Prime Period-6 Solutions On

$x_{n+1} = |x_n| - y_n + 1$ and $y_{n+1} = x_n + |y_n| - 1$

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Abstract

In this paper we consider the system of difference equations in the title, where the initial condition $(x_0, y_0) \in R^2$. We show that there exists a unique equilibrium solution and exactly two prime period-6 solutions, and that except for the unique equilibrium solution, every solution of the system is eventually one of the two prime period-6 solutions.

บทคัดย่อ

ในบทความนี้เราศึกษาระบบของสมการเชิงผลต่างตามชื่อของบทความ โดยเงื่อนไขเริ่มต้น $(x_0, y_0) \in R^2$ เรา แสดงว่ามีจุดสมดุลเพียงจุดเดียวและมีสองผลเฉลยพรามพีเรียด-6 เรายังแสดงว่าผลเฉลยของระบบสมการนี้เป็น สองผลเฉลยพรามพีเรียด-6 ดังกล่าวในที่สุด ยกเว้นจุดสมดุล

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1. Introduction

In this paper we consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, \qquad n = 0, 1, 2, \ldots \tag{1}$$

where the initial condition $(x_0, y_0) \in R^2$. We show that every solution of System(1) is either (from the beginning) the unique equilibrium point $(1, 1)$ or else is eventually one of the following period-6 cycles

$$P_6^1 = \begin{pmatrix} x_0 = 3 \ , y_0 = 3 \\ x_1 = 1 \ , y_1 = 5 \\ x_2 = -3, y_2 = 5 \\ x_3 = -1, y_3 = 1 \\ x_4 = 1 \ , y_4 = -1 \\ x_5 = 3 \ , y_5 = 1 \end{pmatrix}, \quad P_6^1 = \begin{pmatrix} x_0 = 3 \ , y_0 = \frac{7}{5} \\ x_1 = \frac{13}{5} \ , y_1 = \frac{17}{5} \\ x_2 = \frac{1}{5} \ , y_2 = 5 \\ x_3 = -\frac{19}{5}, y_3 = \frac{21}{5} \\ x_4 = \frac{3}{5} \ , y_4 = -\frac{3}{5} \\ x_5 = -\frac{11}{5}, y_5 = \frac{1}{5} \end{pmatrix}$$

In particular, the pre-image of the equilibrium point is itself.

System(1) was motivated by Devaney's Gingerbread man map [1, 2]

$$x_{n+1} = |x_n| - x_{n-1} + 1$$

Its equivalent system of piecewise linear difference equations is [3, 4]

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n \end{cases} \qquad n = 0, 1, 2, \ldots$$

We believe that the methods and techniques used in this paper will be useful in discovering the global behavior of similar systems.

2. The Global Behavior To The Solutions Of System (1)

Set

$$L_1 = \{(x, y): x \geq 0 \text{ and } y = 0\}$$

$$L_2 = \{(x, y): x = 0 \text{ and } y \geq 0\}$$

$$L_3 = \{(x, y): x \leq 0 \text{ and } y = 0\}$$

$$L_4 = \{(x, y): x = 0 \text{ and } y \leq 0\}$$

$$Q_1 = \{(x, y): x > 0 \text{ and } y > 0\}$$

$$Q_2 = \{(x, y): x < 0 \text{ and } y > 0\}$$

$$Q_3 = \{(x, y): x < 0 \text{ and } y < 0\}$$

$$Q_4 = \{(x, y): x > 0 \text{ and } y < 0\}.$$

Theorem 1 Let $(x_0, y_0) \in R^2$ be the solution of System (1). Then, there exists an integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System (1) is eventually the prime period-6 solution (P_6^1 or P_6^2).

The proof is a direct consequence of the following lemmas.

Lemma 2 Assume that there is a positive integer N such that $x_N = -y_N \leq 0$. Then, $\{(x_n, y_n)\}_{n=N+1}^{\infty} \in P_6^1$.

Proof. Suppose that (x_N, y_N) satisfies the hypothesis, then

$$x_{N+1} = |x_N| - y_N + 1 = -x_N + x_N + 1 = 1$$
$$y_{N+1} = x_N + |y_N| - 1 = x_N - x_N - 1 = -1.$$

Hence, $(x_{N+1}, y_{N+1}) = (1, -1) \in P_6^1$. □

Lemma 3 Assume that there is a positive integer N such that $y_N = x_N - 2 \leq 0$ and $x_N \geq 0$. Then, $\{(x_n, y_n)\}_{n=N+1}^{\infty} \in P_6^1$.

Proof. Suppose that (x_N, y_N) satisfies the hypothesis, then

$$x_{N+1} = |x_N| - y_N + 1 = x_N - x_N + 2 + 1 = 3$$
$$y_{N+1} = x_N + |y_N| - 1 = x_N - x_N + 2 - 1 = 1.$$

Hence, $(x_{N+1}, y_{N+1}) = (3, 1) \in P_6^1$. □

Lemma 4 Let $L := \{(3, y) \mid y \in \mathbf{R}\}$. Then every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with initial condition in L is eventually prime period-6 solution $(P_6^1 \text{ or } P_6^2)$.

Proof. Let $(x_0, y_0) \in L$ and $y_0 \geq 0$. Then,

$$x_1 = |x_0| - y_0 + 1 = 3 - y_0 + 1 = -y_0 + 4$$
$$y_1 = x_0 + |y_0| - 1 = 3 + y_0 - 1 = y_0 + 2 > 0.$$

If $x_1 = -y_0 + 4 \leq 0$, then $(x_3, y_3) = (1, -1) \in P_6^1$. Suppose that $x_1 = -y_0 + 4 > 0$. Then

$$x_2 = |x_1| - y_1 + 1 = -2y_0 + 3$$
$$y_2 = x_1 + |y_1| - 1 = 5.$$

Cases 1: $x_2 = -2y_0 + 3 \leq 0 \left(\frac{3}{2} \leq y_0 < 4\right)$, Then

$$x_3 = |x_2| - y_2 + 1 = 2y_0 - 7$$
$$y_3 = x_2 + |y_2| - 1 = -2y_0 + 7.$$

If $x_3 = -y_3 = 2y_0 - 7 \leq 0 \left(\frac{3}{2} \leq y_0 \leq \frac{7}{2}\right)$, Then we apply Lemma 2 and $\{(x_n, y_n)\}_{n=3}^{\infty}$ is eventually prime period-6 solution(P_6^1) solution.

Suppose that $x_3 = -y_3 = 2y_0 - 7 > 0\left(\frac{7}{2} < y_0 < 4\right)$. We will prove that for $y_0 \in \left(\frac{7}{2}, 4\right)$ the solution is eventually prime period-6 solution by mathematical induction.

