

## A Fractional Four-Step Finite Element Method for Analysis of Conjugate Heat Transfer Problems

Atipong Malatip<sup>1\*</sup> and Sutthisak Phongthanapanich<sup>2</sup>

### บทคัดย่อ

การประยุกต์ใช้วิธีการคำนวณแบบแยกส่วนสี่ขั้นตอนกับระเบียบวิธีไฟไนต์เอลิเมนต์สำหรับปัญหาการถ่ายเทความร้อนแบบคอนจูเกตระหว่างของแข็งและของไหลแบบหนืดที่สภาวะไม่คงตัวได้ถูกนำเสนอในบทความนี้ การอินทิเกรตที่เกี่ยวข้องกับเวลากับวิธีแครง-นิโคลสัน ซึ่งจะทำให้สมการในรูปแบบที่ไม่เชิงเส้นอยู่ในรูปแบบเชิงเส้นได้โดยยังคงให้ค่าความถูกต้องในแต่ละช่วงเวลา สำหรับฟังก์ชันถ่วงน้ำหนักจะใช้วิธีสตรีมไลน์อัปวินด์ เพทروف-กาเลอร์คินประยุกต์เข้ากับสมการนาเวียร์-สโตกส์ วิธีดังกล่าวใช้เอลิเมนต์สามเหลี่ยมแบบสามจุดต่อกับใช้ฟังก์ชันการประมาณภายในสำหรับความเร็ว ความดันและอุณหภูมิที่อันดับเท่ากันได้ ข้อได้เปรียบของวิธีดังกล่าวคือการถ่ายเทความร้อนบริเวณผิวรอยต่อระหว่างของแข็งและของไหลมีความสอดคล้องกัน ปัญหาที่ใช้ในการตรวจสอบความถูกต้องของวิธีดังกล่าวประกอบด้วย 2 ปัญหาคือปัญหาการพาความร้อนแบบอิสระในช่วงปิดสี่เหลี่ยมจตุรัสและปัญหาพาความร้อนแบบอิสระในช่องปิดสี่เหลี่ยมโดยที่มีผนังกันความร้อน

**คำสำคัญ:** ระเบียบวิธีไฟไนต์เอลิเมนต์ การคำนวณแบบแยกส่วนสี่ขั้นตอน การถ่ายเทความร้อนแบบคอนจูเกต

### Abstract

A fractional four-step finite element method for analyzing conjugate heat transfer between solid and unsteady viscous flow is presented. The second-order semi-implicit Crank-Nicolson scheme is used equations are linearized without losing the overall time accuracy. The streamline upwind Petrov-Galerkin method (SUPG) is applied for the weighted formulation of the Navier-Stokes equations. The method uses a three-node triangularelement with equal-order interpolation functions for all the variables of the velocity components, the pressure and the temperature. The main advantage of the method presented is to consistently couple heat transfer along the fluid-solid interface. Two test cases, which are natural convection in a square cavity and conjugate natural convection in a square cavity with a conducting wall, are selected to evaluate the efficiency of the method presented.

**Keywords:** Finite Element Method, Fractional Four-Step, Conjugate Heat Transfer

<sup>1</sup> Researcher, National Metal and Materials Technology (MTEC)

<sup>2</sup> Lecturer, Mechanical Engineering Technology Department, College of Industrial Technology, King Mongkut's University of Technology North Bangkok

\* Corresponding Author, Tel. 0-2564-6500 Ext. 4581, E-mail: atipongm@mtec.or.th

## 1. Introduction

Conjugate heat transfer between solid and fluid flow, where heat conduction in a solid region is closely coupled with heat convection in an adjacent fluid, is encountered in many practical applications. There are many engineering problems where conjugate heat transfer should be considered, such as in biomedical engineering, design of air-cooled packaging, heat transfer enhancement by finned surfaces, design of thermal insulation, design of solar equipment, heat transfer in a cavity with thermally conducting wall or internal baffle, etc. Most of the research works in this area employ the finite difference method, the finite volume method, the finite element method, and the meshless collocation method as the numerical tools. Convection heat transfer between the solid and the fluid flow is one of the most challenging problems for computational methods due to its inherent coupling between the governing equations of the fluid motion and the energy equation of the solid. This coupling effect can be seen noticeably at high Rayleigh numbers in free convection problems and at high Reynolds numbers in forced convection problems. Another main reason which increases the for time integration and the resulting nonlinear difficulty in solving the convection heat transfer problems is due to the non-linear phenomenon of the convection terms presented in both the momentum equations and the energy equation. Some algorithms have been proposed and applied to analyze these problems, such as the velocity-pressure segregated method [1]-[3] based on the SIMPLE algorithm and the unsteady algorithm based on the fractional step method [4]-[8]. These two algorithms are similar in that they correct the computed velocity components by using the pressure derived from the continuity equation.

