A Case Study of Feedback Controller and Estimator Designs for Nonlinear Systems via the State-dependent Riccati Equation Technique

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บทความนี้ได้นำเสนอกรณีศึกษาของการออกแบบตัวควบคุมแบบป้อนกลับและตัวประมาณค่าที่ไม่เป็นเชิงเส้นโดยใช้วิธีสมการริกกาตีสถานะไม่อิสระ โดยกล่าวถึงเสถียรภาพและคุณสมบัติของความเหมาะที่สุดของตัวควบคุมที่ถูกออกแบบโดยวิธีสมการริกกาตีสถานะไม่อิสระ ผู้เขียนบทความยังได้กล่าวถึงความยากและข้อจำกัดของการใช้งานตัวควบคุมชนิดนี้และได้แนะนำวิธีการที่อาจจะนำไปใช้หาค่าฟังก์ชันเมทริกซ์ในการออกแบบตัวควบคุมชนิดนี้ นอกจากนี้ยังได้ศึกษาการประยุกต์วิธีสมการริกกาตีสถานะไม่อิสระ เพื่อออกแบบตัวประมาณค่าที่ไม่เป็นเชิงเส้นซึ่งสามารถยืนยันได้ว่าค่าผิดผลาดของการประมาณจะลู่เข้าสู่ค่าศูนย์ ในท้ายสุดผู้เขียนบทความได้นำเสนอตัวอย่างเพื่อใช้อธิบายการออกแบบควบคุมแบบป้อนกลับและตัวสังเกตโดยใช้วิธีสมการริกกาตีสถานะไม่อิสระ

คำสำคัญ: วิธีสมการริกกาตีสถานะไม่อิสระ ตัวควบคุมแบบป้อนกลับที่ไม่เป็นเชิงเส้น ตัวประมาณค่าที่ไม่เป็นเชิงเส้น

Abstract

This paper presents a case study of the design procedure of state-feedback controllers and estimators for nonlinear systems using the state-dependent Riccati equation (SDRE) method. The stability and the optimal property of the controller designed by using the SDRE control laws are discussed. Difficulty and limitation of using this kind of controller are mentioned. Approaches for calculating matrix-valued function in the design procedure of this kind of controller are introduced. In addition, application of state-dependent Riccati equation to design of nonlinear estimator that ensures the local convergence of the state estimation error is investigated. Eventually, explanation and some examples of feedback controller and estimator design basing on SDRE method are given.

Keyword: State-dependent Riccati Equation Method, Nonlinear Feedback Controller, Nonlinear Estimator

1. Introduction

The state-dependent Riccati equation (SDRE) approach is well-known and has become very popular in the control research community during the last few decades. The SDRE method was introduced as a general method for the regulatory control of nonlinear systems [1]. The SDRE strategy provides a very effective procedure for synthesizing nonlinear feedback controls by permitting nonlinearities in the system states while additionally giving great design

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flexibility through the weighting matrices. However, a potential drawback of the SDRE method is that the resulting control is not necessarily optimal with respect to the performance index but instead is usually suboptimal. Thus, this method is suitable to use for suboptimal controller designs since it has explicit formulae for the design procedure. The SDRE method requires the factorization of the nonlinear dynamics into the state vector and the product of a matrix-valued function depending on the state itself. This matrix-valued function plays an important role for the stability of the controlled system under the action of SDRE controller.

The procedures of generating the SDRE controller were presented in Banks and Mhana [1], and Xu [2]. Shamma and Cloutier [3] studied the nonuniqueness of state-dependent representations. The novel application of the S-procedure was used to prove an existence of SDRE stabilizing feedback controllers. In Erdem and Alleyne [4] the infinite-horizon nonlinear regulation of second-order systems using SDRE method is considered. The SDRE control law obtained yields the global asymptotic stability of the closed-loop systems. Other applications of the SDRE techniques for implementations of optimal control can be founded in [5] and [6]. In this paper, the extension of the SDRE approach to the nonlinear state estimation problem is reviewed. The nonlinear estimator (observer) that ensures the local convergence of the state estimation error is addressed. We give several approaches that may be used to find the matrix-valued function and indicate the difficulties of employing the SDRE approach. Further, a case study of the designs of SDRE controller and estimator is proposed.