For each $n \geq 0$, let P(n) be the following statement : For $y_0 \in (a_n, b_n)$,

$$x_{6n+4} = 2^{4n+2} y_0 - (\delta_n - 2) > 0, y_{6n+4} = 2^{4n+2} y_0 - \delta_n$$

such that if $y_0 \in (a_n, c_n]$, then $y_{6n+4} \leq 0$, and so the solution is eventually prime period-6 solution.

If $y_0 \in (c_n, b_n)$, then $y_{6n+4} > 0$, and so

$$x_{6n+5} = 3, \qquad y_{6n+5} = 2^{4n+3} y_0 - (2\delta_n - 1) > 0$$

$$x_{6n+6} = -2^{4n+3} y_0 + (2\delta_n + 3) > 0, \quad y_{6n+6} = 2^{4n+3} y_0 - (2\delta_n - 3) > 0$$

$$x_{6n+7} = -2^{4n+4} y_0 + (4\delta_n + 1), \qquad y_{6n+7} = 5$$

such that if $y_0 \in [b_{n+1}, b_n)$,then $x_{6n+7} \leq 0$ and so the solution is eventually prime period-6 solution (P_6^1).

If $y_0 \in (c_n, b_{n+1})$, then $x_{6n+7} > 0$, and so

$$x_{6n+8} = -2^{4n+4} y_0 + (4\delta_n - 3) < 0, \quad y_{6n+8} = -2^{4n+4} y_0 + (4\delta_n + 5) > 0$$

$$x_{6n+9} = 2^{4n+5} y_0 - (8\delta_n + 1), \qquad y_{6n+9} = -2^{4n+5} y_0 + (8\delta_n + 1)$$

such that if $y_0 \in (c_n, a_{n+1}]$, then $x_{6n+9} = -y_{6n+9} \leq 0$, and so $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually prime period-6solution (P_6^1).

If $y_0 \in (a_{n+1}, b_{n+1})$, then $x_{6n+9} = -y_{6n+9} = 2^{4n+5} y_0 - (8\delta_n + 1) > 0$ where

$$a_n = \frac{19 \times 2^{4n+1} - 3}{5 \times 2^{4n+1}}, b_n = \frac{19 \times 2^{4n} + 1}{5 \times 2^{4n}}, c_n = \frac{19 \times 2^{4n+2} - 1}{5 \times 2^{4n+2}}, \delta_n = \frac{19 \times 2^{4n+2} - 1}{5}$$

We shall show that P(0) is true . For $y_0 \in (a_0, b_0) = \left(\frac{7}{2}, 4\right)$ and $x_3 = -y_3 = 2y_0 - 7 > 0$,

$$x_{6(0)+4} = x_4 = 4y_0 - 13, \ = 2^{4(0)+2} y_0 - (\delta_0 - 2) > 0$$

$$x_{6(0)+4} = y_4 = 4y_0 - 15, \ = 2^{4(0)+2} y_0 - \delta_0.$$

If $y_0 \in (a_0, c_0] = \left(\frac{7}{2}, \frac{15}{4}\right]$, then $y_4 = 4y_0 - 15 \leq 0$. Applying Lemma 3. $\{(x_n, y_n)\}_{n=4}^{\infty}$ is eventually prime period-6 solution (P_6^1).

If $y_0 \in (c_0, b_0) = \left(\frac{15}{4}, 4\right)$, then $y_4 = 4y_0 - 15 > 0$, thus we have

$$x_{6(0)+5} = 3, \qquad y_{6(0)+5} = 2^{4(0)+3} y_0 - (2\delta_0 - 1) > 0,$$

$$x_{6(0)+6} = -2^{4(0)+3} y_0 + (2\delta_0 + 3) \ = -8y_0 + 33 > 0$$

$$y_{6(0)+6} = 2^{4(0)+3} y_0 - (2\delta_0 - 3) > 0,$$

$$x_{6(0)+7} = -2^{4(0)+4} y_0 + (4\delta_0 + 1), \quad y_{6(0)+7} = 5.$$

If $y_0 \in (b_1, b_0] = \left[\frac{61}{16}, 4\right)$, then $x_{6(0)+7} = -2^{4(0)+4} y_0 + (4\delta_0 + 1) = -16y_0 + 61 \le 0$, so

$x_{6(0)+8} = -y_{6(0)+8} = 16y_0 - 65 < 0$ by Lemma 2 , and so $\{(x_n, y_n)\}_{n=9}^{\infty}$ is prime period-6 solution.

If $y_0 \in (c_0, b_1) = \left(\frac{15}{4}, \frac{61}{16}\right)$, then $x_{6(0)+7} = -2^{4(0)+4} y_0 + (4\delta_0 + 1) = -16y_0 + 61 > 0$, thus we have

$$x_{6(0)+8} = -2^{4(0)+4} y_0 + (4\delta_0 - 3) = -16y_0 + 57 < 0$$

$$y_{6(0)+8} = -2^{4(0)+4} y_0 + (4\delta_0 + 5) = -16y_0 + 65 > 0,$$

$$x_{6(0)+9} = 2^{4(0)+5} y_0 - (8\delta_0 + 1), \quad y_{6(0)+9} = -2^{4(0)+5} y_0 + (8\delta_0 + 1).$$

If $y_0 \in (c_0, a_1] = \left(\frac{15}{4}, \frac{121}{32}\right]$. Then $x_{6(0)+9} = -y_{6(0)+9} = 32y_0 - 121 \le 0$, by Lemma 2 and so $\{(x_n, y_n)\}_{n=10}^{\infty}$ is eventually prime period-6 solution (P_6^1).

If $y_0 \in (a_1, b_1) = \left(\frac{121}{32}, \frac{61}{16}\right)$, then $x_{6(0)+9} = -y_{6(0)+9} = 32y_0 - 121 > 0$.

Hence P(0) is true.