Some of the studies in this research area, however, employ the finite difference and the finite volume methods as the numerical tools. He, et al. [9] studied the conjugate problem using an iterative FDM/BEM method for analysis of parallel plate channel with constant outside temperature. Sugavanam, et al. [10] investigated the conjugate heat transfer from a flush heat source on a conductive board in laminar channel flow. Chen and Han [11] showed the solution of a conjugate heat transfer problem using a finite difference SIMPLE-like algorithm. Schäfer and Teschauer [12] used the finite volume method to analyze both the fluid flow behavior and the solid heat transfer together. Aydin [13] studied a conjugate heat transfer phenomenon through a double pane window by using the finite difference technique. Results from these problems showed that both the finite difference and the finite volume methods can perform very well on the problems of interest, but some assumptions on heat transfer coefficients have to be made in order to compute the temperatures along the fluid-solid interface. Furthermore, the unknown temperature and the heat flux at the fluid-solid interface are normally determined in an iterative way, usually through the use of an artificial heat transfer coefficient.

At present, very few computational procedures using the finite element method have been proposed in the literature to analyze such conjugate heat transfer problems. Misra and Sarkar [14] used the standard Galerkin formulation to solve the continuity, momentum and energy equations simultaneously. Malatip, et al. [15] developed a combined SUPG and segregate finite element method for analyzing steady conjugate heat transfer

problems. Al-Amiri, et al. [16] used finite element method to study the steady-state natural convection in a fluid-saturated porous cavity of a conducting vertical wall.

The objective of this paper is to develop a second-order time accurate numerical algorithm for analyzing conjugate heat transfer between solid and unsteady viscous thermal flow. The paper extends the splitting finite element algorithm proposed by Choi, et al. [5] to conjugate heat transfer problem [15]. Triangular finite element is employed herein for deriving the associated finite element equations. These triangular finite elements are used together with an adaptive meshing technique to improve the solution accuracy and computational efficiency. The finite element algorithm employs the four-step fractional method with an equal-order triangular finite element. The idea of the consistent SUPG [17], [18] is included in the formulation as an upwind scheme. The time integration method is based on a semi-implicit fractional step method and the resulting nonlinear momentum and energy equations are linearized without losing the overall time accuracy.

The paper starts from describing the set of the partial differential equations that satisfy the law of conservation of mass, momentums and energy. Corresponding finite element equations are derived and the element matrices are presented. The computational procedure used in the development of the computer program is then briefly described. Finally, the finite element formulation and the computer program are then verified by solving several examples that have benchmark solutions and numerical solutions obtained from other algorithms.

## 2. Theoretical Formulation and Solution Procedure

### 2.1 Searching an Element in the Previous Mesh for a New Nodal Point

The governing equations for the conjugate heat transfer between the solid and fluid flow are presented briefly in this section. For unsteady incompressible viscous thermal flow where the physical properties of the fluid and solid are independent of the temperature, the governing equations for flow and heat transfer in the solid can be written as follows,

Continuity equation,

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (1a)$$

Momentum equations,

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_i u_j) = -\frac{1}{\rho_f} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \nu \frac{\partial u_i}{\partial x_j} \right) - g_i^n (1 - \beta(T^n - T_0)) \quad (1b)$$

Energy equation for fluid,

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j} (u_j T) = \frac{\partial}{\partial x_j} \left( \alpha_f \frac{\partial T}{\partial x_j} \right) + \tilde{Q} \quad (1c)$$

Energy equation for solid,

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x_j} \left( \alpha_s \frac{\partial T}{\partial x_j} \right) + \tilde{Q} \quad (1d)$$

where  $i, j$  is 1, 2,  $x_j$  are the Cartesian coordinates,  $u_i$  are the corresponding velocity components,  $t$  is time,  $\rho$  is the density,  $p$  is the pressure,  $\nu$  is the kinematic viscosity,  $g_i$  is the gravitational acceleration constant in the  $x$  and  $y$  direction, respectively,  $\beta$  is the volumetric coefficient of thermal expansion,  $T$  is

the temperature,  $\alpha T_0$  is the reference temperature,  $\alpha$  is thermal diffusivity,  $\tilde{Q}$  is the internal heat generation rate per unit volume and subscripts  $s$  and  $f$  are solid and fluid.

