BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL INCLUSION: SIMULATION OF LEAF SHAPE AND ITS AREA

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ABSTRACT

The classical methods for modeling leaf shape are always given symmetric upper and lower shapes. The aim of this paper is to investigate the simulation of asymmetric leaf shape. By deducting the traditional model coupled with causality conditions in sense of multivalued maps. We obtain the boundary value problems (BVPs) of fractional differential inclusion. Existent results verify that the problems are given positive solutions, associate to shape figures in nature. We illustrate an example to investigate the simulation of the leaf shape and its area. According to the results, we obtain a new alternative method for the simulation of leaves shape and their area can be computed by integration in closed form.

KEYWORDS: three point boundary valued problem, multivalued mapping, Fractional calculus, fixed point index theorem, asymmetric leaf

1. Introduction

Leaf area index is an important tool to indicate atmosphere – biosphere interaction, atmosphere – vegetation interaction and exchanging between land surface and atmosphere [1-4]. Leaf area measurement is a procedure for evaluating leaf area index. Most of the research measured by instrument and use empirical approach to validate and improve accuracy of parameters.

In mathematics, integration method is the well-known tool for measuring area of simple closed curves. So the area of leaf can be computed by integrating leaf shape along its length. Mathematical model of leaf shape was introduced as investigating formulas of leaf shape functions and its area under two parameters: leaf length and maximum leaf width [5-6]. The conditions of leaf shape function must satisfy the three point conditions and the characteristic factor is added to adjust shape figure [7]. According to [8], the leaf shape problem can be described by fourth order three point boundary value problems.

Recently, the concept of fractional calculus expand in many research fields, especially computational biology and physics. Fractional order models better interpret behavior in nature than integer order, because of persistence memory. In other words, fractional order concepts associate to convolution in domain [9].

Most of the leaf shape research is modeled with classical calculus and symmetry of geometric shapes. In this paper, we investigate a method for simulating asymmetric leaf shape via concept of multivalued maps. To adjust the shape figure, we apply fractional calculus, coupled with the characteristic parameter β. By theoretical approach, the sufficient conditions of positive solutions are given and applied to the leaf shape of Chinese kale including its simulation.

2. Preliminary

We briefly state definitions and theorems, which are used in this paper.

2.1 A concepts of fixed point index and fractional calculus

Lemma 2.1.1 [10] Let K be a cone in X and Θ be bounded open set of X. For each $x \in \mathrm{K} \cap \partial\Theta$ and $x \notin Tx$, the fixed point index $i_{\mathrm{K}}(T, \mathrm{K} \cap \Theta)$ satisfies:

(H1) $i_{\mathrm{K}}(T, \mathrm{K} \cap \Theta) = 0$, if there is $e \notin \mathrm{K} \setminus \{0\}$ for all $x \in \mathrm{K} \cap \partial\Theta$ and $k > 0$ such that $x \notin Tx + ke$.

(H2) $i_{\mathrm{K}}(T, \mathrm{K} \cap \Theta) = 1$, if $kx \notin Tx$ for $x \in \mathrm{K} \cap \partial\Theta$ and $k > 0$.

(H3) Let V_1 and V_2 are open subset of $\mathrm{K} \cap \Theta$ such that $V_1 \cap V_2 \neq \varnothing$ and $x \notin Tx$ for all $x \in \left(\mathrm{K} \cap \overline{\Theta}\right) \setminus \left(V_1 \cup V_2\right)$ then $i_{\mathrm{K}}(T, \mathrm{K} \cap \Theta) = i_{\mathrm{K}}(T, V_1) + i_{\mathrm{K}}(T, V_2)$.

If $i_{\mathrm{K}}(T, \mathrm{K} \cap \Theta) \neq 0$ then T has a fixed point in $\mathrm{K} \cap \Theta$.

Definition 2.1.2 The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as $I^{\alpha} y(x) = \int_0^x (x-s)^{\alpha-1} y(s)ds \Big/ \Gamma(\alpha)$. The Riemann-Liouville derivative, defined by $D^{\alpha} y(x) = D^{\lceil \alpha \rceil} I^{\lceil \alpha \rceil - \alpha} y(x) = \left(\frac{d}{dx}\right)^{\lceil \alpha \rceil} \int_0^x (x-s)^{\lceil \alpha \rceil - \alpha - 1} y(s)ds \Big/ \Gamma(\lceil \alpha \rceil - \alpha)$.

