ระบบพลวัตสำหรับระบบของปัญหาเซตย่อยแปรผัน
DYNAMICAL SYSTEM FOR THE SYSTEM OF VARIATIONAL INCLUSION PROBLEM

จิตติพร ตังควิเวชกุล[1]* และนรินทร์ เพชร์โรจน์[2]
Jittiporn Tangkhawiwetkul[1]*, and Narin Petrot[2]

[1]Faculty of Science and Technology, Pibulsongkram Rajabhat University
[2]Faculty of Science, Naresuan University
*corresponding author e-mail: j_suwannawit@psru.ac.th
Received: 13 February 2019; Revised: 24 May 2019; Accepted: 24 July 2019

บทคัดย่อ

ในงานวิจัยนี้เราได้พิจารณาระบบของเซตย่อยแปรผันและได้แนะนำสมการแก้ปัญหา ซึ่งสมมูลกับระบบของเซตย่อยแปรผัน ระบบพลวัตที่สอดคล้องกับเซตย่อยแปรผันได้ถูกนำเสนอ ยิ่งไปกว่านั้น ผลเฉลยของระบบพลวัตดังกล่าวนั้นได้ถูกพิสูจน์ ซึ่งผลลัพธ์ในงานวิจัยชิ้นนี้ได้พัฒนาและขยายปัญหาเซตย่อยแปรผันที่ได้ศึกษามาแล้วในอดีต

คำสำคัญ: ระบบพลวัต เซตย่อยแปรผัน ตัวดำเนินการแก้ปัญหา อสมการของ Gronwall

Abstract

In this paper, we consider the system of variational inclusion and introduce the resolvent equation which is equivalent to the system of variational inclusion. The dynamical system associated with the system of variational inclusion is presented. Furthermore, the solution of such dynamical system is proved. The results in this paper improve and extend the variational inclusion problems which have been appeared in literature.

Keywords: dynamical system, variational inclusion, resolvent operator, Gronwall's inequality

Introduction

Variational inclusion is the generalization of the variational inequality problem which the class of variational inclusions include variational inequalities, complementarity problems, convex optimization and saddle point problems as special cases. Then, the variational inclusion problem is used to study and apply in fields of optimization and control, economics and transportation equilibrium, engineering science, see (Verma, 2004; Agarwal & Verma, 2009; Lin, 2009). The interesting of the variational inclusion problem implies many researches using such problem to develop problem in various fields. A system of variational inclusion is the generalized of the variational inclusion which the system of variational inclusion is applied in traffic equilibrium problem, Nash equilibrium which is more extension than the variational inclusion problem, see (Agarwal, 2004; Fang & Huang, 2004; Yan et al., 2005).

On the other hand, the dynamical system is well known theory, which is applied for considering some problems related to time and is used in many fields such as in economics, physics, engineering, medicine and mathematics etc. see (Bahiana & Oono, 1990; Dong et al., 1996; Scrimali, 2008; Biswas & Chakraborty, 2015). But not only that, the dynamical system is applied with variational inequality problem because the dynamical systems implies the mathematical problems for close to the real world problem. The methodology which is used to consider the dynamical system and variational inequality problem is the projected dynamical system for considering by P. Dupuis and A. Nagurney (Dupuis & A. Nagurney, 1993), in 1993. P. Dupuis et al. presented the basic theory of the projected dynamical systems and considered the relation of the variational inequality theory and the dynamical system theory, that is, the set of stationary points of dynamical system coincides with the set of the solutions of variational inequality problem. So, the projected dynamical system is used to solve some problems on linear and nonlinear in variational inequality. By the previous reasons, the dynamical system has been used to apply in financial equilibrium problems, optimization problems, fixed point problems, complementarity problem and all problems in the framework of variational inequalities (see Dong et al., 1996; Nagurney & Zhang, 1996; Bliemer & Bovy, 2003; Isac & Cojocaru, 2002; Ansari et al., 2013 and the reference therein). The attention of authors to develop the dynamical system and variational inequality, in 2002, M. A. Noor (Noor, 2002a, b) introduced the dynamical system for variational inclusion which extended from the aspects of dynamical system for variational inequality. By this reasons, the dynamical system for variational inclusion is interesting to study because it can be apply in the various real world problems.