Next, we assume that $P(N)$ is true . We shall show that $P(N+1)$ is true . Since P (N) is true, $x_{6N+9} = -y_{6N+9} = 2^{4N+5} y_0 - (8\delta_N + 1) > 0$ where $y_0 \in (a_{N+1}, b_{N+1}) = \left(\frac{19 \times 2^{4N+5} - 3}{5 \times 2^{4N+5}}, \frac{19 \times 2^{4N+4} + 1}{5 \times 2^{4N+4}}\right)$. Then

$$x_{6(N+1)+4} = x_{6N+10} = 2^{4N+6} y_0 - 16\delta_N - 1 = 2^{4(N+1)+2} y_0 - (\delta_{N+1} - 2)$$

$$= 2^{4N+6} y_0 - \left(\frac{19 \times 2^{4N+6} - 11}{5}\right) > 0$$

$$y_{6(N+1)+4} = y_{6N+10} = 2^{4N+6} y_0 - 16\delta_N - 3 = 2^{4(N+1)+2} y_0 - \delta_{N+1}.$$

Note that

$$\delta_{N+1} = \frac{19 \times 2^{4N+6} - 1}{5} = \frac{19 \times 2^{4N+6} - 16}{5} + \frac{15}{5} = 16\delta_N + 3$$

If $y_0 \in (a_{N+1}, c_{N+1}] = \left(\frac{19 \times 2^{4N+5} - 3}{5 \times 2^{4N+5}}, \frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}\right]$, then

$$y_{6N+10} = 2^{4(N+1)+2} y_0 - \delta_{N+1} = 2^{4N+6} y_0 - \left(\frac{19 \times 2^{4N+6} - 1}{5}\right) \le 0.$$

Applying Lemma 3, $\{(x_n, y_n)\}_{n=6N+10}^{\infty}$ is eventually prime period-6 solution (P_6^1).

If $y_0 \in (c_{N+1}, b_{N+1}) = \left(\frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}, \frac{19 \times 2^{4N+4} + 1}{5 \times 2^{4N+4}}\right)$, then

$$y_{6N+10} = 2^{4(N+1)+2} y_0 - \delta_{N+1} = 2^{4N+6} y_0 - \left(\frac{19 \times 2^{4N+6} - 1}{5}\right) > 0.$$

Thus, we have

$$x_{6(N+1)+5} = x_{6N+11} = 3$$

$$x_{6(N+1)+5} = x_{6N+11} = 2^{4(N+1)+3} y_0 - \left(2\delta_{N+1} - 1\right) > 0$$

$$x_{6(N+1)+6} = x_{6N+12} = -2^{4(N+1)+3} y_0 + \left(2\delta_{N+1} + 3\right)$$

$$= -2^{4N+7} y_0 + \left(\frac{19 \times 2^{4N+7} + 13}{5}\right) > 0$$

$$y_{6(N+1)+6} = y_{6N+12} = 2^{4(N+1)+3} y_0 - \left(2\delta_{N+1} - 3\right) > 0$$

$$y_{6(N+1)+7} = y_{6N+13} = -2^{4(N+1)+4} y_0 + \left(4\delta_{N+1} + 1\right)$$

$$y_{6(N+1)+7} = y_{6N+13} = 5$$

If $y_0 \in \left[b_{N+2}, b_{N+1}\right) = \left[\frac{19 \times 2^{4N+8} + 1}{5 \times 2^{4N+8}}, \frac{19 \times 2^{4N+4} + 1}{5 \times 2^{4N+4}}\right)$, then

$$x_{6N+13} = -2^{4(N+1)+4} y_0 + 4\delta_{N+1} + 1 = -2^{4N+8} y_0 + \frac{19 \times 2^{4N+8} + 1}{5} \le 0,$$

and so

$$x_{6N+14} = 2^{4N+8} y_0 - \left(4\delta_{N+1} + 5\right) = 2^{4N+8} y_0 - \left(\frac{19 \times 2^{4N+8} + 21}{5}\right) < 0$$

$$y_{6N+14} = -2^{4N+8} y_0 + \left(4\delta_{N+1} + 5\right) > 0.$$

Applying Lemma 2, $\left\{\left(x_n, y_n\right)\right\}_{n=6N+14}^{\infty}$ is eventually prime period-6 solution.

If $y_0 \in \left(c_{N+1}, b_{N+2}\right) = \left(\frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}, \frac{19 \times 2^{4N+8} + 1}{5 \times 2^{4N+8}}\right)$, then

$$x_{6N+13} = -2^{4(N+1)+4} y_0 + 4\delta_{N+1} + 1 = -2^{4N+8} y_0 + \frac{19 \times 2^{4N+8} + 1}{5} > 0$$

Thus, we have

$$x_{6(N+1)+8} = x_{6N+14} = -2^{4(N+1)+4} y_0 + \left(4\delta_{N+1} - 3\right)$$

$$= -2^{4N+8} y_0 + \left(\frac{19 \times 2^{4N+8} - 19}{5}\right) < 0$$

$$y_{6(N+1)+8} = y_{6N+14} = -2^{4(N+1)+4} y_0 + \left(4\delta_{N+1} + 5\right)$$

$$= -2^{4N+8} y_0 + \left(\frac{19 \times 2^{4N+8} + 21}{5}\right) > 0$$

$$x_{6(N+1)+9} = x_{6N+15} = 2^{4(N+1)+5} y_0 - (8\delta_{N+1} + 1)$$

$$y_{6(N+1)+9} = y_{6N+15} = -2^{4(N+1)+5} y_0 + (8\delta_{N+1} + 1).$$

If $y_0 \in \left(c_{N+1}, a_{N+2}\right] = \left(\frac{19 \times 2^{4N+6} - 1}{5 \times 2^{4N+6}}, \frac{19 \times 2^{4N+9} - 3}{5 \times 2^{4N+9}}\right]$, then

$$x_{6N+15} = -y_{6N+15} = 2^{4(N+1)+5} y_0 - (8\delta_{N+1} + 1) = 2^{4N+9} - \left(\frac{19 \times 2^{4N+9} - 3}{5}\right) \le 0.$$

Applying Lemma 2, $\{(x_n, y_n)\}_{n=6N+15}^{\infty}$ is eventually prime period-6 solution $\left(P_6^1\right)$.

If $y_0 \in (a_{N+2}, b_{N+2}) = \left(\dfrac{19\times 2^{4N+9}-3}{5\times 2^{4N+5}}, \dfrac{19\times 2^{4N+8}+1}{5\times 2^{4N+8}}\right)$, then

$$x_{6N+15} = -y_{6N+15} = -2^{4(N+1)+5}y_0 + (8\delta_{N+1}+1) = -2^{4N+9} + \left(\frac{19\times 2^{4N+9}-3}{5}\right) > 0$$

Hence, $(N+1)$ is true. By mathematical induction, P(n) is true for all $n \geq 0$.

Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \frac{19}{5}.$$

We also note that if $(x_0, y_0) = \left(3, \dfrac{19}{5}\right)$, then $(x_3, y_3) = \left(\dfrac{3}{5}, -\dfrac{3}{5}\right) \in P_6^2$.