The governing differential equations above are to be solved together with the interface conditions. These include the non-slip condition on the solid wall, while the temperature and heat flux along the fluid/solid interface must be continuous,

$$u_{f, \text{int}} = u_{s, \text{int}} \quad (2a)$$

$$T_{f, \text{int}} = T_{s, \text{int}} \quad (2b)$$

$$\alpha_f \left. \frac{\partial T}{\partial n} \right|_{f, \text{int}} = \alpha_s \left. \frac{\partial T}{\partial n} \right|_{s, \text{int}} \quad (2c)$$

where  $n$  denotes the normal direction of the interface

## 2.2 Fractional Four-step Method

The governing differential equations are integrated in time using the fully implicit four-step fractional method [5]. The pressure gradient terms are first decoupled from those of the convection, diffusion and the external force terms. The second-order fully implicit time-advancement scheme of Crank-Nicolson is applied for both the convective and the viscous terms in Eqs. (1b-c). The pressure is then obtained from the continuity equation and the velocity is corrected by the pressure, as follows,

Step1,

$$\frac{\hat{u}_i - u_i^n}{\Delta t} + \frac{1}{2} \frac{\partial}{\partial x_j} (\hat{u}_i \hat{u}_j + u_i^n u_j^n) = -\frac{1}{\rho} \frac{\partial p^n}{\partial x_i} + \frac{1}{2} \nu \frac{\partial}{\partial x_j} \left( \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial u_i^n}{\partial x_j} \right) + g_i^n (1 - \beta (T^n - T_0)) \quad (3a)$$

$$\text{Step 2,} \quad \frac{u_i^* - \hat{u}_i}{\Delta t} = \frac{1}{\rho} \frac{\partial p^n}{\partial x_i} \quad (3b)$$

$$\text{Step 3,} \quad \frac{\partial}{\partial x_i} \frac{\partial p^{n+1}}{\partial x_i} = \frac{\rho}{\Delta t} \frac{\partial u_i^*}{\partial x_i} \quad (3c)$$

$$\text{Step 4,} \quad \frac{u_i^{n+1} - u_i^*}{\Delta t} = -\frac{1}{\rho} \frac{\partial p^{n+1}}{\partial x_i} \quad (3d)$$

$$\text{Step 5,} \quad \frac{T^{n+1} - T^n}{\Delta t} + \frac{1}{2} \frac{\partial}{\partial x_j} (T^{n+1} u_j^{n+1} + T^n u_j^n) = \frac{1}{2} \tilde{k} \frac{\partial}{\partial x_j} \left( \frac{\partial T^{n+1}}{\partial x_j} + \frac{\partial T^n}{\partial x_j} \right) + \tilde{Q}^n \quad (3e)$$

where  $\Delta t$  is the time increment,  $\hat{u}_i$  and  $u_i^*$  are the intermediate velocities,  $\tilde{k}$  is the thermal conductivity, and superscript  $n$  denotes the time level. The time increment of the fully implicit method is restricted to achieve a desired solution accuracy, not by the numerical stability.

## 2.3 Finite Element for Mulutions

### 2.3.1 Streamline Upwind Petrov-Galerkin Method

In the streamline upwind Petrov-Galerkin method, a modified weighting function,  $W_\alpha$ , is applied to the convection terms for suppressing the non-physical spatial oscillation that may occur in the numerical solution. The weighting function is given by,

$$W_\alpha = N_\alpha + \frac{\Delta t_e}{2} u_j \frac{\partial N_\alpha}{\partial x_j} \quad (4)$$

when

$$\Delta t_e = \frac{\sigma h}{|U|} \quad (4a)$$

$$\sigma = \coth \text{Pe} - \frac{1}{\text{Pe}} \quad (4b)$$

$$\text{Pe} = \frac{|U|h}{2\nu} \quad \text{and} \quad |U| = \sqrt{u^2 + v^2} \quad (4c)$$

where  $Pe$  is the Peclet numbers and  $h$  is the minimum element size.

