

# Identifying Bullwhip Effect in Supply Chains with correlated market demands

Po-ngarm Somkun

Department of Industrial Engineering, Naresuan University, Phitsanulok 65000, E-mail: pongarmr@nu.ac.th

**Abstract** – This paper identifies the Bullwhip Effect related to market demands, which are correlated both to their previous value and amongst the markets. An analytical model is used to represent stochastic and dynamic natures of supply chains' distribution networks. Control engineering tools such as block diagram and z-transform are applied in order to obtain closed form solutions that are crucial in the analytical analysis. A spreadsheet model is used to validate the analytical solution.

The results show that the correlations in the demands have a significant impact on the magnitude of the Bullwhip and consequently supply chains' costs.

**Keywords** – Supply chain management, Bullwhip effect, VAR(1) demand process

## 1. NOMENCLATURE

VAR(1) – Vector Auto-Regressive of the first order  
DC – Distribution centre  
MMSE – Minimum Mean Square Error  
OUT – Order-Up-To

## 2. INTRODUCTION

The Bullwhip Effect [1] is a phenomenon in which the variation of the order is higher when the order is passed to the upstream levels of a supply chain. This increase of the variation is one of the major foundations for inefficient operations. High operational costs arise resulted from the fluctuating inventory status, unstable capacity requirement and ineffective transportation. Four main causes of the bullwhip effect includes anticipation of demands over lead-times, accumulation of orders in periodic inventory system and to achieve economy of scale in shipment, large amount and future buying due to price discounts, and cancellation of orders placed for shortage gaming [1, 2].

Many researchers have proposed models to measure the bullwhip effect. The following expression by Chen et al. [3] is commonly used.

$$\frac{\sigma_{order}^2}{\sigma_{demand}^2} \quad (1)$$

Equation (1) basically expresses the magnitude of the bullwhip effect by the ratio between the variances of the

order and the market demand. Thus, the bullwhip effect does not occur when the ratio is less than or equal to 1. Otherwise, the bullwhip effect exists.

This study will also consider the ratio between the variances of the inventory level and the demand described as

$$\frac{\sigma_{inventory}^2}{\sigma_{demand}^2} \quad (2)$$

This addition is in order to fully capture the dynamics of the inventory system [4, 5].

Customer demands have been widely modelled as first order autoregressive processes in bullwhip effect studies [2, 5, 6, 7, 8]. The correlation of the demands between retailers has often been neglected. Erkip [9] asserted that real product demands possess this complexity. Ratanachote [10] analysed point of sales data and statistically confirmed that the correlated demands can be modelled as VAR(1) processes. Reflecting to the lack of consideration of the demand correlation in most studies while its existence has already been proved, this study will analyse the magnitude of the bullwhip effect in the supply chain's distribution network when the market demands follow the VAR(1) process.

The main technique used in this study is analytical modelling. The model describes the characteristic of the demand process, inventory replenishment, ordering policy and forecasting method of each location in the supply chain. The modelling is explained in Section 3 from Subsections 3.1 to 3.2. The closed form solutions of the orders and inventory variances are subsequently obtained. This is done by applying control engineering tools such as block diagram and z-transform as shown in Subsection 3.3. Due to the complexity of the mathematical results, a spreadsheet model is also used to cross-check the analytical solution. The bullwhip and the variances of inventory level are quantified and analysed in Sections 4 and 5 respectively. Finally, the results are discussed and the conclusions are drawn in Section 6.

## 3. THE SUPPLY CHAIN MODEL

The author intended to show a model of a simple supply chain in this study. This is to allow a meaningful exposition and to avoid lengthy equations. Therefore, a

two-level supply chain, consisting of two retailers and two DCs with simple information and material flows, is considered as shown in Fig. 1. Also, unit lead-times at all locations are assumed.