In this paper, we will study the system of variational inclusion by using the resolvent operator and introduce the implicit resolvent equation of such system of variational inclusion. After that we can introduce the dynamical system associated with such implicit resolvent equation. Furthermore, the existence solution of such dynamical system is considered. Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. Let 2^H be denoted for the class of all nonempty subsets of H.

Now, we will recall the fundamental concepts of stability in dynamical system. Let the general dynamical system as follows:

$$\frac{dx(t)}{dt} = f(x(t)) \qquad \text{----------------------(1)}$$

for $x(t) \in H, t$ is a real number and f is a continuous function from H into itself.

<u>**Definition 1**</u> (Ha et al., 2018) A point x^* is an ***equilibrium point*** for (1) if $f(x^*) = 0$.

<u>**Lemma 1**</u> **(Gronwall's Lemma)** (Ansari et al., 2013)

Let $\hat{u}$ and $\hat{v}$ be real valued nonnegative continuous functions with domain $\{t|t \geq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If for all $t \geq t_0$,

$$\hat{u}(t) \leq \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)\, ds,$$

then,

$$\hat{u}(t) \leq \alpha(t)e^{\int_{t_0}^{t} \hat{v}(s)ds}.$$

Now, we will introduce the properties of mappings which are used in our results.

<u>**Definition 2**</u> (Suwannawit & Petrot, 2012) A mapping $S: H \times H \to H$ is said to be a δ_1-strongly monotone in the first argument if there exists a real number $\delta_1 > 0$ such that for all $x, y \in H$,

$$\langle S(x,\cdot) - S(y,\cdot), x - y \rangle \geq \delta_1 \|x - y\|^2.$$

Definition 3 (Suwannawit & Petrot, 2012) A mapping $S: H \times H \to H$ is said to be a β_1-Lipschitzain in the first argument if there exists a real number $\beta_1 > 0$ such that for all $x, y \in H$,

$$\|S(x,\cdot) - S(y,\cdot)\| \le \beta_1 \|x - y\|.$$

Definition 4 (Noor, 2000) If T is a maximal monotone operator on H, then for a constant $\rho > 0$, the resolvent operator associated with T is defined by

$$J_T^{\rho}(u) = (I + \rho T)^{-1}(u)$$

for all $u \in H$, where I is an identity operator. Also, the resolvent operator J_T is a single valued and nonexpansive mapping, that is,

$$\left\|J_T^{\rho}(u) - J_T^{\rho}(v)\right\| \le \|u - v\|$$

for all $u, v \in H$.

Main Results

Throughout in this paper, we let H_1 and H_2 be two real Hilbert spaces. We will consider the system of nonlinear variational inclusion **(SNVI)** which was studied by R. U. Verma in (Verma, 2004) as follows. Let $M: H_1 \to 2^{H_1}$ and $N: H_2 \to 2^{H_2}$ be nonlinear mappings. Let $S: H_1 \times H_2 \to H_1$ and $T: H_1 \times H_2 \to H_2$ be nonlinear mappings. To find $(x, y) \in H_1 \times H_2$ such that

$$0 \in S(x,y) + M(x)$$

$$0 \in T(x,y) + N(y). \qquad \text{----------------------}\textbf{(SNVI)}$$

Verma considered the existence of the problem (SNVI) by using A-monotonicity. So, in this paper we will present the existence of the problem (SNVI) on the resolvent operator J_T.

Lemma 2 Let H_1 and H_2 be two real Hilbert spaces. Let $S: H_1 \times H_2 \to H_1$ and $T: H_1 \times H_2 \to H_2$ be nonlinear mappings. Let $M: H_1 \to 2^{H_1}$ and $N: H_2 \to 2^{H_2}$ be a maximal monotone operators. Then, the following are true:

1. If $(x, y) \in H_1 \times H_2$ is a solution to the problem (SNVI) then, for any $\rho_1, \rho_2 > 0$ such that

$$x = J_M^{\rho_1}[x - \rho_1 S(x,y)] \text{ and } y = J_N^{\rho_2}[y - \rho_2 T(x,y)].$$

2. If there exist $\rho_1, \rho_2 > 0$ such that

$$x = J_M^{\rho_1}[x - \rho_1 S(x,y)] \text{ and } y = J_N^{\rho_2}[y - \rho_2 T(x,y)].$$

then, (x, y) is a solution to the problem (SNVI).