Theorem 2.1.3 [9] Let $\alpha > 0$ and $y \in C([0,L]) \cap L^1([0,L])$ then $D^{\alpha} y(x) = 0$ has a unique solution $y(x) = \sum_{i=1}^{i=\lceil \alpha \rceil} c_i x^{\alpha - i}$.

2.2 Modelling leaf shape and its area

The causality conditions of leaf shape functions are given by four observations: the first and the second, both initial point and end point are zero. The third condition and the last one are that the maximum leaf width occurs at some point in $u \in (0,L)$ [8]. The three point boundary conditions subject to

$$y(0) = 0,\ y(L) = 0,\ y(u) = w/2 \text{ and } y'(u) = 0, \tag{1}$$

see figure 1 and its area is determined by $2\int_0^L y(x)dx$.

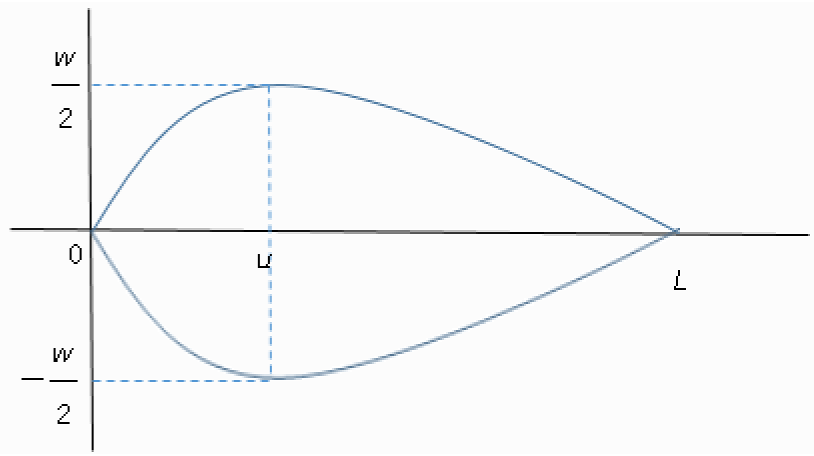


Figure 1 symmetric leaf shape preserve the three point conditions (1)

3. Main results

Let $F : [0,L] \times [0,\infty] \to CC[0,\infty]$ be an upper-Caratheodory map and also be the set of all functions $f(x, y(x)) \in C\big([0,L],[0,\infty]\big) \cap L^1([0,L])$. The generalized asymmetric leaf shape which satisfies the causality conditions in (1), associate to the following figure.

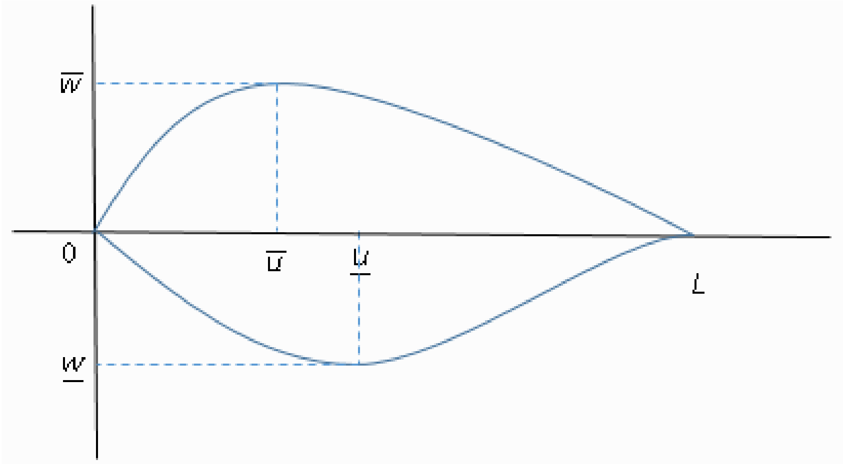


Figure 2 asymmetric leaf shape preserve the conditions in (2)

The solution of boundary value problem of fractional differential inclusion yields the simulation in the figure 2, defined by

$$\begin{cases} D^{\alpha} y(x) \in F(x,y), \ \alpha \in (3,4] \\ y(0) = y(L) = 0, \ y(\overline{u}) = \overline{w} \ (y(\underline{u}) = \underline{w}), \ (y')^{-1} 0 \in \{\overline{u}, \underline{u}\} \end{cases}, \quad (2)$$

where $\overline{u}, \underline{u}, \overline{w}, \underline{w} \in \mathbb{R}^{+}$.