### 3.1 The market demand model

The demand at each market is assumed to follow a VAR(1) process. Specifically, it is the mean-centred VAR(1) process given by

$$\begin{aligned} D_{1,t} &= \mu_1 + \phi_{11}(D_{1,t-1} - \mu_1) + \phi_{12}(D_{2,t-1} - \mu_2) + \varepsilon_{1,t}, \\ D_{2,t} &= \mu_2 + \phi_{21}(D_{1,t-1} - \mu_1) + \phi_{22}(D_{2,t-1} - \mu_2) + \varepsilon_{2,t}, \end{aligned} \quad (3)$$

where  $D_{i,t}$  is the mean-centred demand for retailer  $i$  at time  $t$ , given by the sum of four parts. The first part is the mean demand of retailer  $i$ ,  $\mu_i$ . The second and third parts are autoregressive terms of retailer  $i$  with its own and with retailer  $j$  mean-centred demands, which are  $\phi_{ii}$  and  $\phi_{ij}$  respectively. The final part is independently and identically distributed random process,  $\varepsilon_{i,t}$ . It is assumed to have zero mean and unit variance. Also,  $\varepsilon_{1,t}$  and  $\varepsilon_{2,t}$  are uncorrelated.

The VAR(1) process will be stationary when the following criteria is hold

$$\left| \frac{(\phi_{11} + \phi_{22}) \pm \sqrt{(\phi_{11} + \phi_{22})^2 + 4\phi_{12}\phi_{21}}}{2} \right| < 1. \quad (4)$$

From this point forward, the demand model will be simplified by the following assumptions. (a) The auto-regression in time at all markets is the same, i.e.  $\phi_{11} = \phi_{22} = \beta$ . (b) The auto-regression between the two markets is identical, i.e.  $\phi_{12} = \phi_{21} = \gamma$ .

For stationary processes, the variance of the demand at each retailer can be found as

$$\sigma_D^2 = \frac{1 - \beta^2 - \gamma^2}{(\beta^2 - 1)^2 - 2\gamma^2(1 + \beta^2) + \gamma^4} \sigma_\varepsilon^2, \quad (5)$$

where  $\sigma_\varepsilon^2$  is the variance of the corresponding error term.

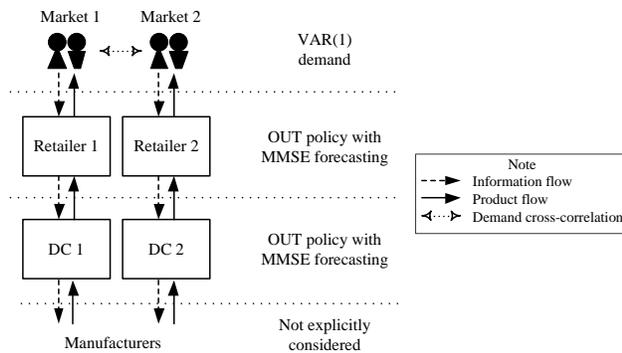


Figure 1 The distribution network

### 3.2 The replenishment policies and forecasting model

All retailers and DCs operate an OUT replenishment policy with MMSE forecasting. The replenishment order at retailer  $i$  at time  $t$ ,  $O_{i,t}$ , is given by

$$O_{i,t} = F_{i,t} + S_i - I_{i,t} - W_{i,t}, \forall i = 1, 2, \quad (6)$$

where  $(F_{i,t} + S_i)$  is the OUT level at time  $t$ .  $F_{i,t}$  is the forecast of demand at time  $t$  and  $S_i$  is the targeted safety stock level of retailer  $i$ . For MMSE forecasting  $F_t$  is the conditional expectation of demand over the lead-time and review period. For a unit lead-time, the forecast is

$$F_{i,t} = 2\mu_i + (\beta + \beta^2 + \gamma^2)(D_{i,t} - \mu_i) + (\gamma + 2\beta\gamma)(D_{j,t} - \mu_j), \quad (7)$$

where for retailer 1,  $i = 1$  and  $j = 2$ ; for retailer 2,  $i = 2$  and  $j = 1$ .

