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Analysis of AHW and EAHW Time-Series Forecasting Methods: A Mathematical and Computational Perspective

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Abstract

Recently, an Extended Additive Holts-Winter method (EAHW) as a modified version of the well-known Additive Holt-Winters method (AHW) has been introduced. Many works in the research literature applied such a method to forecast time-series data, and results indicated that the EAHW provided optimal forecasting accuracy. However, we have found that there is no work in the literature analyzing the AHW and the EAHW in terms of the mathematical operations, which refer to the computation time and complexity of the methods. Since the AHW and EAHW can be implemented on hardware platform for using in many related systems and applications, in this paper, analysis of both methods from a mathematical and computational perspective is presented, where the mathematical operations of the EAHW compared with the AHW is provided. We show that the major difference between the EAHW and the AHW is the calculation of the level of time-series data, while the calculation of the trend and the seasonal factors are the same. With such a difference, the EAHW provided different possible solutions for forecasting. Also, in the worst case scenario, the EAHW has more multiplication operations (i.e., operation ×) than the case of the AHW method (but it is not too high); $2N$ for the AHW and $3N$ for the EAHW, where N is the number of data samples. By our finding, the EAHW is one of the appropriate forecasting methods for implementation (i.e., both on computers and embedded hardware platforms), since it provided good accuracy and computational complexity.

Keywords: Time-series, Forecasting, Holt-Winters, AHW, EAHW, Mathematical operation.

1. INTRODUCTION

In time-series forecasting, exponential forecasting methods as the well-known techniques are widely used. The exponential forecasting methods include the simple exponential smoothing, the Holt's linear exponential smoothing (or the double exponential smoothing), and the Holt-Winters (or the triple exponential smoothing). Here, the Holt-Winters are appropriately applied for time-series data that trend and seasonality behaviors are present. Although the mentioned methods are not new as introduced in the research literature, they are often used in practice and several real-word applications as reported in (Booranawong & Booranawong, 2017; Brown, 1956; Gardner, 1985; Gelper et al., 2010; Hunter, 1986; Kalekar, 2004; Montgomery et al., 2008; Suppalakpanya et al., 2019; Tratar & Srmcnik, 2016; Ventura et al., 2019; Winters, 1960). Here, many studies apply such methods due to their accuracy, simplicity, low complexity, and efficiency (Booranawong & Booranawong, 2017; Brown, 1956; Gardner, 1985; Gelper et al., 2010; Hunter, 1986; Kalekar, 2004; Montgomery et al., 2008; Suppalakpanya et al., 2019; Tratar & Srmcnik, 2016; Ventura et al., 2019; Winters, 1960).

Based on the research literature, recent works based on the Holt-Winters forecasting were introduced here, and they are summarized in Table 1. In (Siregar et al., 2017) an investigation of exponential smoothing methods for prediction of Indonesia agricultural production was introduced. Five-year time-series data were used with the Holt's linear and the AHW methods. The results showed that the AHW gave better forecasting accuracy. The comparison of the AHW and the Box–Jenkins for software failures prediction was presented in (Yakovyna & Bachkai, 2018). They were selected to estimate the angular software failures on a weekly basis. The results showed that the ARIMA performance was almost twice worse than the AHW. For the work in (Ventura et al., 2019) to estimate pollution episodes in rural, industrial, and urban areas, the AHW method and an Artificial Neural Network (ANN) were applied, using PM2.5 concentration time-series. The authors reported that, to predict PM2.5 in the industrial area, the AHW showed the best results as indicated by the forecasting error, while the ANN was the most suitable solution for the urban area and the rural area.

The EAHW method was first proposed in (Tratar et al., 2016). It was the modified version of the AHW method, where an additional parameter as the smoothing parameter in the level equation was introduced. The authors demonstrated that, by the simulation study with business and economic time series data, the forecasting errors decreased substantially. The EAHW method produced more accurate short-term forecasts than the original Holt-Winters. In (Tratar & Srmcnik, 2016), the performance comparison of the multiple regression method and the Holt-Winters method was analyzed. The results revealed that multiple regression was appropriate for daily and weekly short-term heat load forecasting, while the EAHW method showed the best results for both long-term and monthly short-term heat load forecasting.