Case 2: $x_2 = -2y_0 + 3 > 0\left(y_0 < \dfrac{3}{2}\right)$, then

$$x_3 = |x_2| - y_2 + 1 = -2y_0 - 1 < 0$$
$$y_3 = x_2 + |y_2| - 1 = -2y_0 + 7 > 0,$$
$$x_4 = |x_3| - y_3 + 1 = 4y_0 - 5$$
$$y_4 = x_3 + |y_3| - 1 = -4y_0 + 5.$$

If $x_4 = -y_4 = 4y_0 - 5 \leq 0$, then we apply Lemma 2 and so $\{(x_n, y_n)\}_{n=4}^{\infty}$ is eventually prime period-6 solution $\left(P_6^1\right)$.

Suppose that $x_4 = -y_4 = 4y_0 - 5 > 0\left(\dfrac{5}{4} < y_0 < \dfrac{3}{2}\right)$. Then,

$$x_5 = 8y_0 - 9 > 0 \text{ and } y_5 = 8y_0 - 11$$

If $y_5 = 8y_0 - 11 \leq 0\left(\dfrac{5}{4} < y_0 \leq \dfrac{11}{8}\right)$, then we apply Lemma 3 and so $\{(x_n, y_n)\}_{n=4}^{\infty}$ is eventually prime period-6 solution $\left(P_6^1\right)$.

Suppose that $y_5 = 8y_0 - 11 > 0\left(\dfrac{11}{8} < y_0 < \dfrac{3}{2}\right)$. We will prove that for $y_0 \in \left(\dfrac{11}{8}, \dfrac{3}{2}\right)$ the solution is eventually prime period-6 solution by mathematical induction.

For each $n \geq 0$, let $Q(n)$ be the following statement : For $y_0 \in (a_n, b_n)$,

$$x_{6n+5} = 2^{4n+2}y_0 - (\delta_n - 2) > 0, y_{6n+5} = 2^{4n+2}y_0 - \delta_n$$

such that if $y_0 \in (a_n, c_n]$, then $y_{6n+5} \leq 0$, and so $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually prime period-6 solution $\left(P_6^1\right)$.

If $y_0 \in (c_n, b_n)$, then $x_{6n+5} > 0$. Thus, we have

$$x_{6n+6} = 3, \qquad y_{6n+6} = 2^{4n+3}y_0 - (2\delta_n - 1) > 0$$
$$x_{6n+7} = -2^{4n+3}y_0 + (2\delta_n + 3) > 0 \qquad y_{6n+7} = 2^{4n+3}y_0 - (2\delta_n - 3) > 0$$
$$x_{6n+8} = -2^{4n+4}y_0 + (4\delta_n + 1), \qquad y_{6n+8} = 5$$

such that if $y_0 \in (b_{n+1}, b_n]$, then $y_{6n+8} \le 0$, and so $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually prime period-6 solution(P_6^1).

If $y_0 \in (c_n, b_{n+1})$, then $x_{6n+8} > 0$. Thus, we have

$$x_{6n+9} = -2^{4n+4}y_0 + (4\delta_n - 3) < 0, \quad y_{6n+9} = -2^{4n+4}y_0 + (4\delta_n + 5) > 0$$
$$x_{6n+10} = 2^{4n+5}y_0 - (8\delta_n + 1), \qquad y_{6n+10} = -2^{4n+5}y_0 + (8\delta_n + 1).$$

such that if $y_0 \in (c_n, a_{n+1}]$, then $x_{6n+10} = -y_{6n+10} \le 0$, and so the solution is eventually prime period-6 solution (P_6^1).

If $y_0 \in (a_{n+1}, b_{n+1})$, then $x_{6n+10} = -y_{6n+10} > 0$, where

$$a_n = \frac{7\times 2^{4n+2} - 3}{5\times 2^{4n+2}}, b_n = \frac{7\times 2^{4n+1} + 1}{5\times 2^{4n+1}}, c_n = \frac{7\times 2^{4n+3} - 1}{5\times 2^{4n+3}}, \delta_n \frac{7\times 2^{4n+3} - 1}{5}.$$

The proof is similar above case and it will be omitted. We can conclude that $Q(n)$ is true for all $n \ge 0$.

Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \frac{7}{5}.$$

We also note that $\left(3, \frac{7}{5}\right) \in P_6^2$.

We already prove that for initial condition $(x_0, y_0) \in L$ and $y_0 > 0$, every solution $\{(x_n, y_n)\}_{n=2}^{\infty}$ is eventually prime period-6 solution $(P_6^1 \text{ or } P_6^2)$. The remain case $(x_0, y_0) \in L$ and $y_0 < 0$. Then, $x_0 = 3, y_0 < 0$. Thus,

$$x_1 = |x_0| - y_0 + 1 = -y_0 + 4 > 0$$
$$y_1 = x_0 + |y_0| - 1 = -y_0 + 2 > 0,$$
$$x_2 = |x_1| - y_1 + 1 = 3$$
$$y_2 = x_1 + |y_1| - 1 = -2y_0 + 5 > 0.$$

We see that $(x_2, y_2) \in L$ and $y_2 > 0$. Applying the above case, every solution $\{(x_n, y_n)\}_{n=2}^{\infty}$ is eventually prime period-6 solution $(P_6^1 \text{ or } P_6^2)$. □

Lemma 5 Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1) with initial condition in $l_2 \cup l_3 \cup Q_2$ is eventually prime period-6 solution $(P_6^1 \text{ or } P_6^2)$.

Proof. Let $(x_0, y_0) \in l_2 \cup l_3 \cup Q_2$. Then $x_0 \le 0$ and $y_0 \ge 0$. Then

$$x_1 = |x_0| - y_0 + 1 = -x_0 - y_0 + 1$$
$$y_1 = x_0 + |y_0 - 1| = x_0 + y_0 - 1.$$

If $x_1 = -x_0 - y_0 + 1 < 0$, then $y_1 = x_0 + y_0 - 1 > 0$. We apply Lemma 2, and so $(x_2, y_2) \in P_6^1$.

Suppose that $x_1 = -x_0 - y_0 + 1 \geq 0$, then $y_1 = x_0 + y_0 - 1 \leq 0$. Thus,

$$x_2 = |x_1| - y_1 + 1 = -2x_0 - 2y_0 + 3 > 0$$
$$y_2 = x_1 + |y_1| - 1 = -2x_0 - 2y_0 + 1$$
$$x_3 = |x_2| - y_2 + 1 = 3.$$

Applying Lemma 4, every solution $\{(x_n, y_n)\}_{n=3}^{\infty}$ is eventually prime period-6 solution $(P_6^1 \text{ or } P_6^2)$.

□

Lemma 6 Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1) with initial condition in $l_1 \cup l_4 \cup Q_4$ is eventually prime period-6 solution $(P_6^1 \text{ or } P_6^2)$.