The inventory level of retailer  $i$  at time  $t$ ,  $I_{i,t}$ , is written as

$$I_{i,t} = I_{i,t-1} + O_{i,t-2} - D_{i,t}, \forall i = 1, 2. \quad (8)$$

The work in progress of retailer  $i$  at time  $t$ ,  $W_{i,t}$ , is given by

$$W_{i,t} = O_{i,t-1}, \forall i = 1, 2. \quad (9)$$

For DCs, the math expressions of the inventory replenishment system are similar to Equations (6), (8) and (9). A superscripted "dc" is used for DC's variables such as  $O_{i,t}^{dc}$ ,  $I_{i,t}^{dc}$  and  $W_{i,t}^{dc}$ . The demand for DC  $i$  at time  $t$ ,  $D_{i,t}^{dc}$ , is in fact the order that is placed by retailer  $i$  at time  $t$ . It is assumed that orders from the retailer are passed to the DC without delay. The forecast of demand at DC  $i$ ,  $F_{i,t}^{dc}$ , is expressed as

$$F_{i,t}^{dc} = 2\mu_i + (\beta^3 + \beta^4 + \gamma^4 + 3\beta\gamma^2 + 6\beta^2\gamma^2)(D_{i,t} - \mu_i) + (\gamma^3 + 4\beta\gamma^3 + 3\beta^2\gamma + 4\beta^3\gamma)(D_{j,t} - \mu_j), \quad (10)$$

where for DC 1,  $i = 1$  and  $j = 2$ ; for DC 2,  $i = 2$  and  $j = 1$ .

### 3.3 Obtaining the variances of the orders and the inventory level

According to the definition of the Bullwhip Effect in Equations (1) and (2), we need to quantify the order variance and the inventory variance. The block diagram in Fig. 2 represents our replenishment decisions using the discrete time z-transform notation. Interested readers can find background reading on Control Theory in Nise [11]

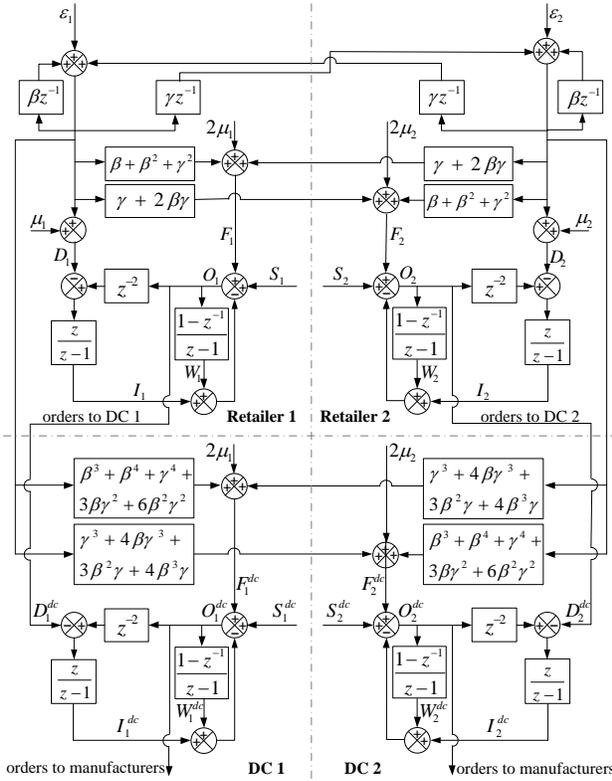


Figure 2 Block diagram represents the distribution network

We first manipulate the block diagram to obtain the transfer function, which is a mathematical expression of the relationship between the system's input and output. In our case, the input is the error term and the output is the orders and the inventory level. We then take the inverse z-transform of the transfer function. This results in the time domain impulse response for the state variables. Next, we can sum its square to determine the long run variance of the orders and inventory level. This operation is known as Tsytsin's Relation [12]. The full equations along the process are omitted due to their excessive length.