3.1 Existence results for the boundary valued problems

Throughout this paper, we note that

$$y(x) \in \mathrm{K} = \left\{ y \in C([0,L]) : \min_{x \in [0,L]} y(x) \geq g\|y\|, y(x) \geq 0 \right\}.$$

To study the positive solution of integral inclusion:

$$y(x) \in p_1(x) + \int_0^L G_1(x,s) F(s, y(s)) ds + \int_0^u G_2(x,s) F(s, y(s)) ds, \quad (3)$$

where, $p_1(x)$ is power function and $G_1(x,s)$ and $G_2(x,s)$ are green's functions.

We will construct the green's functions, by using theorem 2.1.3 the solution of the problems (2) can be written as

$$y(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + c_3 x^{\alpha-3} + c_4 x^{\alpha-4} + I^{\alpha} f(x, y(x)), \qquad (4)$$

where, $c_1, c_2, c_3, c_4 \in \mathbb{R}$. The constant c_4 is vanished by the condition $y(0) = 0$.

Applying the remaining conditions yield the algebraic systems **Ac = b** where,

$$\mathbf{A} = \begin{bmatrix} l^{\alpha-1} & l^{\alpha-2} & l^{\alpha-3} \\ u^{\alpha-1} & u^{\alpha-2} & u^{\alpha-3} \\ (\alpha-1)u^{\alpha-2} & (\alpha-2)u^{\alpha-3} & (\alpha-3)u^{\alpha-4} \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \text{ and}$$

$$\mathbf{b} = \begin{bmatrix} -I^{\alpha} f(L, y(L)) & W - I^{\alpha} f(u, y(u)) & -I^{\alpha-1} f(u, y(u)) \end{bmatrix}^T$$

Here, we get

$$c_i = \frac{1}{\det \mathbf{A}} \left(-M_{1i}(\mathbf{A}) I^{\alpha} f(L, y(L)) + (-1)^i M_{2i}(\mathbf{A})(W - I^{\alpha} f(u, y(u)) - M_{3i}(\mathbf{A}) I^{\alpha-1} f(u, y(u)) \right)$$

Note that, $M_{ij}(\mathbf{A})$ is a minor of matrix **A** for $i, j = 1, 2, 3$.

Putting c_1, c_2 and c_3 in the equation (4), we obtain the following integral equations

$$y(x) = p_1(x) + \int_0^L G_1(x,s) f(s, y(s)) ds + \int_0^u G_2(x,s) f(s, y(s)) ds,$$

where, $p_i(x)$ are power functions for each $i = 1, 2, 3, 4$,

$$G_1(x,s) = \begin{cases} \dfrac{p_2(x)}{\Gamma(\alpha)} + \dfrac{(x-s)^{\alpha-1}}{2\Gamma(\alpha)}, & s < x \\ \dfrac{p_2(x)(L-s)^{\alpha-1}}{\Gamma(\alpha)}, & x < s \end{cases}, \text{ and}$$

$$G_2(x,s)=\begin{cases}\dfrac{p_3(x)(u-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{p_4(x)(u-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\dfrac{(x-s)^{\alpha-1}}{2\Gamma(\alpha)}, & s<x\\ \dfrac{p_3(x)(u-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{p_4(x)(u-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & s>x\end{cases}$$

It is strength forward to show that $h_i(s)\le G_i(x,s)\le H_i(s)$, for $i=1,2$.

Now, we will prove existence of positive solutions for the integral inclusion (3).

Theorem 3.1.1 Assume that there are functions $v,V\in C([0,L],[0,\infty])\cap L^1([0,L])$ such that $v(x)y(x)\le f(x,y(x))\le V(x)y(x)$ and the following integral inequalities $1>g\left(\int_0^L h_1(s)v(s)ds+\int_0^u h_2(s)v(s)ds\right)$ and $k>\int_0^L H_1(s)V(s)ds+\int_0^u H_2(s)V(s)ds$ hold for some $g,k\in(0,1]$ then the problem (2) has at least one positive solution on $[0,L]$.