2. Problem Formation

Consider an input-affine nonlinear system of the form

$$\dot{x} = f(x) + G(x)u, \quad x(0) = x_o, \tag{1}$$

that minimizes the performance index

$$J = \int_0^\infty \left(x^T Q(x)x + u^T R(x)u \right) dt, \tag{2}$$

where $x \in \Re^n$, $u \in \Re^m$, $Q(x) \in \Re^{n\times n}$ is positive symmetric semi-definite, $R(x) \in \Re^{m\times m}$ is symmetric positive definite, *R(x)*, *f(x)* and *G(x)* are all s ufficiently smooth functions of the state vector *x(t)* , and *x*(0) is the initial condition of the process. It is assumed that *f*(0)=0 and *G(x)* and $\neq 0$ for all *x*.

The aim is to determine control signals to solve the system (1) and minimize the performance index (2). Next the procedure to derive the Hamiltonian-Jacobi-Bellman (HJB) equation presented in [1], [2] is briefly restated.

For the given infinite-time optimal control problem (1) and (2), the Hamiltonian function is

$$H(x,u,J_x) = (x^T Q(x)x + u^T R(x)u) + J_x^T (f(x) + G(x)u), \tag{3}$$

where J_x is the derivative of *J* with respect to *x*. Letting $\lambda = J_x$ the necessary conditions for optimality are

$$\dot{x} = \frac{\partial H}{\partial \lambda}, \quad \dot{\lambda} = -\frac{\partial H}{\partial x}, \quad 0 = -\frac{\partial H}{\partial u}. \tag{4}$$

Solving (4) yields the optimal control input

$$u^* = -\frac{1}{2} R^{-1}(x) G^T(x) \frac{\partial J}{\partial x}. \tag{5}$$

It should be noted that $\frac{\partial J}{\partial t}=0$ for the infinite-time optimal control problem. Thus, the HJB equation for the above problem becomes

$$x^T Q(x)x + u^T R(x)u + J_x^T (f(x) + G(x)u) = 0. \quad (6)$$

Substituting the optimal control u^* into (6) the HJB equation becomes

$$\begin{aligned} &x^T Q(x)x + J_x^T f(x) \\ &\quad -\frac{1}{4} J_x^T G(x) R^{-1}(x) G^T(x) J_x = 0, \end{aligned} \quad (7)$$

where $J(x)$ represents the value function of the infinite-time optimal control problem. Solving the HJB equation (7) leads to the suboptimal control law u^*.

3. State-Dependent Coefficient Parameterization

To avoid solving (7) directly, the function $f(x)$ is required to be decomposed as

$$f(x) = A(x)x, \quad (8)$$

where $A(x) \in R^{n\times n}$. Thus, an important step in the SDRE method is to find a good choice of the matrix $A(x)$ and the method to complete this task is called state-dependent coefficient (SDC) parameterization [9]. This concept has alternatively been called apparent linearization [10] or extended linearization [7], [11]. Substituting (8) into the system (1), yields

$$\dot{x} = A(x)x + G(x)u, \qquad x(0) = x_0, \quad (9)$$

which has a linear structure with the state dependent coefficient (SDC) matrices $A(x)$, $G(x)$.

Basically, the SDRE approach puts nonlinear systems into a state-dependent coefficient linear structure. Although the system remains nonlinear, linear methods such as the optimal linear regulator can be used for control.

Although theory exists on whether a parameterization will work, it does not assist in finding a parameterization. Thus, in this section we introduce several approaches that may be used to find a good choice of the matrix $A(x)$. One such approach uses the basic concepts of Lie algebra in [12]. In this technique the system (1) can be written as the form (9) where

$$A(x) = \frac{1}{n}\begin{bmatrix} \frac{f_1(x)}{x_1} & \frac{f_1(x)}{x_2} & \cdots & \frac{f_1(x)}{x_n} \\ \frac{f_2(x)}{x_1} & \frac{f_2(x)}{x_2} & \cdots & \frac{f_2(x)}{x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{f_n(x)}{x_1} & \frac{f_n(x)}{x_2} & \cdots & \frac{f_n(x)}{x_n} \end{bmatrix} \quad (10)$$

Once a good choice of the matrix $A(x)$ has been found (this is usually done off-line), the SDRE method can be applied to obtain a stabilizing control. On the other hand a basic form of partial differentiation under the integral sign presented in [13] may be used to find a good choice of the matrix $A(x)$. In this technique the explicit formula to find a parameterization is

$$A(x) = \int_0^1 \left(\frac{\partial}{\partial x} f(\lambda x)\right) d\lambda, \quad (11)$$

which is guaranteed to exist if $f(0)=0$ and $f(x)$ in C^1.