The variance of the orders for each retailer, which is the summation of variances resulted from the two errors, is described as

$$\sigma_o^2 = \frac{1}{4} \left( 8\beta(1 + \beta + \beta^2) + 8\gamma^2(1 + 3\beta) - \frac{4(\beta^2 + \gamma^2 - 1)}{\beta^4 - 2\beta^2(1 + \gamma^2) + (\gamma^2 - 1)^2} \right) \sigma_\epsilon^2. \quad (11)$$

The variance of the inventory level for each retailer is expressed as

$$\sigma_i^2 = (2 + \beta(2 + \beta) + \gamma^2) \sigma_\epsilon^2. \quad (12)$$

For DCs' level, the variances of orders and inventory level are given in Equations (13) and (14) respectively.

$$\sigma_{o,dc}^2 = \frac{1}{4} \left( 8\gamma^6(1 + 7\beta) - \frac{4(\beta^2 + \gamma^2 - 1)}{\beta^4 - 2\beta^2(1 + \gamma^2) + (\gamma^2 - 1)^2} + 8\beta(1 + \beta^2)(1 + \beta + \beta^2 + \beta^3 + \beta^4) + 8\gamma^4(2 + 5\beta(2 + \beta(3 + 7\beta))) + 8\gamma^2(1 + \beta(6 + \beta(12 + \beta(20 + 3\beta(5 + 7\beta)))) \right) \sigma_\epsilon^2. \quad (13)$$

$$\sigma_{i,dc}^2 = \left( (1 + \beta + \beta^2 + \gamma^2)^2 + (1 + \beta + 3\beta\gamma^2 + \beta^2 + \beta^3 + \gamma^2)^2 + \gamma^2((1 + 2\beta)^2 + (1 + \gamma^2 + \beta(2 + 3\beta))^2) \right) \sigma_\epsilon^2. \quad (14)$$

These variance expressions are quickly cross-checked by a numerical example in the spreadsheet model.

#### 4. BULLWHIP IN A TWO LEVEL SUPPLY CHAIN

We can now quantify the Bullwhip Effect using the variances obtained from Section 3.3. By substituting Equations (5) and (11), we will have the comprehensive forms of the Bullwhip magnitude for each retailer as

$$\begin{aligned} \text{Bullwhip}[\text{Retailer}] &= \frac{\sigma_o^2}{\sigma_D^2} \\ &= \frac{1}{1 - \beta^2 - \gamma^2} \left( 1 + \beta \left( 2 + \beta(1 + 2\beta(1 + \beta)(-1 - \beta + \beta^3)) \right) + \gamma^2 + 2\beta \left( 1 + \beta(-4 + \beta(-10 + (-1 + \beta)\beta)) \right) \right) \gamma^2 - 2(2 + \beta(5 + \beta + 5\beta^2))\gamma^4 + 2(1 + 3\beta)\gamma^6. \end{aligned} \quad (15)$$

By substituting Equations (5) and (13), the Bullwhip magnitude for each DC is written as

$$\begin{aligned} \text{Bullwhip}[DC] &= \frac{\sigma_{o,dc}^2}{\sigma_D^2} \\ &= \frac{\beta^4 + (-1 + \gamma^2)^2 - 2\beta^2(1 + \gamma^2)}{4(1 - \beta^2 - \gamma^2)} \left( 8\gamma^6(1 + 7\beta) + \frac{4(1 - \beta^2 - \gamma^2)}{\beta^4 + (-1 + \gamma^2)^2 - 2\beta^2(1 + \gamma^2)} + 8\beta(1 + \beta^2)(1 + \beta + \beta^2 + \beta^3 + \beta^4) + 8\gamma^4(2 + 5\beta(2 + \beta(3 + 7\beta))) + 8\gamma^2(1 + \beta(6 + \beta(12 + \beta(20 + 3\beta(5 + 7\beta)))) \right). \end{aligned} \quad (16)$$

The analytical examination can be performed by investigating the closed form expressions in Equations (15) and (16). There are some  $\beta$ 's with power of odd numbers while  $\gamma$ 's only have power of even numbers. This fact can be appreciated that the sign of  $\beta$ , which can be either positive or negative, dominates the resulted value of the Bullwhip. It is also demonstrates that the sign of  $\gamma$  does not affect the Bullwhip magnitude. For instance, the demand with  $\gamma = 0.4$  and  $\gamma = -0.4$  will resulted in the same Bullwhip magnitude.