In (Tratar & Strmčnik, 2019), forecasting methods in engineering were presented, where the forecasting performances of the AHW and EAHW were analyzed. The authors showed that the EAHW was the most optimal forecasting solution for quarterly data, in terms of forecasting accuracy. In (Suppalakpanya et al., 2019, p. 44-55), several exponential time-series methods including the Holt's linear, the AHW, and the EAHW methods were used to forecast Thailand oil palm prices, crude palm oil prices, and crude palm oil production. The authors demonstrated that the EAHW provided the lowest error indicated by the Mean Absolute Percentage Error (MAPE). In (Suppalakpanya et al., 2019, p.13-22), to forecast Thailand crude palm oil production and prices, an evaluation of the AHW and the EAHW with different initial trend values were investigated. The results reported that the EAHW achieved good accuracy. The results also revealed that, the forecasting error was significantly different when the different initial trend values were used. Finally, in (Suppalakpanya et al., 2019, p.123-139), a study of the Holt's linear, the AHW, and the EAHW for forecasting Thailand crude palm oil productions was presented, where the main contribution of such a work was that different input data (i.e., 3-year data to 12-year data) were used and tested. The results indicated that both AHW and EAHW gave the lowest error, when 12-year input data were applied.

Although the AHW and the EAHW provide appropriate forecasting accuracy in practice as introduced in the existing works presented above, an analysis of the AHW and the EAHW in terms of their mathematical operations which refer to the computational complexity of their forecasting algorithms is not investigated. Since the AHW and EAHW methods can be applied and implemented on hardware platforms for using in any related systems, the balance between the forecasting accuracy and the computation complexity is significantly required. Thus, in this paper, analysis of the original AHW and the recently proposed EAHW methods in terms of their mathematical operations is investigated, and the difference between the EAHW and the AHW functions is reported. We also show that, in the worst case scenario, the EAHW has more multiplication operations than the case of the traditional method; $2N$ for the AHW and $3N$ for the EAHW, where N is the number of data samples. By our study, we can conclude that since the EAHW computational complexity is not much high, the EAHW method is one of the appropriate methods for time-series forecasting and implementation on computers and embedded hardware platforms.

The structure of this paper is as follows. Section 2 introduces the details of the AHW and the EAHW forecasting methods. Section 3 provides mathematical analysis and discussion. Finally, we conclude the paper in Section 4.

Table 1 The summary of the related works

Ref.	Time-series forecasting methods	Application	Results (The best accuracy)
(Siregar et al., 2017)	- Holt's linear exponential smoothing - AHW	Agricultural production; palm oil production in Indonesia	AHW
(Yakovyna & Bachkai, 2018)	- AWH - Box-Jenkins method	Software failures prediction	AHW
(Ventura et al., 2019)	- AWH - Artificial Neural Network (ANN)	Air pollution; PM2.5 concentration	- AWH Prediction of PM2.5 in the industrial area - ANN For urban and rural areas
(Tratar et al., 2016)	- AHW - Multiplicative Holt-Winters (MHW) - EAHW	Business and economic time series	EAHW
(Tratar & Srmcnik, 2016)	- Multiple regression - Holt-Winters	Heat load forecasting	- Multiple regression Daily and weekly short-term heat load forecasting - EAHW Long-term and monthly short-term heat load forecasting
(Tratar & Strmčnik, 2019)	- AHW - EAHW	Engineering data	EAHW

Table 1 (Cont.)

Ref.	Time-series forecasting methods	Application	Results (The best accuracy)
(Suppalakpanya et al., 2019, p. 44-55)	- Double exponential smoothing - AHW - MHW - EAHW	- Oil palm price - Crude palm oil price - Crude palm oil production in Thailand	EAHW
(Suppalakpanya et al., 2019, p.13-22)	- AHW - MHW - EAHW	- Crude palm oil price - Crude palm oil production in Thailand	EAHW with the optimal initial trend value
(Suppalakpanya et al., 2019, p.123-139)	- Double exponential smoothing - AHW - MHW - EAHW	- Crude palm oil production in Thailand	AHW and EAHW with 12-year input data

2. AHW AND EAHW FORECASTING METHOD

Details of the AHW and the EAHW are introduced here. In time-series data forecasting, the exponential forecasting methods are widely used. The exponential methods include the simple exponential smoothing method, the Holt's linear method, and the Holt-Winters methods. Here, the Holt-Winters are suitably applied for time-series data when trend and seasonality behaviors (Kalekar, 2004; Montgomery et al., 2008; Holt, 2004) are presented.

