$$\begin{aligned}
&\psi(\|y_n - q\|) \\
&= \psi(\|(1-\alpha_n)f(x_n) + \alpha_n T(x_n) - f(q)\|) \\
&\leq \psi((1-\alpha_n)\|f(x_n) - f(q)\| + \alpha_n\|T(x_n) - f(q)\|) \\
&= \psi((1-\alpha_n)\|f(x_n) - f(q)\|) + \psi(\alpha_n\|T(x_n) - T(q)\|) \\
&= (1-\alpha_n)\psi(\|f(x_n) - f(q)\|) + \alpha_n\psi(\|T(x_n) - T(q)\|) \\
&\leq (1-\alpha_n)\psi(\|f(x_n) - f(q)\|) + \alpha_n[\psi(\|f(x_n) - f(q)\|) - \phi(\|f(x_n) - f(q)\|)] \\
&= \psi(\|f(x_n) - f(q)\|) - \alpha_n\psi(\|f(x_n) - f(q)\|) + \alpha_n\psi(\|f(x_n) - f(q)\|) \\
&\quad -\alpha_n\phi(\|f(x_n) - f(q)\|)) \\
&= \psi(\|f(x_n) - f(q)\|) - \alpha_n\phi(\|f(x_n) - f(q)\|)) \quad \text{.....................(6)} \\
&\leq \psi(\|f(x_n) - f(q)\|).
\end{aligned}$$

Since $\psi(\|y_n - q\|) \leq \psi(\|f(x_n) - f(q)\|)$ and ψ is nondecreasing function, we have

$$\|y_n - q\| \leq \|f(x_n) - f(q)\| = \|y_{n-1} - q\|.$$

This implies that $\|y_n - q\|$ is a nonincreasing sequence.

Let $\lim_{n\to\infty}\|y_n - q\| = r \geq 0$. Assume that $r > 0$. For all $n \in N$, by (6) we get

$$\psi(\|y_n - q\|) \leq \psi(\|f(x_n) - f(q)\|) - \alpha_n \phi(\|f(x_n) - f(q)\|)$$
$$= \psi(\|y_{n-1} - f(q)\|) - \alpha_n \phi(\|y_{n-1} - f(q)\|)$$
$$= \psi(\|y_{n-1} - q\|) - \alpha_n \phi(\|y_{n-1} - q\|).$$

This implies that

$$\alpha_n \phi(\|y_{n-1} - q\|) \leq \psi(\|y_{n-1} - q\|) - \psi(\|y_n - q\|) \quad \text{.......(7)}$$

Since $\{\|y_n - q\|\}$ is a nonincreasing sequence, we obtain that

$$\phi(\|y_n - q\|) \leq \phi(\|y_{n-1} - q\|),$$

and$\alpha_n \phi(\|y_n - q\|) \leq \alpha_n \phi(\|y_{n-1} - q\|)$. So,

$$\sum_{n=N}^{\infty} \alpha_n \phi(\|y_n - q\|) \leq \sum_{n=N}^{\infty} \alpha_n \phi(\|y_{n-1} - q\|) \quad \text{.......(8)}$$

By using (2) and (7), we have

$$\sum_{n=N}^{\infty} \alpha_n \phi(r) \leq \sum_{n=N}^{\infty} \alpha_n \phi(\|y_n - q\|)$$
$$\leq \sum_{n=N}^{\infty} [\psi(\|y_{n-1} - q\|) - \psi(\|y_n - q\|)]$$
$$\leq \psi(\|y_{n-1} - q\|).$$

It is a contradiction with α_n. Thus, $r = 0$. This implies that, $\lim_{n\to\infty} \|y_n - q\| = 0$.

Therefore, $\lim_{n\to\infty} y_n = q$.

□

Next, we will present the generalized modified Mann iterative scheme as follows: the sequences $\{y_n\}$ and $\{z_n\}$ define by

$$z_n = f(x_{n+1}) = (1 - \alpha_n) f(x_n) + \alpha_n T(v_n)$$
$$y_n = f(v_n) = (1 - \beta_n) f(x_n) + \beta_n T(x_n), \ \ n = 0, 1, 2, \ldots \quad \text{.......(9)}$$

where $0 \leq \alpha_n, \beta_n \leq 1$, and $x_0 \in X$. The following theorem show that the iterative scheme (9) converges to a common fixed point of T and f.

Theorem 9 Let X be a normed space and a mapping $T: X \to X$ be a weakly contractive mapping with respect to f, as $f: X \to X$. If T and$f$ are weakly compatible and $T(X) \subseteq f(X)$ and $f(X)$ is complete subspace of X. Suppose that $\{y_n\}$ and $\{z_n\}$ are define as (9) and $\sum \alpha_n \beta_n = \infty$. Then, the iterative scheme $\{z_n\}$ converges to a common fixed point of T and f.