Although the supply chain is simple, the closed form expressions are still complex. Thus, we also use the graphical method to identify the Bullwhip. The contour plots in Fig. 3 and Fig. 4 show the impact of the auto-

regression of the demand both in time ( $\beta$ ) and between markets ( $\gamma$ ) on the Bullwhip magnitude. To show how to read the plots; if your demand has  $\beta = 0.5$  and  $\gamma = 0.1$ , the Bullwhip magnitudes at the retailer and the DC are 2.3 and 3.0. The exact number can certainly get from the equations. Note that only the area that meets the stationary criteria as in Equation (4) is presented. The bold black line separates the areas where the Bullwhip does and does not exist.

Almost all demand patterns with negative  $\beta$  does not introduce Bullwhip. This is true for both retailer and DC echelons. However, high cross correlation ( $\gamma$ ) can cause Bullwhip even when the demand has negative  $\beta$ .

Demand patterns with positive  $\beta$  clearly establish Bullwhip. Once the correlated demand is passed up the supply chain as the order, its Bullwhip magnitude is significantly greater. For instance, a retailer that faces the demand with  $\beta = 0.4$  and  $\gamma = -0.4$  has Bullwhip scale of 2.03341. This measurement increases by 64% to be 3.33493 in the DC level.

## 5. VARIANCE AMPLIFICATION IN A TWO LEVEL SUPPLY CHAIN

Again we can have the complete forms for the inventory variance amplification by substituting Equations (5), (12) and (14). Let the inventory variance amplification for each retailer be given by

$$\begin{aligned} InvAmp[Retailer] &= \frac{\sigma_r^2}{\sigma_D^2} \\ &= 2 + \gamma^2 + \frac{\beta(2+\beta)(\beta^4+(-1+\gamma^2)^2-2\beta^2(1+\gamma^2))}{1-\beta^2-\gamma^2}, \end{aligned} \quad (17)$$

and for each DC is given by

$$\begin{aligned} InvAmp[DC] &= \frac{\sigma_{I,dc}}{\sigma_D^2} \\ &= (1 + \beta + \beta^2 + \gamma^2)^2 + (1 + \beta + \beta^2 + \beta^3 + 3\beta\gamma^2 + \gamma^2)^2 + \gamma^2 \left( \frac{\beta^4+(-1+\gamma^2)^2-2\beta^2(1+\gamma^2)}{1-\beta^2-\gamma^2} \right) ((1 + 2\beta)^2 + (1 + \gamma^2 + \beta(2 + 3\beta))^2). \end{aligned} \quad (18)$$

Investigating the closed form expressions analytically, the domination of the impact of  $\beta$  is still valid in the inventory variance amplification cases. The symmetric of the inventory variance amplification resulted from positive or negative  $\gamma$  is also valid.

The contour plots in Fig. 5 and Fig. 6 show the impact of the auto-regression of the demand both in time ( $\beta$ ) and between markets ( $\gamma$ ) on the amplification of the inventory variance. The amplification of the inventory variance is guaranteed to occur at the retailer level as entire combinations of  $\beta$  and  $\gamma$  have the plot magnitude greater than 1. The demand patterns with negative  $\beta$  likewise introduce smaller inventory variance amplification than positive  $\beta$  in most cases.