The Holt-Winters incorporate three equations: the level, the trend, and seasonality, respectively. Generally, there are two methods of Holt-Winters: the Multiplicative Holt-Winters (MHW) and the Additive Holt-Winters (AHW) methods. The MHW is used when the seasonal variations are changing proportionally to the level of the data series. For the AHW, it is preferred in the case the seasonal variations are roughly constant through the data series.

AHW: The AHW method is described by (1) to (4), where L_i is the estimation of the level of the data series (at the sample number *i*), b_i is the estimation of the trend, S_i is the estimation of the seasonal factor, X_i is the input, n represents the number of periods in a year (or seasonality length), α , β , and γ refers to the weighting factors $(0 \le \alpha, \beta, \gamma \le 1)$, and Y_{i+m} is the forecasted value with m periods ahead.

$$L_i = \alpha(X_i - S_{i-m}) + (1-\alpha)(L_{i-1} + b_{i-1}) \tag{1}$$

$$b_i = \beta(L_i - L_{i-1}) + (1-\beta)b_{i-1} \tag{2}$$

$$S_i = \gamma(X_i - L_i) + (1-\gamma)S_{i-n} \tag{3}$$

$$Y_{i+m} = L_i + mb_i + S_{i-1+m} \tag{4}$$

EAHW: The EAHW was developed by Tratar in the year 2016 (Tratar & Srmcnik, 2016; Tratar et al., 2016). It is the revised version of the original AHW, where EAHW equations are shown in (5) to (8). Here, the only difference between the EAHW and the original AHW is the equation (5) and (1) (i.e., the equation for the level). The EAHW allows to smooth the seasonal factor (i.e., S_{i-m} in (5)) more or less than the AHW, depending on the δ value (δ is between 0 and 1).

$$L_i = \alpha X_i - \delta S_{i-m} + (1-\alpha)(L_{i-1} + b_{i-1}) \tag{5}$$

$$b_i = \beta(L_i - L_{i-1}) + (1-\beta)b_{i-1} \tag{6}$$

$$S_i = \gamma(X_i - L_i) + (1-\gamma)S_{i-n} \tag{7}$$

$$Y_{i+m} = L_i + mb_i + S_{i-1+m} \tag{8}$$

To initialize b_1 (or the trend value) in (2) and (6) for both AHW and EAHW, the works in (Kalekar, 2004; Montgomery et al., 2008; Tratar & Srmcnik, 2016; Tratar et al., 2016; Holt, 2004) suggested that the equation (9) including four different options can be used. However, by the study and test in (Suppalakpanya et al., 2019, p.13-22; Booranawong & Booranawong, 2019), options 1 and 2 significantly provided good results in terms of the forecasting error.

$$b_1 = \begin{cases} 0, & Option \quad 1 \\ (X_n - X_1)/(n-1), n = 12 & Option \quad 2 \\ X_2 - X_1, & Option \quad 3 \\ ((X_2 - X_1) + (X_3 - X_2) + (X_4 - X_3))/3, & Option \quad 4 \end{cases}$$

To initialize L_i and S_i (or the level and the seasonal factor) for both AHW and EAHW, as suggested by the pioneer works in (Kalekar, 2004; Montgomery et al., 2008;

Tratar & Srmcnik, 2016; Tratar et al., 2016; Holt, 2004), equations (10) and (11) are used, where $i = 1,2,3,\ldots,12$.

$$L_n = (X_1 + X_2 + \ldots + X_n)/n \tag{10}$$

$$S_i = X_i - L_n \tag{11}$$

Finally, for α, β, γ, and δ as the weighting factors, their optimal values are automatically determined during the process by minimizing the forecasting error (e.g. MAPE, Mean Square Error (MSE), and Mean Absolute Error (MAE)) (Booranawong & Booranawong, 2017; Tratar & Srmcnik, 2016; Suppalakpanya et al., 2019; Suppalakpanya et al., 2019; Suppalakpanya et al., 2019, Booranawong & Booranawong, 2017). We note that summary of the AHW and the EAHW are also provided in Table 2.