<u>Proof</u> Assume that q is a common fixed point of T and f, that is, $T(q) = f(q) = q$ for some $q \in X$. Since ψ is a linear mapping and let $\{y_n\}$ and $\{z_n\}$ be sequences which satisfy (9), we see that

$$\psi(\|z_n - q\|)$$
$$= \psi(\|(1-\alpha_n)f(x_n) + \alpha_n T(v_n) - q\|)$$
$$\leq \psi((1-\alpha_n)\|f(x_n) - q\| + \alpha_n\|T(v_n) - q\|)$$
$$= \psi((1-\alpha_n)\|f(x_n) - q\| + \alpha_n\|T(v_n) - T(q)\|)$$
$$= \psi((1-\alpha_n)\|f(x_n) - q\|) + \psi(\alpha_n\|T(v_n) - T(q)\|)$$
$$= (1-\alpha_n)\psi(\|f(x_n) - q\|) + \alpha_n\psi(\|T(v_n) - T(q)\|)$$
$$\leq (1-\alpha_n)\psi(\|f(x_n) - q\|) + \alpha_n[\psi(\|f(v_n) - f(q)\|) - \phi(\|f(v_n) - f(q)\|)]$$
$$= (1-\alpha_n)\psi(\|f(x_n) - q\|) + \alpha_n\psi(\|(1-\beta_n)f(x_n) + \beta_n T(x_n) - q\|)$$
$$-\alpha_n\phi(\|f(v_n) - q\|)$$
$$\leq (1-\alpha_n)\psi(\|f(x_n) - q\|) - \alpha_n\phi(\|f(v_n) - q\|)$$
$$+\alpha_n\psi(\|(1-\beta_n)(f(x_n) - q)\| + \beta_n\|T(x_n) - T(q)\|)$$
$$\leq (1-\alpha_n)\psi(\|f(x_n) - q\|) - \alpha_n\phi(\|f(v_n) - q\|) + (1-\beta_n)\alpha_n\psi(\|f(x_n) - q\|)$$
$$+\beta_n\alpha_n\psi(\|T(x_n) - T(q)\|)$$
$$\leq (1-\alpha_n)\psi(\|f(x_n) - q\|) - \alpha_n\phi(\|f(v_n) - q\|) + (1-\beta_n)\alpha_n\psi(\|f(x_n) - q\|)$$
$$+\beta_n\alpha_n[\psi(\|f(x_n) - f(q)\|) - \phi(\|f(x_n) - f(q)\|)]$$
$$= (1-\alpha_n)\psi(\|f(x_n) - q\|) - \alpha_n\phi(\|f(v_n) - q\|) + (1-\beta_n)\alpha_n\psi(\|f(x_n) - q\|)$$
$$+\beta_n\alpha_n[\psi(\|f(x_n) - q\|) - \phi(\|f(x_n) - q\|)]$$
$$= \psi(\|f(x_n) - q\|) - \alpha_n\psi(\|f(x_n) - q\|) - \alpha_n\phi(\|f(v_n) - q\|) + \alpha_n\psi(\|f(x_n) - q\|)$$
$$-\beta_n\alpha_n\psi(\|f(x_n) - q\|) + \beta_n\alpha_n\psi(\|f(x_n) - q\|) - \beta_n\alpha_n\phi(\|f(x_n) - q\|)$$
$$= \psi(\|f(x_n) - q\|) - \alpha_n\phi(\|f(v_n) - q\|) - \beta_n\alpha_n\phi(\|f(x_n) - q\|) \quad \text{.....................(10)}$$
$$\leq \psi(\|f(x_n) - q\|)$$
$$= \psi(\|z_{n-1} - q\|).$$

Since ψ is nondecreasing function, we obtain that $\{\|z_{n-1} - q\|\}$ is a nonincreasing sequence. Let $\lim_{n\to\infty}\|z_n - q\| = r \geq 0$. Assume that $r > 0$. For all $n \in N$, by (10) we get

$$\psi(\|z_n - q\|) \leq \psi(\|f(x_n) - q\|) - \alpha_n\phi(\|f(v_n) - q\|) - \beta_n\alpha_n\phi(\|f(x_n) - q\|)$$
$$= \psi(\|z_{n-1} - q\|) - \alpha_n\phi(\|y_n - q\|) - \beta_n\alpha_n\phi(\|z_{n-1} - q\|)$$

$\leq \psi(\|z_{n-1} - q\|) - \beta_n \alpha_n \phi(\|z_{n-1} - q\|)$.

Then, $\beta_n \alpha_n \phi(\|z_{n-1} - q\|) \leq \psi(\|z_{n-1} - q\|) - \psi(\|z_n - q\|)$.....................(11)

Since $\{\|z_n - q\|\}$ is a nonincreasing sequence, we have

$$\|z_n - q\| \leq \|z_{n-1} - q\|.$$

Then, $\beta_n \alpha_n \phi(\|z_n - q\|) \leq \beta_n \alpha_n \phi(\|z_{n-1} - q\|)$.

This implies that

$$\sum_{n=N}^{\infty} \beta_n \alpha_n \phi(\|z_n - q\|) \leq \sum_{n=N}^{\infty} \beta_n \alpha_n \phi(\|z_{n-1} - q\|) \quad \text{.............................(12)}$$

By using (11) and(12),we have

$$\begin{aligned}
\sum_{n=N}^{\infty} \beta_n \alpha_n \phi(r) &\leq \sum_{n=N}^{\infty} \beta_n \alpha_n \phi(\|z_n - q\|) \\
&\leq \sum_{n=N}^{\infty} \beta_n \alpha_n \phi(\|z_{n-1} - q\|) \\
&\leq \sum_{n=N}^{\infty} [\psi(\|z_{n-1} - q\|) - \psi(\|z_n - q\|)] \\
&\leq \psi(\|z_{N-1} - q\|).
\end{aligned}$$

It is a contradiction with $\sum \alpha_n \beta_n = \infty$. Then $r > 0$.
This implies that $\lim_{n\to\infty} \|z_n - q\| = 0$. Therefore, $\lim_{n\to\infty} z_n = q$.

□

Conclusion

In this paper, we present the convergence sequences of both modified Mann iterative scheme and generalized modified Mann iterative scheme on the concept of the generalized weakly contractive mapping. Moreover, we show that these sequences converge to a common fixed. The results in this paper extend (Beg & Abbas, 2006).

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