### 2.3.2 Discretization of the Momentum and Energy Equations

The three-node triangular element is used in this study. The element assumes linear interpolation functions for the velocity components, the pressure, and the temperature as:

$$\phi(x, y) = \sum_{\alpha=1}^3 N_{\alpha}(x, y) \phi_{\alpha} \quad (5)$$

where  $\phi$  denotes the transport property ( $u$ ,  $v$ ,  $p$  and  $T$ ) and  $N_{\alpha}$  are the element interpolation functions. The method of weighted residuals with the streamline upwind Petrov-Galerkin method is employed to discretize the finite element equations by multiplying Eqs. (4) by the weighting function. Integration by parts is then performed using the Gauss theorem to yield the element equations in the form,

$$\begin{aligned} \text{Step 1,} \quad & \left( \frac{[M]}{\Delta t} + \frac{1}{2}([C] + [K_m]) \right) \{ \hat{u}_i \} = \\ & \left( \frac{[M]}{\Delta t} - \frac{1}{2}([C] + [K_m]) \right) \{ u \} - [G_i] \{ p \}^n + \\ & \{ R_{gi} \}^n + \{ R_{bi} \}^n \end{aligned} \quad (6a)$$

Step 2,

$$[M] \{ u_i \}^* = [M] \{ \hat{u}_i \} + \Delta t [G_i] \{ p \}^n \quad (6b)$$

Step 3,

$$[K_p] \{ p \}^{n+1} = \{ R_u \}^* + \{ R_v \}^* + \{ R_b \}^*$$

Step 4,

$$[M] \{ u_i \}^{n+1} = [M] \{ u_i \}^* - \Delta t [G_i] \{ p \}^{n+1} \quad (6d)$$

Step 5,

$$\begin{aligned} & \left( \frac{[M]}{\Delta t} + \frac{1}{2}([C] + [K_T]) \right) \{ T \}^{n+1} = \\ & \left( \frac{[M]}{\Delta t} - \frac{1}{2}([C] + [K_T]) \right) \{ T \}^n + \{ R_c \}^n + \\ & \{ R_q \}^n + \{ R_Q \}^n \end{aligned} \quad (6e)$$

In the above equations, the element matrices written in the integral form are,

$$[M] = \int_{\Omega} \{ N \} [N] d\Omega \quad (7a)$$

$$[C] = \int_{\Omega} \{ W \} \left( u_j \left[ \frac{\partial N}{\partial x_j} \right] \right) d\Omega \quad (7b)$$

$$[K_m] = \nu \int_{\Omega} \left\{ \frac{\partial W}{\partial x_j} \right\} \left[ \frac{\partial N}{\partial x_j} \right] d\Omega \quad (7c)$$

$$[K_T] = \tilde{k} \int_{\Omega} \left\{ \frac{\partial W}{\partial x_j} \right\} \left[ \frac{\partial N}{\partial x_j} \right] d\Omega \quad (7d)$$

$$[G_i] = \frac{1}{\rho} \int_{\Omega} \{ W \} \left[ \frac{\partial N}{\partial x_i} \right] d\Omega \quad (7e)$$

$$[K_p] = \int_{\Omega} \left\{ \frac{\partial W}{\partial x_j} \right\} \left[ \frac{\partial N}{\partial x_j} \right] d\Omega \quad (7f)$$

$$\{ R_g \} = g_i \int_{\Omega} \{ W \} (\beta T - (1 + \beta T_0)) d\Omega \quad (7g)$$

$$\{ R_u \} = \frac{\rho}{\Delta t} \int_{\Omega} \left\{ \frac{\partial W}{\partial x} \right\} [N] \{ u \} d\Omega \quad (7h)$$

$$\{ R_v \} = \frac{\rho}{\Delta t} \int_{\Omega} \left\{ \frac{\partial W}{\partial y} \right\} [N] \{ v \} d\Omega \quad (7i)$$

$$\{R_v\} = \frac{\rho}{\Delta t} \int_{\Omega} \left\{ \frac{\partial W}{\partial y} \right\} [N] \{v\} d\Omega \quad (7j)$$

$$\{R_b\} = -\frac{\rho}{\Delta t} \int_{\Gamma} \{W\} (u_j \hat{n}_k) d\Gamma \quad (7k)$$

$$\{R_c\} = \tilde{k} \int_{\Gamma} \{W\} \left( \frac{\partial T}{\partial x_j} \hat{n}_k \right) d\Gamma \quad (7l)$$

$$\{R_q\} = \frac{1}{\rho c} \int_{\Gamma} \{W\} q_s d\Gamma \quad (7m)$$

$$\{R_Q\} = \frac{1}{\rho c} \int_{\Omega} \{W\} Q d\Omega \quad (7n)$$

where  $\Omega$  is the element area and  $\Gamma$  is the element boundary. The local time step is assumed as the minimum between the convective local time step and diffusive local time step,