<u>Proof</u> 1. Assume that $(x,y) \in H_1 \times H_2$ is a solution to the problem (SNVI). This implies that, for any $\rho_1, \rho_2 > 0$,

$$x - \rho_1 S(x,y) \in (I + \rho_1 M)(x) \text{ and } y - \rho_2 T(x,y) \in (I + \rho_2 N)(y).$$

Hence, $x = J_M^{\rho_1}[x - \rho_1 S(x,y)]$ and $y = J_N^{\rho_2}[y - \rho_2 T(x,y)]$, where $J_M^{\rho_1} = (I + \rho_1 M)^{-1}$ and $J_N^{\rho_2} = (I + \rho_2 N)^{-1}$.

2. There exist $\rho_1, \rho_2 > 0$ such that

$$x = J_M^{\rho_1}[x - \rho_1 S(x,y)] \text{ and } y = J_N^{\rho_2}[y - \rho_2 T(x,y)].$$

Since$J_M^{\rho_1} = (I + \rho_1 M)^{-1}$ and $J_N^{\rho_2} = (I + \rho_2 N)^{-1}$, we have

$$x - \rho_1 S(x,y) \in (I + \rho_1 M)(x) \text{ and } y - \rho_2 T(x,y) \in (I + \rho_2 N)(y).$$

Therefore, $0 \in S(x,y) + M(x)$ and $0 \in T(x,y) + N(y)$. We conclude that (x,y) is a solution to the problem (SNVI). This completes the proof. ■

<u>Theorem 1</u> Let H_1 and H_2 be two real Hilbert spaces. Let $M: H_1 \to 2^{H_1}$ and $N: H_2 \to 2^{H_2}$ be a maximal monotone operator. Let $S: H_1 \times H_2 \to H_1$ be a β_1-Lipschitz mapping, δ_1-strongly monotone mapping in a first argument and β_2-Lipschitz mapping in a second argument. Let $T: H_1 \times H_2 \to H_2$ be a κ_1-Lipschitz mapping in a first argument and a κ_2-Lipschitz mapping, ξ_2-strongly monotone mapping in a second argument. If there exist positive constants $\rho_1, \rho_2 > 0$such that

$$\sqrt{1 - 2\rho_1\delta_1 + \rho_1^2\beta_1^2} + \rho_2\kappa_1 < 1 \text{ and } \sqrt{1 - 2\rho_2\xi_2 + \rho_2^2\kappa_2^2} + \rho_1\beta_2 < 1. \text{-----------(3)}$$

Then, the problem (SNVI) admits a unique solution.

<u>Proof</u> For any given $(x, y) \in H_1 \times H_2$, we now define $Q: H_1 \times H_2 \to H_1 \times H_2$ by $Q(x,y) = (f(x,y), g(x,y))$ where

$$f(x,y) = J_M^{\rho_1}[x - \rho_1 S(x,y)] \text{ and } g(x,y) = J_N^{\rho_2}[y - \rho_2 T(x,y)].$$

Next, we will show that the mapping Q is a contraction mapping.

Let $(\tilde{x},\tilde{y})$ and (x^*,y^*) in $H_1 \times H_2$. Since

$\|f(\tilde{x},\tilde{y}) - f(x^*,y^*)\| = \|J_M^{\rho_1}[\tilde{x} - \rho_1 S(\tilde{x},\tilde{y})] - J_M^{\rho_1}[x^* - \rho_1 S(x^*,y^*)]\|$

$\leq \|\tilde{x} - x^* - \rho_1(S(\tilde{x},\tilde{y}) - S(x^*,\tilde{y}))\| + \rho_1\|S(x^*,\tilde{y}) - S(x^*,y^*)\|$

and

$\|g(\tilde{x},\tilde{y}) - g(x^*,y^*)\|$

$\leq \|\tilde{y} - y^* - \rho_2(T(\tilde{x},\tilde{y}) - T(\tilde{x},y^*))\| + \rho_2\|T(\tilde{x},y^*) - T(x^*,y^*)\|.$