3. Mathematical Analysis and Discussion

Analysis and discussion of the AHW and the EAHW mathematical operations is presented in this section. As mentioned in Section 2 before, the only difference between the AHW and the EAHW methods is the estimation of the level in (1) and (5). The EAHW allows to smooth the seasonal factor or S_{i-m} more or less than the traditional AHW, depending on δ, which is between 0 and 1.

From (5), there are four cases of the δ weighting value including, $\delta \neq \alpha$, $\delta = \alpha$, $\delta = 1$, and $\delta = 0$, respectively, as shown in Table 3. If $\delta \neq \alpha$, the EAHW allows to smooth S_{i-m} more or less than (i.e., $\alpha < \delta$, and $\alpha > \delta$) the AHW. If $\delta = \alpha$, the EAHW method then reduces to the traditional AHW method (i.e., equation (1)). If $\delta = 1$, α occurs only at the input X_i and not at the seasonal component S_{i-m}. Therefore, when $\alpha X_i > S_{i-m}$ (the average in its seasonality is lower than the smoothed value) the level increases in comparison with the level in the earlier period. The opposite adjustment occurs when $\alpha X_i < S_{i-m}$. Finally, if $\delta = 0$, the seasonal component S_{i-m} is not considered.

Table 2 Summary of the AHW and the EAHW

Methods	Level, trend, and seasonal components	Initial values	Forecast value
AHW	$L_i = \alpha(X_i - S_{i-m}) + (1-\alpha)(L_{i-1} + b_{i-1})$ $b_i = \beta(L_i - L_{i-1}) + (1-\beta)b_{i-1}$ $S_i = \gamma(X_i - L_i) + (1-\gamma)S_{i-n}$	$b_1 = \begin{cases} 0 \\ (X_n - X_1)/(n-1), n = 12 \\ X_2 - X_1 \\ ((X_2 - X_1) + (X_3 - X_2) + (X_4 - X_3))/3 \end{cases}$	$Y_{i+m} = L_i + mb_i + S_{i-1+m}$
EAHW	$L_i = \alpha X_i - \delta S_{i-m} + (1-\alpha)(L_{i-1} + b_{i-1})$ $b_i = \beta(L_i - L_{i-1}) + (1-\beta)b_{i-1}$ $S_i = \gamma(X_i - L_i) + (1-\gamma)S_{i-n}$	$L_n = (X_1 + X_2 + \ldots + X_n)/n$ $S_i = X_i - L_n$	

Table 3 The δ value of the EAHW method

Cases	L_i	Note
$\delta \neq \alpha$	$L_i = \alpha X_i - \delta S_{i-m} + (1-\alpha)(L_{i-1} + b_{i-1})$	EAHW
$\delta = \alpha$	$L_i = \alpha X_i - \delta S_{i-m} + (1-\alpha)(L_{i-1} + b_{i-1})$ $= \alpha(X_i - S_{i-m}) + (1-\alpha)(L_{i-1} + b_{i-1})$	The EAHW reduces to the AHW.
$\delta = 1$	$L_i = \alpha X_i - S_{i-m} + (1-\alpha)(L_{i-1} + b_{i-1})$	$\alpha X_i > S_{i-m}$ and $\alpha X_i < S_{i-m}$ The smoothed value is higher or lower than the average in its seasonality.

Table 3 (Cont.)

Cases	L_i	Note
$\delta = 0$	$L_i = \alpha X_i + (1-\alpha)(L_{i-1} + b_{i-1})$	The seasonal component S_{i-m} is not considered.