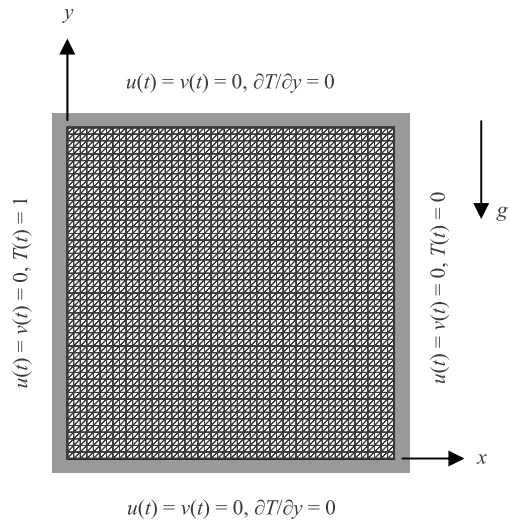
$$\Delta t = \min(\Delta t_a, \Delta t_b) \quad (8)$$

$$\Delta t_a = \frac{h}{|U|}, \quad \Delta t_b = \frac{h^2}{2k} \quad (9)$$

where  $Re$  is the Reynolds number.

### 3. Examples

In this section, two examples are presented. The first example, natural convection in a square cavity, is used to illustrate the efficiency of the scheme presented for the analysis of transient viscous thermal flow. The last example, conjugate natural convection in a square cavity with a conducting wall, is used to illustrate the efficiency of the scheme presented for the analysis of the conjugate heat transfer problem.