By using the δ_1-strongly monotone mapping and β_1-Lipschitz mapping in a first argument of the mapping S, we see that

$\|\tilde{x} - x^* - \rho_1(S(\tilde{x},\tilde{y}) - S(x^*,\tilde{y}))\|^2$

$= \|\tilde{x} - x^*\|^2 - 2\rho_1\langle S(\tilde{x},\tilde{y}) - S(x^*,\tilde{y}), \tilde{x} - x^*\rangle + \rho_1^2\|S(\tilde{x},\tilde{y}) - S(x^*,\tilde{y})\|^2$

$\leq (1 - 2\rho_1\delta_1 + \rho_1^2\beta_1^2)\|\tilde{x} - x^*\|^2$----------------------**(4)**

and using a κ_2-Lipschitz mapping and ξ_2-strongly monotone mapping in a second argument of the mapping T, we obtain that

$\|\tilde{y} - y^* - \rho_2(T(\tilde{x},\tilde{y}) - T(\tilde{x},y^*))\|^2 \leq (1 - 2\rho_2\xi_2 + \rho_2^2\kappa_2^2)\|\tilde{y} - y^*\|^2$. ------**(5)**

Since S is β_2-Lipschitz mapping in a second argument and T is a κ_1-Lipschitz mapping in a first argument, (4) and (5), we have

$$\|f(\tilde{x},\tilde{y}) - f(x^*,y^*)\| \leq \sqrt{1 - 2\rho_1\delta_1 + \rho_1^2\beta_1^2}\|\tilde{x} - x^*\| + \rho_1\beta_2\|\tilde{y} - y^*\|,$$

$$\|g(\tilde{x},\tilde{y}) - g(x^*,y^*)\| \leq \sqrt{1 - 2\rho_2\xi_2 + \rho_2^2\kappa_2^2}\|\tilde{y} - y^*\| + \rho_2\kappa_1\|\tilde{x} - x^*\|.$$

Now, we define the norm $\|\cdot\|^+$ on $H_1 \times H_2$ by

$$\|(x,y)\|^+ = \|x\| + \|y\| \text{ for all } (x,y) \in H_1 \times H_2.$$

It is easy to see that $(H_1 \times H_2, \|\cdot\|^+)$ is a Hilbert space, see (Suantai & Petrot, 2011; Suwannawit & Petrot, 2012). Then

$\|Q(\tilde{x},\tilde{y}) - Q(x^*,y^*)\|^+ = \|f(\tilde{x},\tilde{y}) - f(x^*,y^*)\| + \|g(\tilde{x},\tilde{y}) - g(x^*,y^*)\|$

$\leq \sqrt{1 - 2\rho_1\delta_1 + \rho_1^2\beta_1^2}\|\tilde{x} - x^*\| + \rho_1\beta_2\|\tilde{y} - y^*\| +$

$\sqrt{1 - 2\rho_2\xi_2 + \rho_2^2\kappa_2^2}\|\tilde{y} - y^*\| + \rho_2\kappa_1\|\tilde{x} - x^*\|$

$\leq k(\|\tilde{x} - x^*\| + \|\tilde{y} - y^*\|) = k\|(\tilde{x},\tilde{y}) - (x^*,y^*)\|^+,$

where $k = max\left\{\sqrt{1-2\rho_1\delta_1+\rho_1^2\beta_1^2}+\rho_2\kappa_1, \sqrt{1-2\rho_2\xi_2+\rho_2^2\kappa_2^2}+\rho_1\beta_2\right\}$.