Proof. Let $(x_0, y_0) \in l_1 \cup l_4 \cup Q_4$. Then $x_0 \geq 0$ and $y_0 \leq 0$. We have,

$$x_1 = |x_0| - y_0 + 1 = x_0 - y_0 + 1 > 0$$
$$y_1 = x_0 + |y_0| - 1 = x_0 - y_0 - 1,$$
$$x_2 = |x_0| - y_0 + 1 = x_0 - y_0 + 1 - x_0 + y_0 + 1 + 1 = 3.$$

Applying Lemma 4, every solution $\{(x_n, y_n)\}_{n=3}^{\infty}$ is eventually prime period-6 solution $(P_6^1 \text{ or } P_6^2)$.

□

Lemma 7 Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1) with initial condition in Q_3 is eventually period-6 solution $(P_6^1 \text{ or } P_6^2)$.

Proof. Let $(x_0, y_0) \in Q_3$. Then $x_0 < 0$ and $y_0 < 0$. Thus,

$$x_1 = |x_0| - y_0 + 1 = -x_0 - y_0 + 1 > 0$$
$$y_1 = x_0 + |y_0| - 1 = x_0 - y_0 - 1.$$

If $y_1 = x_0 - y_0 - 1 < 0$, then

$$x_2 = |x_1| - y_1 + 1 = -2x_0 + 3 > 0$$
$$y_2 = x_1 + |y_1| - 1 = -2x_0 + 1$$
$$x_3 = |x_2| - y_2 + 1 = 3.$$

Applying Lemma 4, every solution $\{(x_n, y_n)\}_{n=3}^{\infty}$ is eventually prime period-6 solution $(P_6^1 \text{ or } P_6^2)$

Suppose that $y_1 = x_0 - y_0 - 1 \geq 0 \, (-y_0 > 1)$, then

$$x_2 = |x_1| - y_1 + 1 = -2x_0 + 3 > 0$$
$$y_2 = x_1 + |y_1| - 1 = -2y_0 - 1 > 0$$
$$x_3 = |x_2| - y_2 + 1 = -2x_0 + 2y_0 + 5$$
$$y_3 = x_2 + |y_2| - 1 = -2x_0 - 2y_0 + 1 > 0.$$

If $x_3 = -2x_0 + 2y_0 + 5 < 0$, then

$$x_4 = |x_3| - y_3 + 1 = 4x_0 - 5 < 0$$
$$y_4 = x_3 + |y_3| - 1 = -4x_0 + 5 > 0.$$

Applying Lemma 2, $(x_5, y_5) \in (1, -1) \in P_6^1$.

Suppose that $x_3 = -2x_0 + 2y_0 + 5 \geq 0$, then

$$x_4 = |x_3| - y_3 + 1 = 4y_0 + 5$$
$$y_4 = x_3 + |y_3| - 1 = -4x_0 + 5 > 0$$

If $x_4 = 4y_0 + 5 \geq 0 \left(-\frac{5}{4} \leq y_0 \leq -1 \right)$, then

$$x_5 = |x_4| - y_4 + 1 = 4x_0 + 4y_0 + 1 < 0$$
$$y_5 = x_4 + |y_4| - 1 = -4x_0 + 4y_0 + 9 > 0.$$

We apply Theorem 5. Then, every solution $\{(x_n, y_n)\}_{n=5}^{\infty}$ is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$.

Suppose $x_4 = 4y_0 + 5 < 0$, then

$$x_5 = |x_4| - y_4 + 1 = 4x_0 - 4y_0 - 9$$
$$y_5 = x_4 + |y_4| - 1 = -4x_0 + 4y_0 + 9.$$

If $x_5 \geq 0$, then $y_5 \leq 0$, and so we apply Lemma 6. Then, every solution $\{(x_n, y_n)\}_{n=5}^{\infty}$ is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$.

If $x_5 \leq 0$, then $y_5 \geq 0$, and so we apply Lemma 5. Then, every solution $\{(x_n, y_n)\}_{n=5}^{\infty}$ is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$. □

Lemma 8 Every solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1) with initial condition in Q_1 is eventually prime period-6 $\left(P_6^1 \text{ or } P_6^2\right)$.

Proof. Let $(x_0, y_0) \in Q_1$. Then $x_0 > 0$ and $y_0 > 0$. Then,

$$x_1 = |x_0| - y_0 + 1 = x_0 - y_0 + 1$$
$$y_1 = x_0 + |y_0| - 1 = x_0 + y_0 - 1.$$

The signs of x_1 and y_1 are unknown, but if $(x_1, y_1) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution is eventually prime period-6 solution. For the case that $x_1 = x_0 - y_0 + 1 > 0$ and $y_1 = x_0 + y_0 - 1 > 0$, then

$$x_2 = |x_1| - y_1 + 1 = -2y_0 + 3$$
$$y_2 = x_1 + |y_1| - 1 = 2x_0 - 1.$$

The signs of x_2 and y_2 are unknown, but if $(x_2, y_2) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution is eventually prime period-6 solution. For the case that $x_2 = -2y_0 + 3 > 0$ and

$y_2 = 2x_0 - 1 > 0 \left(x_0 > \frac{1}{2}, y_0 < \frac{3}{2} \right)$, then

$$x_3 = |x_2| - y_2 + 1 = -2x_0 - 2y_0 + 5$$
$$y_3 = x_2 + |y_2| - 1 = 2x_0 - 2y_0 + 1.$$

The signs of x_3 and y_3 are unknown, but if $(x_3, y_3) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution is eventually prime period-6 solution. For the case that $x_3 = -2x_0 - 2y_0 + 5 > 0$ and

$y_3 = 2x_0 - 2y_0 + 1 > 0$, then

$$x_4 = |x_3| - y_3 + 1 = -4y_0 + 5$$
$$y_4 = x_3 + |y_3| - 1 = -4x_0 + 5.$$

The signs of x_4 and y_4 are unknown, but if $(x_4, y_4) \in R^2 \setminus Q_1$, , we apply the above Lemma to conclude that solution is eventually prime period-6 solution. For the case that $x_4 = -4x_0 + 5 > 0$ and

$y_4 = -4y_0 + 5 > 0 \left(\frac{1}{2} < x_0 < \frac{5}{4}, y_0 < \frac{5}{4} \right)$, then

$$x_5 = |x_4| - y_4 + 1 = -4x_0 + 4y_0 + 1$$
$$y_5 = x_4 + |y_4| - 1 = -4x_0 - 4y_0 + 9.$$