4. SDRE Controller Structure

Similar to the linear quadratic regulator (LQR) design, the SDRE controller can be designed by mimicking the LQR formulation for the input-affine system (9) which the performance index (2). The state feedback controller is obtained in the form

$$u(x) = -R^{-1}(x)G^T(x)P(x)x, \tag{12}$$

where *P(x)* is unique, symmetric, positive definite solution of the algebraic state-dependent Riccati equation

$$\begin{aligned} &P(x)A(x) + A^T(x)P(x) \\ &\quad - P(x)G(x)R^{-1}(x)G^T(x)P(x) + Q(x) = 0. \end{aligned} \tag{13}$$

The resulting SDRE controlled trajectory becomes the solution of the closed-loop dynamics

$$\dot{x}(t) = \left[A(x) - G(x)R^{-1}(x)G^T(x)P(x)\right]x(t), \tag{14}$$

such that the state-feedback gain

$$K(x) = R^{-1}(x)G^T(x)P(x) \tag{15}$$

minimizes (2) and under the action of controller (12) the closed-loop dynamics (SDC)

$$A_{CL}(x) = A(x) - G(x)K(x) \tag{16}$$

is pointwise Hurwitz.

The SDRE solution to the infinite-horizon autonomous nonlinear regulator problem (1) and (2) is a true generalization of the infinite-horizon time-invariant LQR problem, where all of the coefficient matrices are state dependent.

5. Stability Analyses

For SDRE gain matrix to exist that results in the closed-loop SDC matrix $A_{CL}(x)$. The following definitions presented by Cloutier, Stansbery and Sznaier [7] are required.

Definition 1 The SDC representation (8) is a stabilizable (controllable) parameterization of nonlinear system (1) in a region $\Omega \in \Re^n$ if the pair $\{A(x), G(x)\}$ $\left\{A(x), Q^{\frac{1}{2}}(x)\right\}$ is pointwise stabilizable (controllable) in the linear sense for all $x \in \Omega$.

Definition 2 The SDC representation (8) is a detectable (observable) parameterization of nonlinear system (1) in a region $\Omega \in \Re^n$ if the pair $\left\{A(x), Q^{\frac{1}{2}}(x)\right\}$ is pointwise detectable (observable) in the linear sense for all $x \in \Omega$.

Definition 3 The SDC representation (8) is pointwise Hurwitz in a region Ω if the eigenvalues of *A(x)* are in the open left haft plane $\text{Re}(s) < 0$ (that is, have negative real parts) for all $x \in \Omega$.

5.1 Local Asymptotic Stability

Associated with the existence of SDRE stabilizing feedback controls, the following assumptions and conditions by Mracek and Cloutier [8] are addressed.

Assumption 1 $A(\cdot)$, $G(\cdot)$, $Q(\cdot)$ and $R(\cdot)$ are $C^1(\Re^n)$ matrix-valued function.

Assumption 2 The respective pairs $\{A(x), G(x)\}$ and $\left\{A(x), Q^{\frac{1}{2}}(x)\right\}$ are pointwise stabilizable and detectable SDC parameterization of nonlinear system (1) for all *x*.

Using assumptions above, the following conditions required for ensuring local asymptotic stability are addressed.

A sufficient test for the stabilizability condition is that the controllability matrix

$$M_C = \left[G(x) \quad A(x)G(x) \quad \ldots \quad A^{\text{n-1}}(x)G(x)\right] \tag{17}$$

has $\text{rank}(M_C) = \text{n} \quad \forall x \in \Re^n$. Similarly a sufficient test for the detectability is that the observability matrix

$$M_O = \left[Q^{\frac{1}{2}}(x) \quad Q^{\frac{1}{2}}(x)A(x) \ \ldots \ Q^{\frac{1}{2}}(x)A^{\text{n-1}}(x) \right] \quad (18)$$

has rank $(M_O) = \text{n} \quad \forall x \in \Re^n$. This can be guaranteed by ensuring that $Q(x)$ is positive definite $\forall x \in \Re^n$.