By using (3), we have $k<1$. This implies that Q is a contraction mapping. Hence there exists $(x,y) \in H_1 \times H_2$ which is the unique fixed point of Q. Thus

$$x = J_M^{\rho_1}[x-\rho_1 S(x,y)] \text{ and } y = J_N^{\rho_2}[y-\rho_2 T(x,y)].$$

Therefore, by Lemma 2, (x,y) is a unique solution of the problem (SNVI). This completes the proof. ■

Next, we will introduce the resolvent equation which is equivalent to the problem (SNVI) as follows. Fixed $(x,y) \in H_1 \times H_2$ and let $S: H_1 \times H_2 \to H_1$ and $T: H_1 \times H_2 \to H_2$ be nonlinear mappings. Let $M: H_1 \to 2^{H_1}$ and $N: H_2 \to 2^{H_2}$ be maximal monotone operators. We consider the problem of finding $(u,v) \in H_1 \times H_2$ such that

$$S(x,y) + \rho_1^{-1} R_M^{\rho_1}(u) = 0$$

$$T(x,y) + \rho_2^{-1} R_N^{\rho_2}(v) = 0 \qquad \text{----------------------}\textbf{(RESNVI)}$$

for some positive constants ρ_1 and ρ_2 and $R_M^{\rho} = I - J_M^{\rho}$ where I is an identity operator and J_M^{ρ} is defined in Definition 4.

__Lemma 3__ Let H_1 and H_2 be two real Hilbert spaces. Let $S: H_1 \times H_2 \to H_1$ and $T: H_1 \times H_2 \to H_2$ be nonlinear mappings. Let $M: H_1 \to 2^{H_1}$ and $N: H_2 \to 2^{H_2}$ be maximal monotone operators. Then, $(x,y) \in H_1 \times H_2$ is a solution of (SVNI) if and only if $(u,v) \in H_1 \times H_2$ is a solution of **(RESNVI)** with

$$x = J_M^{\rho_1}[u] \text{ and } y = J_N^{\rho_2}[v]$$

with $u = x - \rho_1 S(x,y)$ and $v = y - \rho_2 T(x,y)$ and ρ_1, ρ_2 are positive constants.

__Proof__$(\Rightarrow)$ Assume that $(x,y) \in H_1 \times H_2$ is a solution of (SVNI). By Lemma 2, we have

$$x = J_M^{\rho_1}[x-\rho_1 S(x,y)] \text{ and } y = J_N^{\rho_2}[y-\rho_2 T(x,y)].$$

Since $R_M^{\rho_1} = I - J_M^{\rho_1}$, we have

$$R_M^{\rho_1}(x-\rho_1 S(x,y)) = (I - J_M^{\rho_1})(x-\rho_1 S(x,y)) = -\rho_1 S(x,y).$$

This implies that $S(x,y) + \rho_1^{-1} R_M^{\rho_1}(u) = 0$, where $u = x - \rho_1 S(x,y)$. In similarly way, we obtain that $T(x,y) + \rho_2^{-1} R_N^{\rho_2}(v) = 0$, where $v = y - \rho_2 T(x,y)$.

$(\Leftarrow)$Conversely, fixed $(x,y) \in H_1 \times H_2$ and (u,v) is a solution of **(RESNVI)**, that is,

$$S(x,y) + \rho_1^{-1} R_M^{\rho_1}(u) = 0$$
$$T(x,y) + \rho_2^{-1} R_N^{\rho_2}(v) = 0.$$

Since $R_M^{\rho_1} = I - J_M^{\rho_1}$ and $R_N^{\rho_2} = I - J_N^{\rho_2}$, we have

$$x = J_M^{\rho_1}(u) \text{ and } y = J_N^{\rho_2}(v).$$

So, $x = J_M^{\rho_1}(x - \rho_1 S(x,y))$ and $y = J_N^{\rho_2}(y - \rho_2 T(x,y))$. By Lemma 2, we have (x,y) is a solution of the problem (SNVI). This completes the proof. ■

Now, we will present the resolvent dynamical system of the system of nonlinear variational inclusions. By Lemma 3, we have if $(x,y) \in H_1 \times H_2$is a solution of the problem (SNVI) then

$$S(x,y) + \rho_1^{-1} R_M^{\rho_1}(u) = 0 \text{ and } T(x,y) + \rho_2^{-1} R_N^{\rho_2}(v) = 0.$$