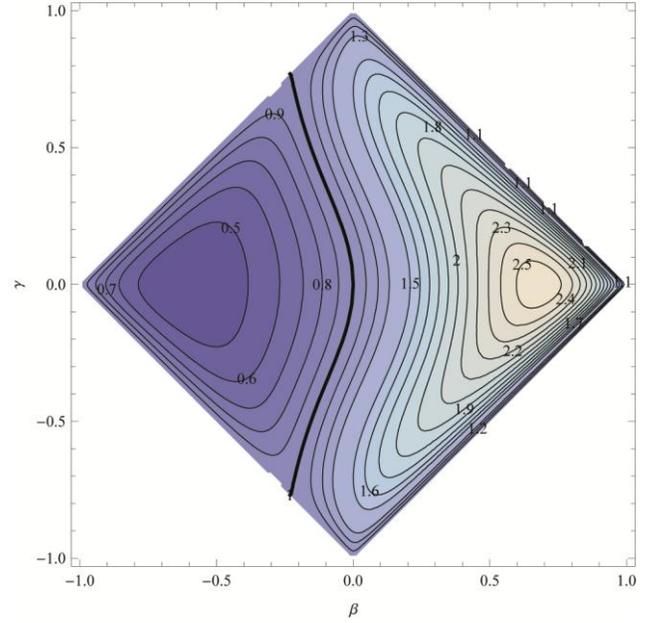


Figure 3 Contour plot exhibiting the Bullwhip at the retailer

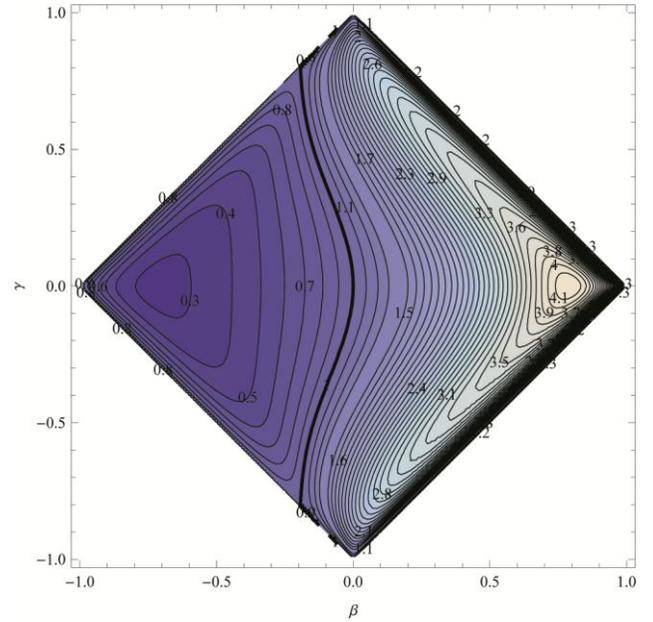


Figure 4 Contour plot exhibiting the Bullwhip at the DC

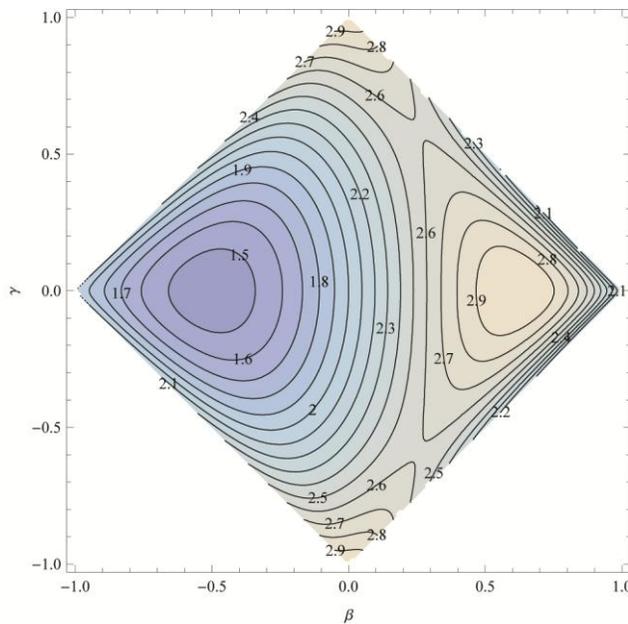
When compare with the Bullwhip, the inventory variance amplification possesses different character in the upper echelons. For  $\beta$  less than  $-0.6$ , there is no amplification at the DC. Otherwise, the magnitude of the inventory variance amplification is sharply increasing when the value of  $\beta$  closer to  $+1$ .