Theorem 1 (Mracek & Cloutier [8]). Consider the nonlinear system (1) with feedback control (7) applied where $x \in \Re^n$ and $P(x)$ is unique, symmetric, positive definite, pointwise stabilizabing solution of the algebraic state-dependent Riccati equation (13). Under Assumptions 1 and 2, the SDRE method produces a closed-loop solution which is locally asymptotically stable.

Proof See Mracek & Cloutier [8].

5.2 Global Asymptotic Stability

Global asymptotic stability of the closed-loop system implies that the states can be regulated to the origin regardless of the initial conditions. This is obviously a very desirable property, however, it is usually difficult to achieve. Owing to the nature of LQR formulation, the origin of the SDRE controlled system is locally asymptotically stable, that is, all eigenvalues of the closed-loop dynamics matrix (15) have negative real parts at x=0. However, this property is not sufficient to ensure the global stability of a nonlinear system. In fact, although all eigenvalues of $A_{CL}(x)$ have negative real parts $\forall x \in \Re^n$, global stability of a nonlinear system still cannot be guaranteed.

Global stability results of the SDRE-closed loop system presented in [4] were studied for two cases. For the first case, the closed-loop coefficient matrix $A_{CL}(x)$ is supposed to be symmetric. The second case examines the system (1) with n=1 and it is usually called the scalar system.

6. Nonlinear Estimations

In this section the SDRE theory is extended to state estimation. Consider a nonlinear system of the form

$$\dot{x} = F(x) = A(x)x, \quad (19)$$

where $x \in \Re^n$ and $F(x)$ is a sufficiently smooth function of the state vector x. The output is given by

$$y = C(x)x, \quad (20)$$

where $C(x) \in \Re^{p \times n}$. From [15] it can be seen that the estimator compensation is formulated as

$$\dot{x}_e = A(x_e)x_e + \Psi(x_e, (C(x)x - C(x_e)x_e)) \quad (21)$$

where $x_e \in \Re^n$ is the estimated state. The work of [15] has shown that the error is asymptotically stable at the zero equilibrium.

The proposed nonlinear estimator is constructed by mimicking estimators for linear systems. If the system is linear, it is well known that the suboptimal observer is given by [15]

$$\Psi(z, (Cx - Cz)) = L(Cx - Cz),$$

where $L = \Gamma C^T \bar{R}^{-1}$ and solves the algebraic Riccati equation for the dual system. We obtain a suboptimal observer for the nonlinear system by exploiting the dual system.

Let the error for the system be defined as $\hat{x} = x - x_e$ and v be the virtual control. To find the suboptimal observer, we consider the performance index

$$J = \int_0^\infty \left(\hat{x}^T \bar{Q} \hat{x} + v^T \bar{R}(x) v \right) dt \quad (22)$$

with associated state dynamics given by

$$\dot{\hat{x}} = A^T(\hat{x})\hat{x} + C(\hat{x})v, \tag{23}$$

where $\hat{x} \in \Re^n$, $v \in \Re^m$, $\bar{Q} \in \Re^{n\times n}$ is symmetric positive semi-definite and $R(x) \in \Re^{m\times m}$ is symmetric positive definite. We choose

$$\Psi(x_e, (C(x)x - C(x_e)x_e)) = L(x_e)(C(x)x - C(x_e)x_e),$$

where

$$L(x_e) = \frac{1}{2}\Gamma(x_e)C^T(x_e)\bar{R}^{-1} \tag{24}$$

and $\Gamma(x_e)$ solves the dual SDRE

$$\begin{aligned}\Gamma(x_e)A^T(x_e) + A(x_e)\Gamma(x_e) + \bar{Q}\\ -\Gamma(x_e)C^T(x_e)R^{-1}(x)C(x_e)\Gamma(x_e) = 0.\end{aligned} \tag{25}$$

Based on the assumptions that the pair $\{A(x), C(x)\}$ is a pointwise detectable parameterization, and *A(x)* and *C(x)* are locally Lipschitz for all $x \in \Omega$, the estimated state given by (21) will converge locally asymptotically to the state [15].