Since $R_M^{\rho} = I - J_M^{\rho}$and $R_N^{\rho_2} = I - J_N^{\rho_2}$, we have

$$x - J_M^{\rho_1}(x - \rho_1 S(x,y)) = 0 \text{ and } y - J_N^{\rho_2}(y - \rho_2 T(x,y)) = 0.$$

So, we suggest the resolvent dynamical system associate with the system of nonlinear variational inclusion: for any real number t,

$$\frac{dx(t)}{dt} = \lambda\left\{J_M^{\rho_1}\left(x(t) - \rho_1 S(x(t), y(t))\right) - x(t)\right\}$$

$$\frac{dy(t)}{dt} = \gamma\{J_N^{\rho_2}(y(t) - \rho_2 T(x(t), y(t))) - y(t)\} \quad \text{-----------}\textbf{(RDSSNVI)}$$

with $x(t_0) = x_0 \in H_1$ and $y(t_0) = y_0 \in H_2$ and λ, γ are positive constants with a positive real number t_0.

Next, we will propose the existence theorem of the problem (RDSSNVI).

<u>**Theorem 2**</u> Assume that all of the assumption of Theorem 1 hold. Then, for each $(x_0, y_0) \in H_1 \times H_2$, there exists a unique continuous solution$(x(t), y(t))$ of the problem (RDSSNVI) with $x(t_0) = x_0$ and $y(t_0) = y_0$ over $[t_0, \infty)$.

<u>**Proof**</u> Let $F: H_1 \times H_2 \to H_1 \times H_2$ define by $F(x(t), y(t)) = (q(x(t), y(t)), p(x(t), y(t)))$ where

$$q(x(t), y(t)) = \lambda\{J_M^{\rho_1}(x(t) - \rho_1 S(x(t), y(t))) - x(t)\}$$

$$p(x(t), y(t)) = \gamma\{J_N^{\rho_2}(y(t) - \rho_2 T(x(t), y(t))) - y(t)\}$$

for all $(x(t), y(t)) \in H_1 \times H_2$. Let $(\bar{x}, \bar{y}), (x', y') \in H_1 \times H_2$ where $(\bar{x}, \bar{y}) = (\bar{x}(t), \bar{y}(t))$ and $(x', y') = (x'(t), y(t)')$. We see that

$$\|F(\bar{x}, \bar{y}) - F(x', y')\|^+$$
$$= \lambda\|J_M^{\rho_1}(\bar{x} - \rho_1 S(\bar{x}, \bar{y})) - \bar{x} - J_M^{\rho_1}(x' - \rho_1 S(x', y')) + x'\|$$
$$+\gamma\|J_N^{\rho_2}(\bar{y} - \rho_2 T(\bar{x}, \bar{y})) - \bar{y} - J_N^{\rho_2}(y' - \rho_2 T(x', y')) + y'\|$$
$$\leq \lambda\{\|\bar{x} - x'\| + \rho_1\|S(x', y') - S(\bar{x}, \bar{y})\| + \|x' - \bar{x}\|\}$$
$$+\gamma\{\|\bar{y} - y'\| + \rho_2\|T(x', y') - T(\bar{x}, \bar{y})\| + \|y' - \bar{y}\|\}$$
$$\leq \lambda\{\|\bar{x} - x'\| + \rho_1\beta_2\|y' - \bar{y}\| + \rho_1\beta_1\|x' - \bar{x}\| + \|x' - \bar{x}\|\}$$
$$+\gamma\{\|\bar{y} - y'\| + \rho_2\kappa_2\|y' - \bar{y}\| + \rho_2\kappa_1\|x' - \bar{x}\| + \|y' - \bar{y}\|\}$$
$$\leq \Delta(2 + \rho\beta + \rho\kappa)\|(\bar{x}, \bar{y}) - (x', y')\|^+$$

where $\Delta = \max\{\lambda, \gamma\}$, $\rho = \max\{\rho_1, \rho_2\}$, $\kappa = \max\{\kappa_1, \kappa_2\}$ and $\beta = \max\{\beta_1, \beta_2\}$. Thus, F is a Lipschitz continuous on $\|\cdot\|^+$. Hence, for each $(x_0, y_0) \in H_1 \times H_2$, there exists a unique continuous solution $(x(t), y(t))$ of the problem (RDSSNVI) defined in a initial $t_0 \leq t \leq \Gamma$ with the initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$.