## 6. DISCUSSION AND CONCLUSION

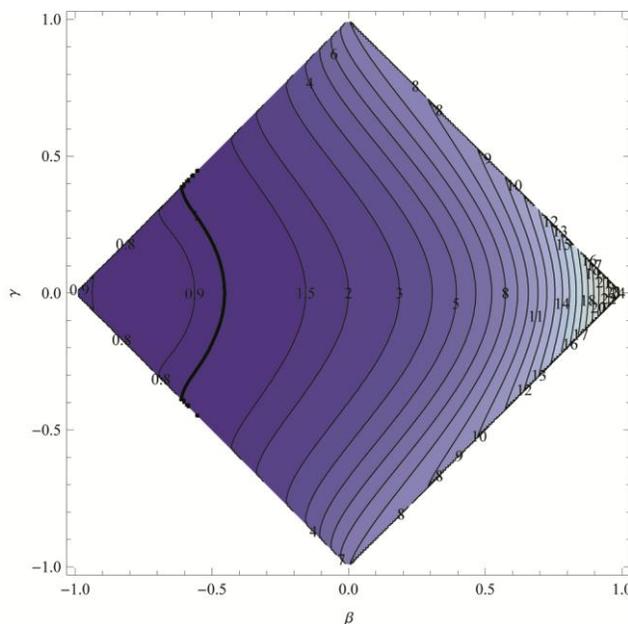
The result has clearly shown the influence of the correlations in the market demand on the variances of orders and inventory levels in a two-level supply chain. The management should understand the effect in order to operate the supply chain efficiently. The situation where

the market demand has positive auto-correlation is more crucial as it almost always introduces Bullwhip and inventory variance amplification. Ratanachote [10] verified that the demands of a consumer product in the real market have positive auto-correlations. This evidence emphasises the need for controlling the Bullwhip Effect.

According to the demand pattern and the inventory replenishment policy adopted in this study, the model is more suitable to analyse the Bullwhip effect of fast-moving consumer goods. Such products have high volume and high stock turnover. These products are groceries, soft drinks and toiletries for example.



**Figure 5** Contour plot for the inventory variance amplification at the retailer



**Figure 6** Contour plot for the inventory variance amplification at the DC

The model is, however, very simplified. Firstly, its auto-correlations of the two markets are assumed to be identical. This assumption is acceptable when two markets that have the similar character are analysed. Otherwise, different values of  $\phi_{11}$ ,  $\phi_{22}$ ,  $\phi_{12}$  and  $\phi_{21}$  should be allowed. In practice, we can obtain the values  $\phi_{ij}$  by statistical processes. Several freeware that are able to validate the VAR process are available for downloading. Secondly, the model application is limited to the situation where all locations have unit lead-times. Arbitrary lead-times should be more attracting. Thirdly, the more flexible number of locations should be considered in future studies.

Then again, allowing the above three constraints will make the model very complex and large. The use of vector notation is recommended. It can present  $n$  locations instead of the current two locations in a single equation. The recursive pattern of the MMSE forecast for  $n$  locations with arbitrary lead-times can also be avoided by using the vector notation.

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## 8. BIOGRAPHY



**Po-ngarm (Ratanachote) Somkun** has been working as a lecturer at the Department of Industrial Engineering, Naresuan University, Thailand, since 1997. She received B. Eng. (Industrial Engineering) from Chiangmai University and M. Eng. (Industrial Engineering) from Asian Institute of Technology. In 2001, she was awarded UNESCO/CHINA one-year scholarship in a field of Operations Research to do research at Northeastern University, China. She joined the Logistics Systems Dynamics Group, Cardiff University, UK, as a PhD student and received her PhD in 2011. Her current research interest concerns Logistics and Supply Chain Management, emphasising on Distribution Network Design, lateral transshipment and the Bullwhip Effect.