The signs of x_5 and y_5 are unknown, but if $(x_5, y_5) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution is eventually prime period-6 solution. For the case that $x_5 = -4x_0 + 4y_0 + 1 > 0$ and

$y_5 = -4x_0 - 4y_0 + 9 > 0$, then

$$x_6 = |x_5| - y_5 + 1 = 8y_0 - 7$$
$$y_6 = x_5 + |y_5| - 1 = -8x_0 + 9.$$

The signs of x_6 and y_6 are unknown, but if $(x_6, y_6) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution is eventually prime period-6 solution. For the case that $x_6 = 8y_0 - 7 > 0$ and

$y_6 = -8x_0 + 9 > 0 \left(\frac{1}{2} < x_0 < \frac{9}{8}, \frac{7}{8} < y_0 < \frac{5}{4} \right)$, then

We will prove that for $x_0 \in \left(\frac{1}{2}, \frac{9}{8} \right)$ and $y_0 \in \left(\frac{7}{8}, \frac{5}{4} \right)$ the solution is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2 \right)$ by mathematical induction.

For each $n \geq 0$, let $P(n)$ be the following statement: For $x_0 \in (a_n, b_n)$ and $y_0 \in (c_n, d_n)$

$$x_{8n+7} = 2^{4n+3} x_0 + 2^{4n+3} y_0 - \delta_n, \quad y_{8n+7} = 2^{4n+3} x_0 + 2^{4n+3} y_0 + 1$$

such that x_{8n+7} and y_{8n+7} are positive. Otherwise, the solution is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$. If x_{8n+7} and y_{8n+7} are positive, then

$$x_{8n+8} = 2^{4n+4}x_0 - \delta_n, \quad y_{8n+8} = 2^{4n+4}y_0 - \delta_n$$

such that x_{8n+8} and y_{8n+8} are positive when $x_0 \in (e_n, b_n)$ and $y_0 \in (e_n, d_n)$. Otherwise, the solution is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$. If x_{8n+8} and y_{8n+8} are positive, then

$$x_{8n+9} = 2^{4n+4}x_0 - 2^{4n+4}y_0 + 1, \quad y_{8n+9} = 2^{4n+4}x_0 + 2^{4n+4}y_0 - (2\delta_n + 1)$$

such that x_{8n+9} and y_{8n+9} are positive. Otherwise, the solution is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$. If x_{8n+9} and y_{8n+9} are positive, then

$$x_{8n+10} = -2^{4n+5}x_0 + (2\delta_n + 3), \quad y_{8n+10} = 2^{4n+5}y_0 - (2\delta_n + 1)$$

such that x_{8n+10} and y_{8n+10} are positive when $x_0 \in (a_{n+1}, b_n)$ and $y_0 \in (e_n, f_n)$. Otherwise, the solution is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$. If x_{8n+10} and y_{8n+10} are positive, then

$$x_{8n+11} = -2^{4n+5}x_0 - 2^{4n+5}y_0 + (4\delta_n + 5), \quad y_{8n+11} = 2^{4n+5}x_0 - 2^{4n+5}y_0 + 1$$

such that x_{8n+11} and y_{8n+11} are positive. Otherwise, the solution is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$. If x_{8n+11} and y_{8n+11} are positive, then

$$x_{8n+12} = -2^{4n+6}x_0 + (4\delta_n + 5), \quad y_{8n+12} = -2^{4n+6}y_0 + (4\delta_n + 5)$$

such that x_{8n+12} and y_{8n+12} are positive when $x_0 \in (a_{n+1}, d_{n+1})$ and $y_0 \in (e_n, d_{n+1})$. Otherwise, the solution is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$. If x_{8n+12} and y_{8n+12} are positive, then

$$x_{8n+13} = -2^{4n+6}x_0 + 2^{4n+6}y_0 + 1, \quad y_{8n+13} = -2^{4n+6}x_0 - 2^{4n+6}y_0 + (8\delta_n + 9)$$

such that x_{8n+13} and y_{8n+13} are positive. Otherwise, the solution is eventually prime period-6 solution $\left(P_6^1 \text{ or } P_6^2\right)$. If x_{8n+13} and y_{8n+13} are positive, then

$$x_{8n+14} = 2^{4n+7}y_0 - (8\delta_n + 7), \quad y_{8n+14} = -2^{4n+7}x_0 + (8\delta_n + 9)$$

such that x_{8n+14} and y_{8n+14} are positive when $x_0 \in (a_{n+1}, b_{n+1})$ and $y_0 \in (c_{n+1}, d_{n+1})$ where

$$a_n = \frac{2^{4n+1} - 1}{2^{4n+1}}, \; b_n = \frac{2^{4n+3} + 1}{2^{4n+3}}, \; c_n = \frac{2^{4n+3} - 1}{2^{4n+3}}, \; d_n = \frac{2^{4n+2} + 1}{2^{4n+2}}, \; e_n = \frac{2^{4n+4} - 1}{2^{4n+4}},$$

$$f_n = \frac{2^{4n+5} + 1}{2^{4n+5}}, \; \text{and } \delta_n = 2^{4n+4} - 1.$$

We shall show that $P(0)$ is true. For $x_0 \in (a_0, b_0) = \left(\frac{1}{2}, \frac{9}{8}\right)$ and $y_0 \in (c_0, d_0) = \left(\frac{7}{8}, \frac{5}{4}\right)$ and

$x_6 = 8y_0 - 7 > 0, \quad y_6 = -8x_0 + 9 > 0.$ Then

$$x_{8(0)+7} = x_7 = |x_6| - y_6 + 1 = 8x_0 + 8y_0 - 15 \quad = 2^{4(0)+3}x_0 + 2^{4(0)+3}y_0 - \delta_0$$
$$y_{8(0)+7} = y_7 = x_6 + |y_6| - 1 = -8x_0 + 8y_0 + 1 \quad = -2^{4(0)+3}x_0 + 2^{4(0)+3}y_0 + 1.$$

The signs of x_7 and y_7 are unknown, but if $(x_7, y_7) \in R^2 \setminus Q_1,$ we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. If x_7 and y_7 are positive, then we have

$$x_{8(0)+8} = x_8 = |x_7| - y_7 + 1 = 16x_0 - 15 \quad = 2^{4(0)+4}x_0 - \delta_0$$
$$y_{8(0)+8} = y_8 = x_7 + |y_7| - 1 = 16y_0 - 15 \quad = 2^{4(0)+4}y_0 - \delta_0.$$