Table 4 The computational cost in terms of the mathematical operation

Methods	Mathematical operation		
	+	–	×
AHW			
$L_i = \alpha(X_i - S_{i-m}) + (1-\alpha)(L_{i-1} + b_{i-1})$	**2**	**2**	**2**
EAHW			
$\delta \neq \alpha$; $L_i = \alpha X_i - \delta S_{i-m} + (1-\alpha)(L_{i-1} + b_{i-1})$	2	2	**3**
$\delta = \alpha$; $L_i = \alpha(X_i - S_{i-m}) + (1-\alpha)(L_{i-1} + b_{i-1})$	2	2	2
$\delta = 1$; $L_i = \alpha X_i - S_{i-m} + (1-\alpha)(L_{i-1} + b_{i-1})$	2	2	2
$\delta = 0$; $L_i = \alpha X_i + (1-\alpha)(L_{i-1} + b_{i-1})$	2	**1**	**2**

Table 4 shows the AHW and EAHW computational cost, where the number of mathematical operations; summations (+), subtractions (–), and multiplications (×) required by each forecasting method are listed in the table. The result demonstrates that the AHW and the EAHW use the same computational cost, when $\delta = \alpha$ and $\delta = 1$. In these cases, both methods require each mathematical operation of $2N$, where N is the number of data samples. For the case $\delta \neq \alpha$, the EAHW requires more × operations than the AHW method; $2N$ for the AHW and $3N$ for the EAHW, as illustrated by an example in Figure 1 (where $n = 1$ to 10,000). However, for the case $\delta \neq \alpha$ as seen in Figure 1, the cost difference is not much. Finally, for the case $\delta = 0$, the EAHW uses – and × operations lower than the case of the AHW; $2N$ for the AHW and $1N$ for the EAHW.

The results provided in this section can be useful for consideration in terms of the hardware implementation (i.e., for both on computers and embedded hardware platforms). As summarized in Table 4, it indicates that the EAHW is also the appropriate forecasting method for implementation, since it provides good forecasting accuracy as tested and reported by several works in the research literature, and its computational complexity is not high in the worst case scenario (i.e., $\delta \neq \alpha$).

4. CONCLUSIONS

In this paper, analysis of the original AHW and the EAHW forecasting methods in terms of the mathematical and computational perspective is presented, and the mathematical operations of the EAHW compared with the AHW is provided. We show that the major difference between these methods is the estimation of the level of the time-series data, while the estimation of the trend and seasonal factor is not different. According to this, the EAHW method provided different possible solutions for forecasting. The EAHW requires more multiplication operations than the case of the AHW method in the worst case scenario, but it is not too high. By our investigation, the EAHW method is one of the appropriate methods for the time-series forecasting, since it can satisfy both the forecasting accuracy and the computational complexity.

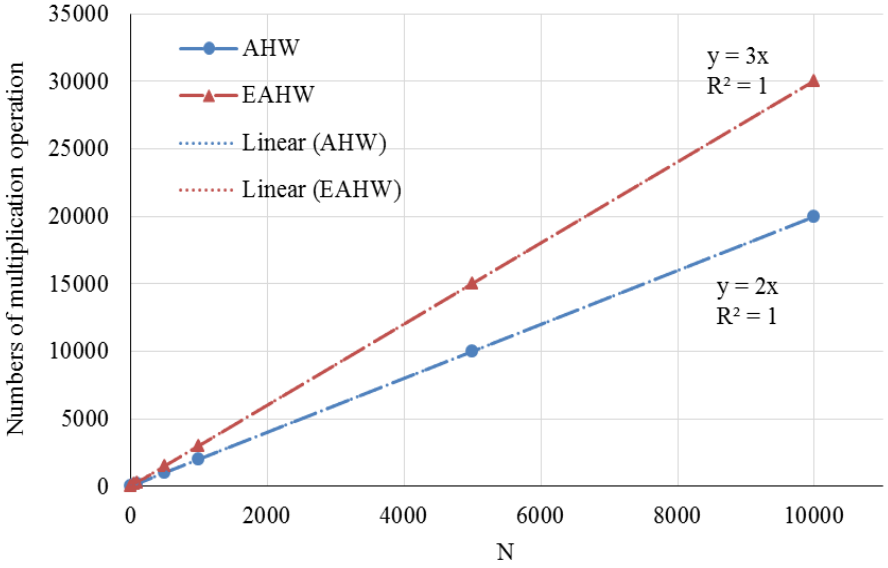

Figure 1 Illustration of the number of multiplication operations required by the AHW and the EAHW when $\delta \neq \alpha$

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