7. Numerical Examples

We present several examples which illustrate the SDRE design procedure for the control of a system and the state estimation. To solve systems of ordinary differential equations in the following examples we use Euler's method in Scilab for the numerical integration.

7.1 Example I

This example is a simple demonstration of the SDRE technique for designing a state feedback control law. Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2) + u.\end{aligned}$$

The open-loop system has unstable eigenvalues equal to 0.5 ± j0.866 at the origin. Using the formula (10) it is pointwise stabilizable for the following choices of *A(x)* and *G(x)*

$$A(x) = \begin{bmatrix} 0 & 1 \\ -1 & 1 - 3x_1^2 - 2x_2^2 \end{bmatrix} \text{ and } G(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In the simulation results we set the initial condition $\begin{bmatrix}0.5 & -0.5\end{bmatrix}^T$ and the weighting matrices *Q, R* are chosen as $Q = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}$ and $R = 1$, respectively.

We found that the controllability matrix

$$M_c = \begin{bmatrix} 0 & 1 \\ 1 & 1 - 3x_1^2 - 2x_2^2 \end{bmatrix}$$

is of full rank for all x. For this example, since *Q* is positive definite, the observability matrix M_0 is of full rank. From Theorem 1 the SDRE method generates a closed-loop solution, which is locally asymptotically stable.

Using the SDRE controller (12), the simulation results are obtained as follow:

As shown in Figure 1 the states reach zero in about 5 seconds. Figure 2 shows that the control response is on the zero line after 6 seconds. From these results we can conclude that the SDRE control law works well for the proposed example.

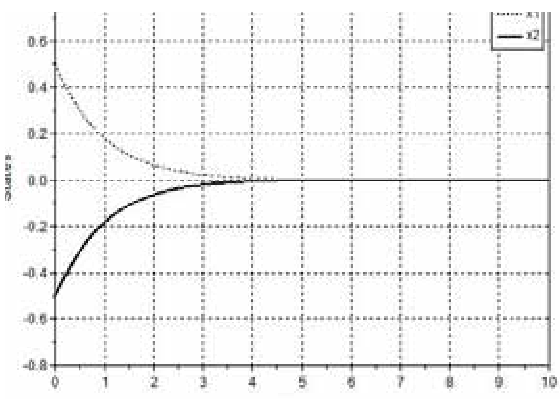

Figure 1 States x_1 and x_2.

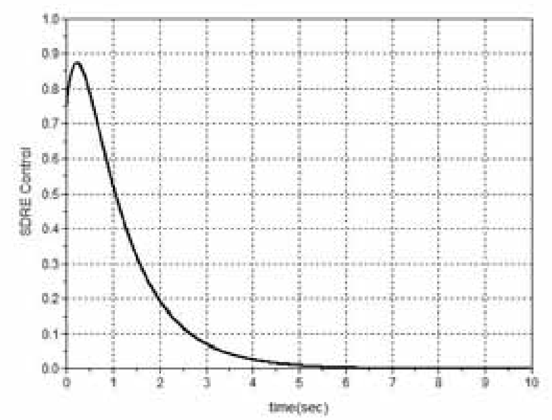

Figure 2 Control u.

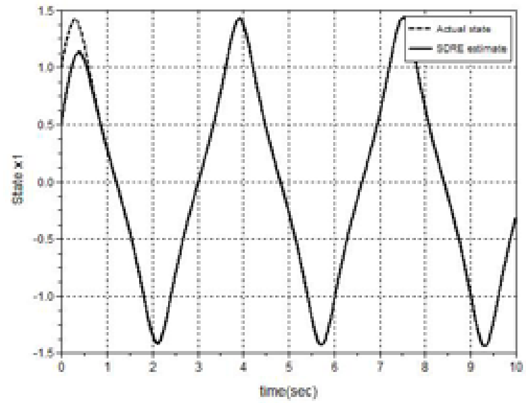

Figure 3 State x_1 and SDRE estimation for x_1.