Next, let $[t_0, \Gamma)$be its maximal interval of existence, we now show that $\Gamma = \infty$. Since the assumption of Theorem 2, we have the problem (SNVI) has a unique solution, that is, $(x^*, y^*) \in H_1 \times H_2$ which

$$x^* = J_M^{\rho_1}(x^* - \rho_1 S(x^*, y^*)) \text{ and } y^* = J_N^{\rho_2}(y^* - \rho_2 T(x^*, y^*)).$$

So, we let $(x, y) \in H_1 \times H_2$ where $(x, y) = (x(t), y(t))$. We obtain that

$$\|F(x, y)\|^+ = \|q(x, y)\| + \|p(x, y)\|$$
$$\leq \Delta\{\|J_M^{\rho_1}(x - \rho_1 S(x, y)) - x\| + \|J_N^{\rho_2}(y - \rho_2 T(x, y)) - y\|\}$$
$$\leq \Delta\begin{Bmatrix} \|J_M^{\rho_1}(x - \rho_1 S(x, y)) - J_M^{\rho_1}(x^* - \rho_1 S(x^*, y^*))\| \\ +\|J_M^{\rho_1}(x^* - \rho_1 S(x^*, y^*)) - x^*\| + \|x^* - x\| + \\ \|J_N^{\rho_2}(y - \rho_2 T(x, y)) - J_N^{\rho_2}(y^* - \rho_2 T(x^*, y^*))\| \\ +\|J_N^{\rho_2}(y^* - \rho_2 T(x^*, y^*)) - y^*\| + \|y^* - y\| \end{Bmatrix}$$
$$= \Delta\begin{Bmatrix} 2\|x - x^*\| + \rho_1\beta_2\|y^* - y\| + \rho_1\beta_1\|x - x^*\| \\ +2\|y - y^*\| + \rho_2\kappa_2\|y^* - y\| + \rho_2\kappa_1\|x^* - x\| \end{Bmatrix}$$
$$\leq \Delta(2 + \rho\beta + \rho\kappa)\{\|x - x^*\| + \|y - y^*\|\}$$
$$\leq \Delta(2 + \rho\beta + \rho\kappa)\{\|x\| + \|x^*\| + \|y\| + \|y^*\|\}$$
$$= \Delta(2 + \rho\beta + \rho\kappa)\|(x, y)\|^+ + \Delta(2 + \rho\beta + \rho\kappa)\|(x^*, y^*)\|^+.$$

Hence,

$$\|(x(t), y(t))\|^+ \leq \|(x(t_0), y(t_0))\|^+ + \int_{t_0}^{t} \|F(x(s), y(s))\|^+ \, ds$$

$$= \left\|\left(x(t_0), y(t_0)\right)\right\|^+ + k_1(t - t_0) + k_2 \int_{t_0}^{t} \left\|\left(x(s), y(s)\right)\right\|^+ ds,$$

where, $k_1 = \Delta(2 + \rho\beta + \rho\kappa)\|(x^*, y^*)\|^+$ and $k_2 = \Delta(2 + \rho\beta + \rho\kappa)$. By Gronwall's Lemma, we get

$$\|(x(t), y(t))\|^+ \leq \left\{\left\|\left(x(t_0), y(t_0)\right)\right\|^+ + k_1(t - t_0)\right\} e^{k_2(t - t_0)}.$$

Thus, we conclude that for each $(x_0, y_0) \in H_1 \times H_2$ there exists a unique solution $(x(t), y(t)) \in H_1 \times H_2$ of the problem (RDSSNVI) over $[t_0, \infty)$. This completes the proof. ■

Conclusion

In this work, we study the problem of the system of variational inclusion which was studied by R. U. Verma (Verma, 2004). We prove the existence solution of such problem by using resolvent operator and present the resolvent equation which is equivalent to the problem (SNVI) and prove that the solution of the resolvent equation is equivalent to the solution of the problem (SNVI). Later, we use such resolvent equation for introducing the resolvent dynamical system associate with the system of nonlinear variational inclusion and consider the existence solution of such resolvent dynamical system. We desire that the results which presented here will be useful and valuable for researchers who study in the fields of variational inclusion.