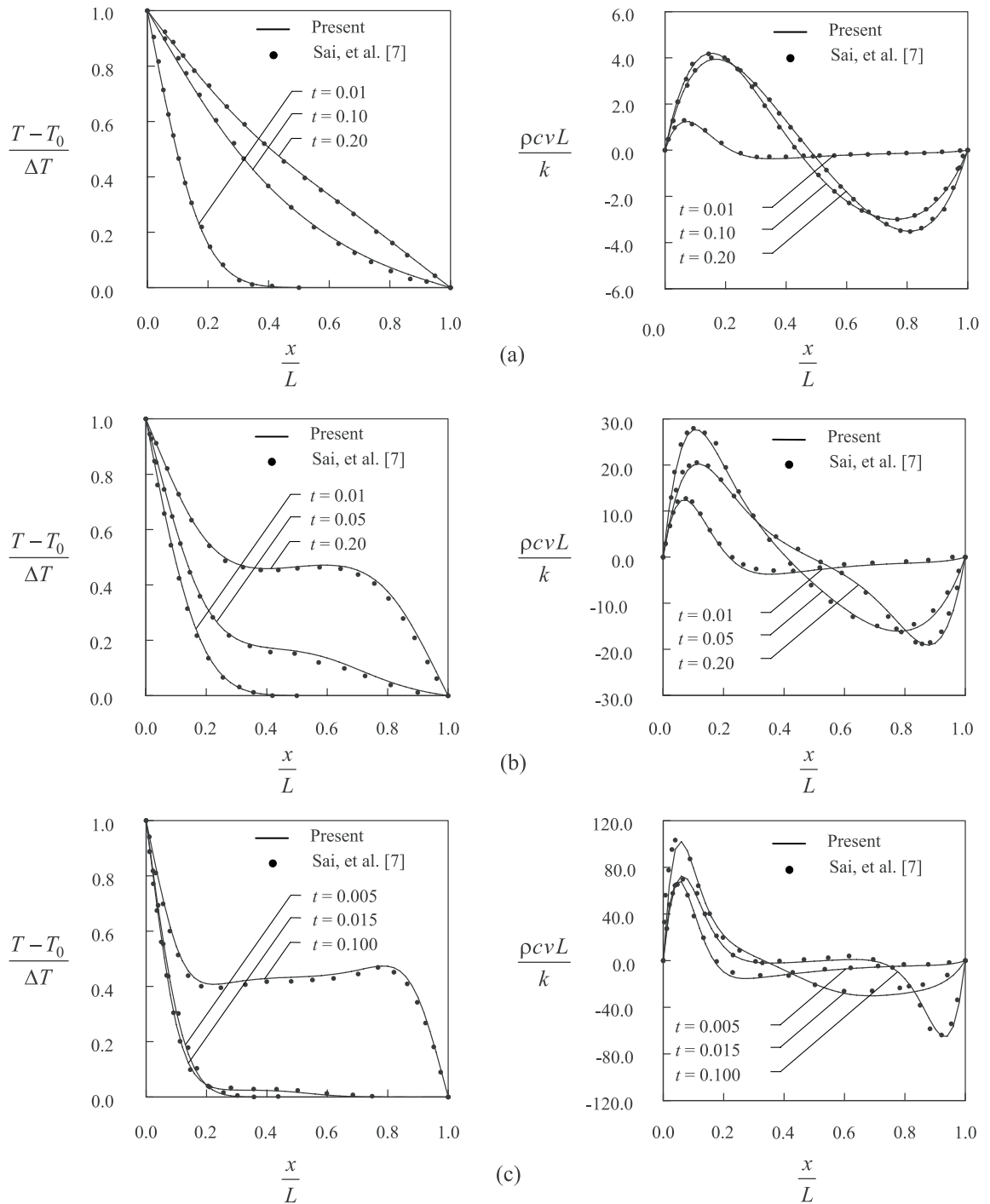


**Figure 1** Problem statement and finite element model of the natural convection in a square cavity problem.

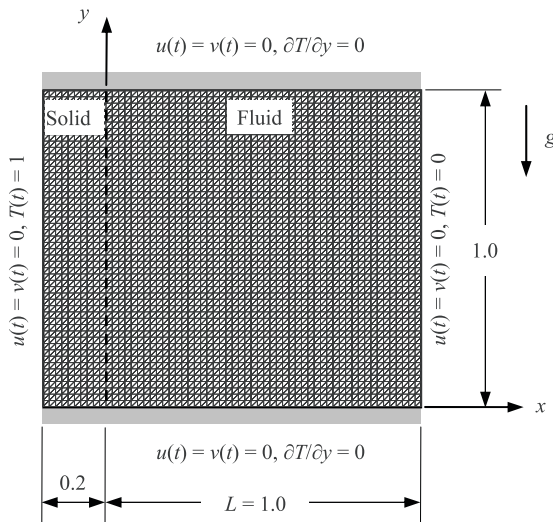
### 3.1 The Natural Convection in a Square Cavity

The first example for evaluating the finite element formulation and validating the developed computer program is the problem of free convection in a square enclosure. The square enclosure as shown in Fig. 1, is bounded by the two vertical walls with specified temperatures of one along the left side and zero along the right side, all other boundaries are insulated. The finite element model, consisting of 2,601 nodes and 5,000 elements, is also shown in the figure.

Fig. 2(a)-(c) shows the predicted temperature and vertical velocity component distributions at the cavity mid-plane ( $y = 0.5$ ) that are compared with the results from Sai, et al. [7]. The figures present the comparisons of the transient solutions for the three cases of Rayleigh number,  $Ra = 10^3$ ,  $10^4$ , and  $10^5$ . These figures highlight good agreement of the predicted solutions and the solutions from Ref. [7].



**Figure 2** Predicted temperature and vertical velocity component distributions of  $x$ -direction on  $y = 0.5$  at (a)  $Ra = 10^3$ , (b)  $Ra = 10^4$ , and (c)  $Ra = 10^5$ .



**Figure 3** Problem statement and finite element model of the conjugate natural convection problem.

Table 1 compares the average Nusselt numbers at the hot wall,  $\overline{Nu}_{x=0}$ , obtained from the presented method and the results from the literatures [6], [7], [19], [20]. The table shows that the solutions from the method presented compare very well with the results from Ref. [19].