The signs of x_8 and y_8 are unknown, but if $(x_8, y_8) \in R^2 \setminus Q_1,$ we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. The case when $x_0 \in (e_0, b_0) = \left(\frac{15}{16}, \frac{9}{8}\right)$ and $y_0 \in (e_0, d_0) = \left(\frac{15}{16}, \frac{5}{4}\right)$, we have $x_8 > 0$ and $y_8 > 0$. Then

$$x_{8(0)+9} = x_9 = |x_8| - y_8 + 1 = 16x_0 - 16y_0 + 1 \quad = 2^{4(0)+4}x_0 - 2^{4(0)+4}y_0 + 1$$
$$y_{8(0)+9} = y_9 = x_8 + |y_8| - 1 = 16x_0 + 16y_0 - 31 \quad = 2^{4(0)+4}x_0 + 2^{4(0)+4}y_0 - (2\delta_0 + 1).$$

The signs of x_9 and y_9 are unknown, but if $(x_9, y_9) \in R^2 \setminus Q_1,$ we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. If x_9 and y_9 are positive, then we have

$$x_{8(0)+10} = x_{10} = |x_9| - y_9 + 1 = -32y_0 + 33 \quad = -2^{4(0)+5}y_0 + (2\delta_0 + 3)$$
$$y_{8(0)+10} = y_{10} = x_9 + |y_9| - 1 = 32x_0 - 31 \quad = 2^{4(0)+5}x_0 - (2\delta_0 + 1).$$

The signs of x_{10} and y_{10} are unknown, but if $(x_{10}, y_{10}) \in R^2 \setminus Q_1,$ we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. The case when $x_0 \in (a_1, b_0) = \left(\frac{31}{32}, \frac{9}{8}\right)$ and $y_0 \in (e_0, f_0) = \left(\frac{15}{16}, \frac{33}{32}\right)$, we have $x_{10} > 0$ and $y_{10} > 0$. Then

$$x_{8(0)+11} = x_{11} = |x_{10}| - y_{10} + 1 = -32x_0 - 32y_0 + 65 \quad = -2^{4(0)+5}x_0 - 2^{4(0)+5}y_0 + (4\delta_0 + 5)$$
$$y_{8(0)+11} = y_{11} = x_{10} + |y_{10}| - 1 = 32x_0 - 32y_0 + 1 \quad = 2^{4(0)+5}x_0 - 2^{4(0)+5}y_0 + 1.$$

The signs of x_{11} and y_{11} are unknown, but if $(x_{11}, y_{11}) \in R^2 \setminus Q_1,$ we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. If x_{11} and y_{11} are positive, then we have

$$x_{8(0)+12} = x_{12} = |x_{11}| - y_{11} + 1 = -64x_0 + 65 \quad = -2^{4(0)+6}x_0 + (4\delta_0 + 5)$$
$$y_{8(0)+12} = y_{12} = x_{11} + |y_{11}| - 1 = -64y_0 + 65 \quad = -2^{4(0)+6}y_0 + (4\delta_0 + 5).$$

The signs of x_{12} and y_{12} are unknown, but if $(x_{12}, y_{12}) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. The case when

$x_0 \in (a_1, d_1) = \left(\frac{31}{32}, \frac{65}{64}\right)$ and $y_0 \in (e_0, d_1) = \left(\frac{15}{16}, \frac{65}{64}\right)$, we have $x_{12} > 0$ and $y_{12} > 0$. Then

$$x_{8(0)+13} = x_{13} = |x_{12}| - y_{12} + 1 = -64x_0 + 64y_0 + 1 = -2^{4(0)+6} x_0 + 2^{4(0)+6} y_0 + 1$$
$$y_{8(0)+13} = y_{13} = x_{12} + |y_{12}| - 1 = -64x_0 - 64y_0 + 129 = -2^{4(0)+6} x_0 - 2^{4(0)+6} y_0 + (8\delta_0 + 9).$$

The signs of x_{13} and y_{13} are unknown, but if $(x_{13}, y_{13}) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. If x_{13} and y_{13} are positive, then we have

$$x_{8(0)+14} = x_{14} = |x_{13}| - y_{13} + 1 = 128y_0 - 127 = 2^{4(0)+7} y_0 - (8\delta_0 + 7)$$
$$y_{8(0)+14} = y_{14} = x_{13} + |y_{13}| - 1 = -128x_0 + 129 = -2^{4(0)+7} x_0 + (8\delta_0 + 9).$$

The signs of x_{14} and y_{14} are unknown, but if $(x_{14}, y_{14}) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. The case when

$x_0 \in (a_1, b_1) = \left(\frac{31}{32}, \frac{129}{128}\right)$ and $y_0 \in (c_1, d_1) = \left(\frac{127}{128}, \frac{65}{64}\right)$, we have $x_{12} > 0$ and $y_{12} > 0$. Hence $P(0)$ is true.

Next, we assume that $P(N)$ is true for some $N \in \mathbb{N}$. We shall show that $P(N+1)$ is true. Since $P(N)$ is true, for

$x_0 \in (a_{N+1}, b_{N+1}) = \left(\frac{2^{4N+5}-1}{2^{4N+5}}, \frac{2^{4N+7}+1}{2^{4N+7}}\right)$ and $y_0 \in (c_{N+1}, d_{N+1}) = \left(\frac{2^{4N+7}-1}{2^{4N+7}}, \frac{2^{4N+6}+1}{2^{4N+6}}\right)$,

$x_{8N+14} = 2^{4N+7} y_0 - (8\delta_N + 7) > 0$ and $y_{8N+14} = -2^{4N+7} x_0 + (8\delta_N + 9) > 0$. Then,

$$x_{8(N+1)+7} = x_{8N+15} = 2^{4N+7} x_0 + 2^{4N+7} y_0 - (16\delta_N + 15)$$
$$= 2^{4(N+1)+3} x_0 + 2^{4(N+1)+3} y_0 - (\delta_{N+1})$$
$$y_{8(N+1)+7} = y_{8N+15} = -2^{4N+7} x_0 + 2^{4N+7} y_0 + 1$$
$$= -2^{4(N+1)+3} x_0 + 2^{4(N+1)+3} y_0 + 1.$$

The signs of $x_{8(n+1)+7}$ and $y_{8(n+1)+7}$ are unknown, but if $(x_{8(n+1)+7}, y_{8(n+1)+7}) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. If $x_{8(n+1)+7}$ and $y_{8(n+1)+7}$ are positive, then we have

$$x_{8(N+1)+8} = x_{8N+16} = 2^{4N+8} x_0 - \delta_{N+1} = 2^{4(N+1)+4} x_0 - \delta_{N+1}$$
$$y_{8(N+1)+8} = y_{8N+16} = 2^{4N+8} y_0 - \delta_{N+1} = 2^{4(N+1)+4} y_0 - \delta_{N+1}.$$