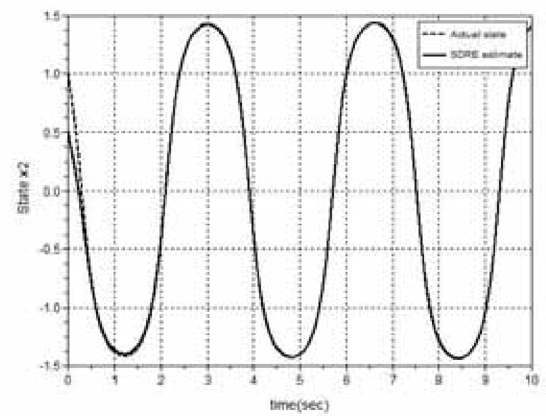

Figure 4 State x_2 and SDRE estimation for x_2.

However, the success of the SDRE approach depends on a good choice of the matrix *A(x)*. It is difficult to obtain global stability because of the limitations of this technique.

7.2 Example II

This example is a simple demonstration of the SDRE technique for state estimation. Consider the system

$$\dot{x}_1 = x_2 x_1^2 + x_2$$
$$\dot{x}_2 = -x_1^3 - x_1$$
$$y = x_1.$$

Using the formula (10) we can obtain the following choices of *A(x)* and *C(x)*

$$A(x) = \begin{bmatrix} 0 & 1+x_1^2 \\ -(1+x_1^2) & 0 \end{bmatrix} \text{ and } C(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The state-dependent observability matrix is

$$M_o = \begin{bmatrix} 1 & 0 \\ 0 & 1+x_1^2 \end{bmatrix}.$$

Since M_o is of full rank for all x, the sufficient condition for the detectability is satisfied. Hence, the system is detectable.

In the simulation results we set the initial condition $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ for the state and $x_e(0) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T$ for the estimated state. The weighting matrices $\bar{Q}$, $\bar{R}$ are chosen as $\bar{Q} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ and $\bar{R} = 1$, respectively.

Using the state estimator (21) we obtain the following results.

Figure 3 and Figure 4 show the fast convergence of each state estimate to x_1 and x_2. As shown in Figure 5 the errors decrease gradually and reach zero after 2.5 seconds. Thus, the SDRE estimator works properly for this example.

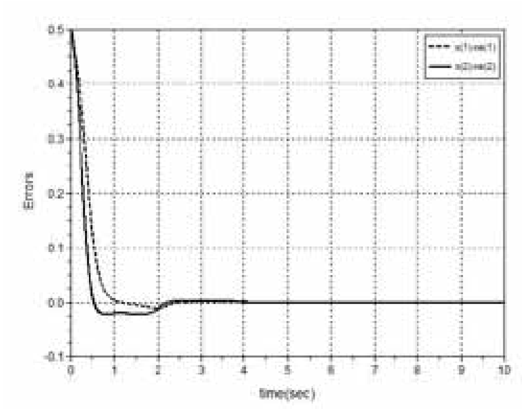

Figure 5 Errors.

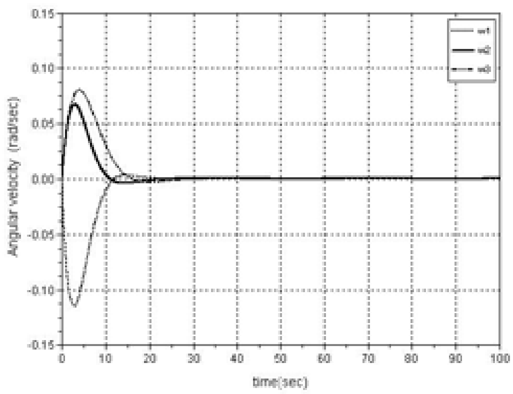

Figure 7 Components of angular velocity vector.

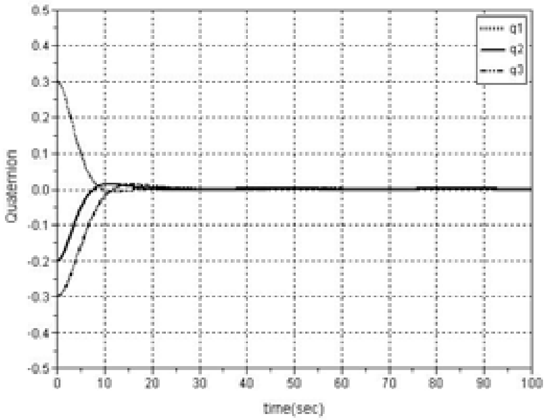

Figure 6 Components of attitude quaternion.