Acknowledgement

This work is supported by the National Research Council of Thailand (Project No. RDI-1-61-6-28) and Pibulsongkram Rajabhat University.

References

Agarwal RP, Huang NJ, Tan MY. Sensitivity analysis for a new system of generalized nonlinear mixed quasi-variational inclusions, *Applied Mathematics Letters.* 2004; 17: 345-352.

Agarwal RP, Verma RU. General implicit variational inclusion problems based on A-maximal (m)-relaxed monotonicity (AMRM) frameworks, *Applied Mathematics and Computation.* 2009; 215(1): 367-379.

Ansari QH, Balooee J, Yao JC. Extended general nonlinear quasi variational inequalities and Projection dynamical systems, *Taiwanese Journal of Mathematics.* 2013; 17(4): 1321-1352.

Bahiana M, Oono Y. Cell dynamical system approach to block copolymers, *Physical Review Journals.* 1990; 41(12): 6763-6771.

Biswas SKr., Chakraborty S. Interacting dark energy in f(T) cosmology: A dynamical system analysis, *International Journal of Modern Physics D*. 2015; 24(07): 21 pages.

Bliemer M, Bovy P. Quasi variational inequality formulation of the multi-class dynamic traffic assignment problem, *Transportation Reseach*. 2003; 37: 501-519.

Dong J, Zhang D, Nagurney A. A projected dynamical systems model of general financial equilibrium with stability analysis, *Mathematical and Computer Modelling.* 1996; 24(2): 35-44.

Dupuis P, Nagurney A. Dynamical systems and variational inequalities, *Annals of Operations Research*. 1993; 44: 9-42.

Fang YP, Huang NJ. H-monotone operators and system of variational inclusions, *Communications on Applied Nonlinear Analysis.* 2004; 11(1): 93-101.

Ha Nguyen TT, Strodiot Jean J, Vuong Phan T. On the global exponential stability of a projected dynamical system for strongly pseudomonotone variational inequalities, *Optimization Letters .* 2018; 12: 1-14.

Isac G, Cojocaru MG. The projection operator in a Hilbert space and its directional derivative. *Consequences for the theory of projected dynamical systems*, 2002.

Lin LJ. Systems of variational inclusion problems and differential inclusion problems with applications, *Journal of Global Optimization*. 2009; 44DOI 10.1007/s10898-008-9359-x: 579–591.

Nagurney A, Zhang AD. *Projected Dynamical Systems and Variational Inequalities with Applications.* Kluwer Academic: Boston; 1996.

Noor MA. Equivalence of Variational Inclusions with Resolvent Equations, *Nonlinear Analysis.* 2000; 14: 963-970.

Noor MA. Resolvent dynamical systems for mixed variational inequalities, *The Korean Journal of Computational & Applied Mathematics.* 2002a; 9(1): 15-26.

Noor MA. Implicit resolvent dynamical systems for quasi variational inclusion, *Journal of Mathematical Analysis and Applications.* 2002b; 269: 216-226.

Scrimali L. The financial equilibrium problem with implicit budget constraints, *Central European Journal of Operations Research*. 2008; 16: 191-203.

Suantai S, Petrot N. Existence and Stability of Iterative Algorithms for the System of Nonlinear Quasi-Mixed Equilibrium Problems, *Applied Mathematics Letters*. 2011; 24: 308-313.

Suwannawit J, Petrot N. Existence and Stability of Iterative Algorithm for a System of Random Set-Valued Variational Inclusion Problems Involving (A, m, η)-Generalized Monotone Operators, *Journal of Applied Mathematics.* 2012; ID 590676: 21 pages.

Verma RU. A-monotonicity and applications to nonlinear variational inclusion problems, *Journal of Applied Mathematics and Stochastic Analysis*. 2004; 2: 193-195.

Yan WY, Fang YP, Huang NJ. A new system of set-valued variational inclusions with H-monotone operators, *Mathematical Inequalities & Applications.* 2005; 8(3): 537-546.