The signs of $x_{8(N+1)+8}$ and $y_{8(N+1)+8}$ are unknown, but if $(x_{8(N+1)+8}, y_{8(N+1)+8}) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. The case when

$x_0 \in \left(e_{N+1}, b_{N+1}\right) = \left(\dfrac{2^{4N+8}-1}{2^{4N+8}}, \dfrac{2^{4N+7}+1}{2^{4N+7}}\right)$ and $y_0 \in \left(e_{N+1}, d_{N+1}\right) = \left(\dfrac{2^{4N+8}-1}{2^{4N+8}}, \dfrac{2^{4N+6}+1}{2^{4N+6}}\right)$, we have

$x_{8N+16} > 0$ and $y_{8N+16} > 0$. Then,

$$x_{8(N+1)+9} = x_{8N+17} = 2^{4(N+1)+4}x_0 - 2^{4(N+1)+4}y_0 + 1$$
$$y_{8(N+1)+9} = y_{8N+17} = 2^{4(N+1)+4}x_0 + 2^{4(N+1)+4}y_0 - \left(2\delta_{N+1} + 1\right).$$

The signs of $x_{8(n+1)+9}$ and $y_{8(n+1)+9}$ are unknown, but if $\left(x_{8(n+1)+9}, y_{8(n+1)+9}\right) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. If $x_{8(n+1)+9}$ and $y_{8(n+1)+9}$ are positive, then we have

$$x_{8(N+1)+10} = x_{8N+18} = -2^{4(N+1)+5}y_0 + \left(2\delta_{N+1} + 3\right)$$
$$y_{8(N+1)+10} = y_{8N+18} = 2^{4(N+1)+5}x_0 - \left(2\delta_{N+1} + 1\right).$$

The signs of $x_{8(N+1)+10}$ and $y_{8(N+1)+10}$ are unknown, but if $\left(x_{8(N+1)+10}, y_{8(N+1)+10}\right) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. The case when $x_0 \in \left(a_{N+2}, b_{N+1}\right) = \left(\dfrac{2^{4N+9}-1}{2^{4N+9}}, \dfrac{2^{4N+7}+1}{2^{4N+7}}\right)$ and $y_0 \in \left(e_{N+1}, f_{N+1}\right) = \left(\dfrac{2^{4N+8}-1}{2^{4N+8}}, \dfrac{2^{4N+9}+1}{2^{4N+9}}\right)$, we have

$x_{8N+18} > 0$ and $y_{8N+18} > 0$. Then,

$$x_{8(N+1)+11} = x_{8N+19} = -2^{4(N+1)+5}x_0 - 2^{4(N+1)+5}y_0 + \left(4\delta_{N+1} + 5\right)$$
$$y_{8(N+1)+11} = y_{8N+19} = 2^{4(N+1)+5}x_0 - 2^{4(N+1)+5}y_0 + 1.$$

The signs of $x_{8(N+1)+11}$ and $y_{8(N+1)+11}$ are unknown, but if $\left(x_{8(N+1)+11}, y_{8(N+1)+11}\right) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. If $x_{8(n+1)+11}$ and $y_{8(n+1)+11}$ are positive, then we have

$$x_{8(N+1)+12} = x_{8N+20} = -2^{4(N+1)+6}x_0 + \left(4\delta_{N+1} + 5\right)$$
$$y_{8(N+1)+12} = y_{8N+20} = -2^{4(N+1)+6}y_0 + \left(4\delta_{N+1} + 5\right).$$

The signs of $x_{8(N+1)+12}$ and $y_{8(N+1)+12}$ are unknown, but if $\left(x_{8(N+1)+12}, y_{8(N+1)+12}\right) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. The case when $x_0 \in \left(a_{N+2}, d_{N+2}\right) = \left(\dfrac{2^{4N+9}-1}{2^{4N+9}}, \dfrac{2^{4N+10}+1}{2^{4N+10}}\right)$ and $y_0 \in \left(e_{N+1}, d_{N+2}\right) = \left(\dfrac{2^{4N+8}-1}{2^{4N+8}}, \dfrac{2^{4N+10}+1}{2^{4N+10}}\right)$, we have

$x_{8N+20} > 0$ and $y_{8N+20} > 0$. Then,

$$x_{8(N+1)+13} = x_{8N+21} = -2^{4(N+1)+6} x_0 + 2^{4(N+1)+6} y_0 + 1$$
$$y_{8(N+1)+13} = y_{8N+21} = -2^{4(N+1)+6} x_0 - 2^{4(N+1)+6} y_0 + \left(8\delta_{N+1} + 9\right).$$

The signs of $x_{8(n+1)+13}$ and $y_{8(n+1)+13}$ are unknown, but if $\left(x_{8(n+1)+13}, y_{8(n+1)+13}\right) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. If $x_{8(n+1)+13}$ and $y_{8(n+1)+13}$ are positive, then we have

$$x_{8(N+1)+14} = x_{8N+22} = 2^{4(N+1)+7} y_0 - \left(8\delta_{N+1} + 7\right)$$
$$y_{8(N+1)+14} = y_{8N+22} = -2^{4(N+1)+7} x_0 + \left(8\delta_{N+1} + 9\right).$$

The signs of $x_{8(N+1)+14}$ and $y_{8(N+1)+14}$ are unknown, but if $\left(x_{8(N+1)+14}, y_{8(N+1)+14}\right) \in R^2 \setminus Q_1$, we apply the above Lemma to conclude that solution that solution is eventually prime period-6 solution. The case when $x_0 \in \left(a_{N+2}, b_{N+2}\right) = \left(\frac{2^{4N+9}-1}{2^{4N+9}}, \frac{2^{4N+11}+1}{2^{4N+11}}\right)$ and $y_0 \in \left(c_{N+2}, d_{N+2}\right) = \left(\frac{2^{4N+11}-1}{2^{4N+11}}, \frac{2^{4N+10}+1}{2^{4N+10}}\right)$, we have $x_{8N+22} > 0$ and $y_{8N+22} > 0$. Hence $P(N+1)$ is true. By mathematical induction $P(n)$ is true for all $n \geq 0$. Note that

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty} b_n = \lim_{x\to\infty} c_n = \lim_{x\to\infty} d_n = \lim_{x\to\infty} e_n = \lim_{x\to\infty} f_n = 1.$$

The proof is complete. □

3. Conclusion

We have presented the complete results concerning the global character of the solutions to System(1). We utilized mathematical induction, and direct computations to show that every solution of System(1) is eventually either the prime period-6 solution P^1_6, or the prime period-6 solution P^2_6 . The proofs involve careful consideration of the various

cases.

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