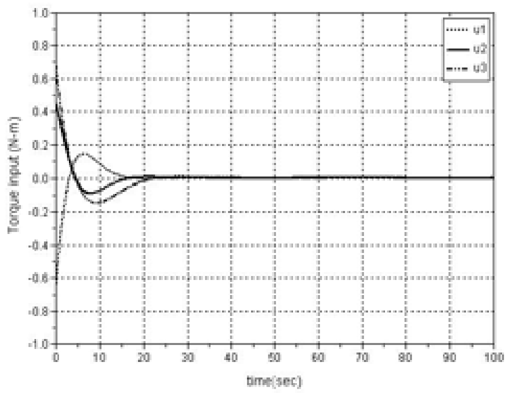

Figure 8 Components of angular velocity vector.

7.3 Example III

In this example we apply the SDRE technique to deal with the problem of attitude control of a rigid spacecraft. A rigid body rotating under the influence of body fixed devices is considered. In [16] the kinematics of the attitude error and the dynamics of error rate are given by

$$J\dot{\omega} = -\omega^{\times} J\omega + u$$

and $$\bar{q} = 0.5\begin{bmatrix} q^{\times} + q_0 I_3 \\ -q^T \end{bmatrix}\omega$$

where $\omega \in \Re^3$ denotes the angular velocity vector of the spacecraft, $\bar{q} \in \Re^4$ is the attitude quaternion defined by $\bar{q} = [q^T \quad q_4]^T$ with $q \in \Re^3$ and $q_4 \in \Re$ and $u \in \Re^3$ represents the control vector.

Here $J \in \Re^{3\times 3}$ denotes the inertia matrix. I_3 is the identity matrix. The skew-symmetric matrix $\omega^{\times}$ is

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

We now consider the optimal control problem (1) and (2). Letting $x = \begin{bmatrix} q^T & \omega^T \end{bmatrix}^T$, the SDC matrix $A(x)$ and $G(x)$ are selected as

$$A(x) = \begin{bmatrix} 0 & -J^{-1}\omega^{\times} J \\ 0 & 0.5(q^{\times} + q_4 I_3) \end{bmatrix}.$$

and $$G(x) = \begin{bmatrix} 0 \\ J^{-1} \end{bmatrix}.$$

In this example the spacecraft is assumed to have the inertia matrix

$$J = \begin{bmatrix} 10 & 1.0 & 0.7 \\ 1.0 & 15 & 0.4 \\ 0.7 & 0.4 & 20 \end{bmatrix} kg \cdot m^2.$$

The weighting matrices are chosen to be $Q = diag\,(1,1,1,5,5,5)$ and $R = I_3$.

The initial condition for the attitude quaternion is $q(0) = [0.3320\ \ 0.4618\ 0.1915\ \ 0.7999]^T$.

A rest-to-rest maneuver is considered and hence, the initial condition for the angular velocity is given as $\omega(0) = [0\ 0\ 0]^T$.

Next, the numerical simulation is carried out and the results are shown below.

As shown in Figs. 7 and 8 the components of quaternion and angular velocity vectors reach zero after 20 seconds. Fig 8 depicts the control torque responses. Clearly, the SDRE technique provides suboptimal control for attitude stabilization and achieves the control objectives to minimize the given performance index. Note that the magnitude of the control law obtain depends on the diagonal components of the weighting matrix Q. If the values of the diagonal components of the matrix Q are increased, the magnitude of the control law will be increased and this leads to the faster convergences of system responses.

8. Conclusions

In this paper a review of synthesizing state-feedback controller and observer designs for nonlinear systems using the SDRE method has been proposed. For practical implementation we have to find the factorization $f(x) = A(x)x$ specifically for each nonlinear system. Several methods that may be used for the factorization have been presented. The success of the SDRE approach depends on a good choice of the matrix $A(x)$. Due to the limitation of this method, it usually provides suboptimal controllers or estimators instead of the optimal ones. Numerical examples are presented and simulation results are given to demonstrate the usefulness of the SDRE feedback controller and SDRE estimator. In Example I-III, to obtain better results the global asymptotic stability of the closed-loop systems is needed to be ensured. Thus, the generation of Lyapunov functions for the stability analysis is required. The future work includes construction of the suitable Laypunov function to guarantee the global asymptotic stability for the controller designs based on the SDRE techniques.

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