### 3.2 Conjugate Natural Convection in a Square Cavity with a Conducting Wall

To further evaluate the efficiency of the

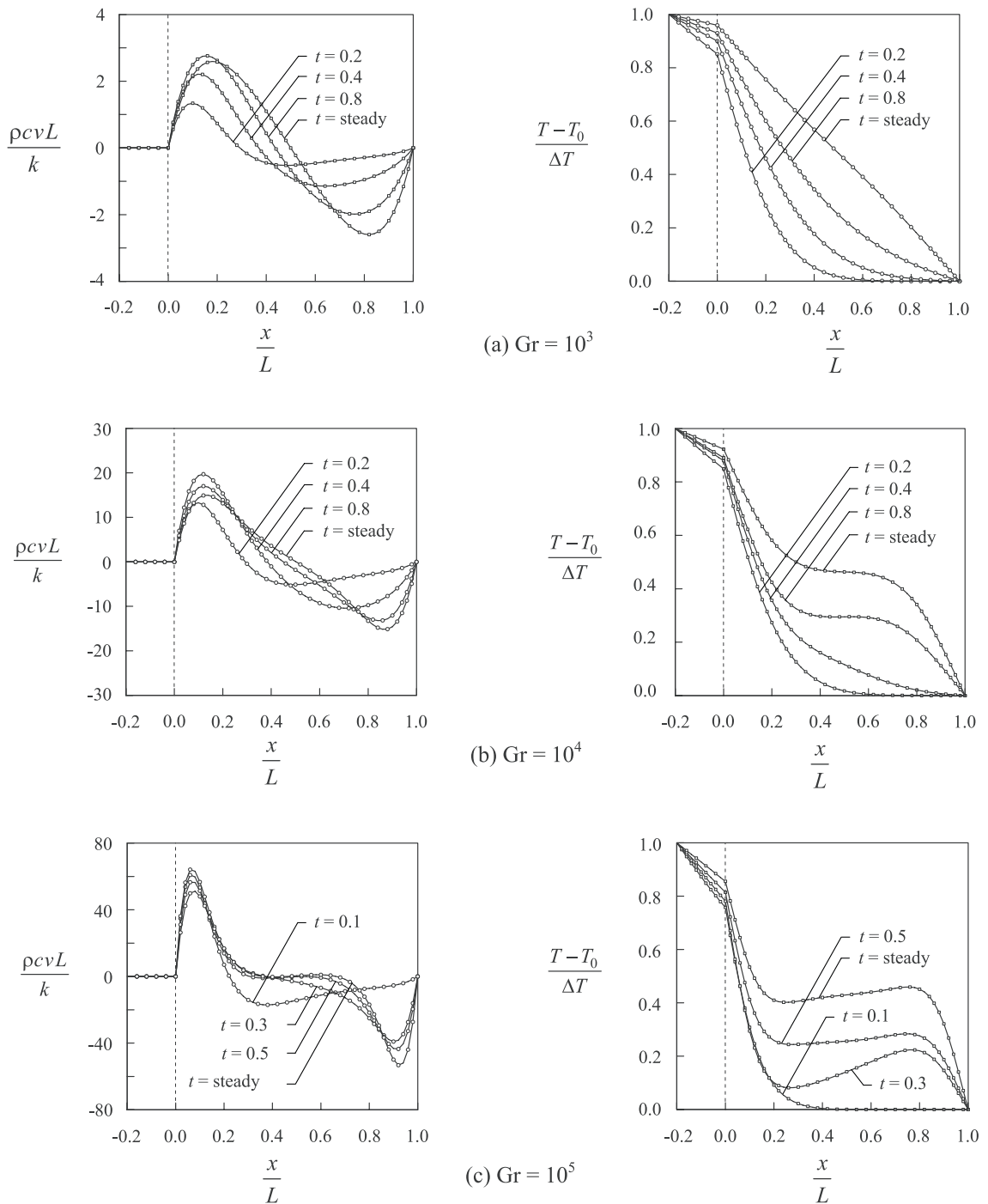
schemes presented, the problem of conjugate natural convection in a square cavity with a conducting wall as shown in Fig. 3 is selected. The fluid in the cavity is heated from the higher temperature solid wall along the left side and maintained at zero temperature along the right side, all other boundaries are insulated. The finite element model for both the solid wall and fluid region consisting of 3,111 nodes and 6,000 elements is also shown in the figure. Fig. 4 shows the predicted vertical velocity component and the temperature contours at the cavity mid-plane ( $y = 0.5$ ) that vary in times for Grashof numbers,  $Gr$ , of  $10^3$ ,  $10^4$  and  $10^5$ , respectively, all at  $Pr = 0.71$ , solid-to-fluid thermal diffusivity ratio,  $\alpha_{sf}$ , and the thermal conductivity ratio between solid and fluid,  $K = k_s/k_f = 5$ . The temperature and the heat flux distributions along the solid-fluid interface with the variation of conduction ratio,  $K$ , are shown in Figs. 5 and 6 for Grashof numbers of  $10^5$  and  $10^7$ , respectively.

In addition, Table 2 compares the predicted average Nusselt numbers along the interface,  $\overline{Nu}_{x=0}$ , with the results using the boundary-domainintegral method by Hriberšek [21]. The table shows good agreement of the average Nusselt numbers for both the temperature and the heat flux.

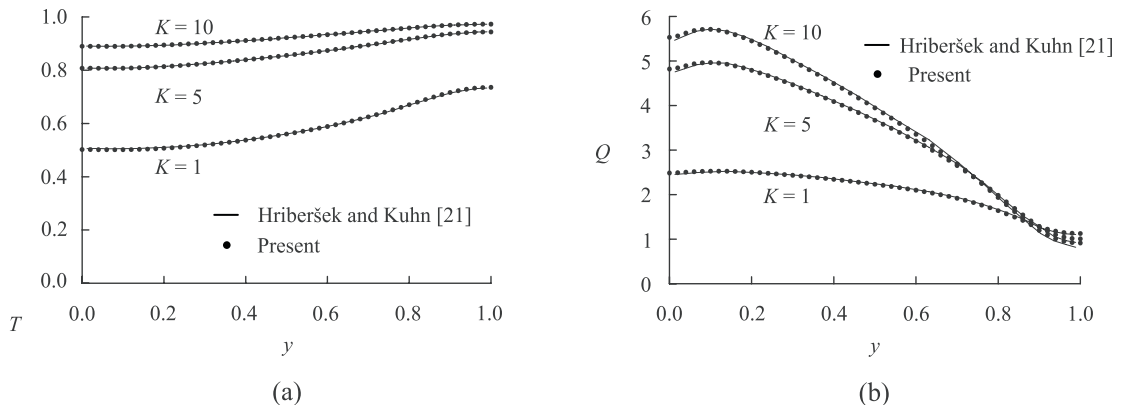
**Table 1** Variation of the overall Nusselt numbers

Average Nusselt number along hot wall (% difference from Ref. [19])			
Ra	$10^3$	$10^4$	$10^5$
de Vahl Davis [19]	1.117	2.238	4.509
Choi, et al. [6]	1.143 (2.33%)	2.264 (1.16%)	4.530 (0.47%)
Sai, et al. [7]	1.131 (1.25%)	2.289 (2.28%)	4.687 (3.95%)
Leal, et al. [20]	1.118 (0.09%)	2.248 (0.44%)	4.562 (1.18%)
Present	1.117 (0.00%)	2.234 (0.18%)	4.466 (0.95%)





**Figure 4** Predicted vertical velocity component and temperature distributions of x-direction on  $y = 0.5$  at (a)  $Gr = 10^3$ , (b)  $Gr = 10^4$ , and (c)  $Gr = 10^5$ , all at  $K = 5$ .



**Figure 5** (a) Interface temperatures and (b) Interface heat fluxes, all at  $Gr = 10^5$ .

**Table 2** Variation of the overall Nusselt numbers

Gr	Average Nusselt number along interface (% difference from Ref. [21])			
	Conductivity ratio, $K = k_s/k_f$	1	5	10
$10^3$	Hriberšek and Kuhn [28]	0.87	1.02	1.04
$10^3$	Present	0.87 (0.0%)	1.02 (0.0%)	1.04 (0.0%)
$10^4$	Hriberšek and Kuhn [28]	1.35	1.83	1.92
$10^4$	Present	1.35 (0.0%)	1.83 (0.0%)	1.91 (0.52%)
$10^5$	Hriberšek and Kuhn [28]	2.08	3.42	3.72
$10^5$	Present	2.08 (0.0%)	3.40 (0.58%)	3.70 (0.54%)
$10^6$	Hriberšek and Kuhn [28]	2.87	5.88	6.78
$10^6$	Present	2.83 (1.39%)	5.80 (1.36%)	6.69 (1.33%)
$10^7$	Hriberšek and Kuhn [28]	3.53	9.07	11.25
$10^7$	Present	3.45 (2.27%)	8.73 (3.75%)	10.88 (3.29%)

#### 4. Conclusion

A combined fractional four-step finite element method and streamline upwind Petrov-Galerkin method (SUPG), for analysis of conjugate heat transfer between solid and unsteady viscous thermal flow, was presented. The method combines a viscous thermal flow analysis in the fluid region and a heat

transfer analysis in the solid region together. The Navier-Stokes equations are solved by the streamline upwind Petrov-Galerkin method in order to suppress the non-physical spatial oscillation in the numerical solutions. All the finite element equations were derived and presented in detail. The efficiency of the

coupled finite element method has been evaluated by several examples that were previously analyzed by using other methods. These examples highlight the benefit of the combined finite element method that can simultaneously model and solve both the fluid and solid regions, as well as to compute the temperatures along the fluid-solid interface directly.

## 5. Acknowledgment

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