Proof Define $\Omega_r=\left\{y\in\mathrm{K}, \min_{x\in[0,L]} y(x)<r\right\}$ and $\mathrm{B}_R=\left\{y\in\mathrm{K},\|y\|<R\right\}$, such that $\Omega_r\subset\mathrm{B}_R$. Let $T:\mathrm{K}\to C([0,L])$ and suppose that there exists a function $y^*\in\mathrm{K}\cap\partial\Omega_r$ such that $y^*-ke\in Ty$, where

$$Ty:=\left\{f\in C([0,L]), f(x)=p_1(x)+\int_0^L G_1(x,s)f(s)ds+\int_0^L G_2(x,s)f(s)ds\right\}$$

Consider, setting $e=1$

$$y^*(x)=p_1(x)+\int_0^L G_1(x,s)f(s,y^*(s))ds+\int_0^u G_2(x,s)f(s,y^*(s))ds+k$$

$$\ge m+gr\left(\int_0^L h_1(s)v(s)ds+\int_0^u h_2(s)v(s)ds\right)+k\ge r+k>r$$

Hence $y^*\notin\partial\Omega_r$, which is contradiction therefore (H1) holds.

Next we will show that (H2) holds, suppose that $ky^*\in Tx$ for some $y^*\in\partial\mathrm{B}_R$.

$$ky^*(x) \le M + R\left(\int_0^L H_1(s)V(s)ds + \int_0^u H_1(s)V(s)ds\right)$$

Hence $y^* \notin \partial B_R$, if we choose $R > M / \left(k - \int_0^L H_1(s)V(s)ds + \int_0^u H_2(s)V(s)ds\right)$.

Consequently, (H1) and (H2) are satisfied. Since $i_K(T, B_R) = 1$ and $i_K(T, \Omega_r) = 0$, we see that $1 = i_K(T, B_R) = i_K(T, B_R \setminus \overline{\Omega}_r) + i_K(T, \Omega_r) = i_K(T, B_R \setminus \overline{\Omega}_r)$. That is the operator T has a fixed point in $B_R \setminus \overline{\Omega}_r$, corresponding to (H3). Therefore the problem (2) has at least one positive solution on $[0, L]$.

3.2 Simulation results of leaves shape and their area

In this section, we will demonstrate simulation of the Chinese kale in the figure 3(a) such that $L = 12$, $u \in \{\underline{u} = 5.69, \overline{u} = 6.87\}$ and $w \in \{\underline{w} = 4.03, \overline{w} = 4.74\}$. We illustrate using the result of theorem 3.1.1 and simulation investigation of Chinese kale leaf shape via the function $\overline{k}(\beta)x^{\beta-4} = f(x, y(x)) \in F(x, y(x))$ where $\alpha = 3.7$, $\beta = \{\underline{\beta}=1.2, \overline{\beta}=2.4\}$ and $\overline{k}(\beta) = \beta(\beta-1)(\beta-2)(\beta-3)$.

Corollary 3.2.1 Given $D^\alpha y(x) \in \left\{\overline{k}(\underline{\beta})x^{\underline{\beta}-4}, \overline{k}(\overline{\beta})x^{\overline{\beta}-4}\right\}$, $\alpha \in (3,4]$ subject to three point boundary conditions in (2). Define

$$\Phi = \overline{k}(\underline{\beta})\Gamma(\underline{\beta}-3)\left(\frac{\min p_2 L^{\alpha+\underline{\beta}-4}}{\Gamma(\alpha+\underline{\beta}-3)} + \frac{\min p_3 u^{\alpha+\underline{\beta}-4}}{\Gamma(\alpha+\underline{\beta}-3)} + \frac{\min p_4 u^{\alpha+\underline{\beta}-5}}{\Gamma(\alpha+\underline{\beta}-4)}\right) \tag{5}$$

and

$$\Psi = \overline{k}(\overline{\beta})\Gamma(\overline{\beta}-3)\left(\frac{\|p_2\| L^{\alpha+\overline{\beta}-4}}{\Gamma(\alpha+\overline{\beta}-3)} + \frac{\|p_3\| u^{\alpha+\overline{\beta}-4}}{\Gamma(\alpha+\overline{\beta}-3)} + \frac{\|p_4\| u^{\alpha+\overline{\beta}-5}}{\Gamma(\alpha+\overline{\beta}-4)}\right) \tag{6}$$

If there are $g < \Phi^{-1}$ and $k > \Psi$ then the problem has positive solution.

Proof Here, $v(x)=\overline{k}(\underline{\beta})x^{\underline{\beta}-4}\le f\le \overline{k}(\overline{\beta})x^{\overline{\beta}-4}=V(x)$. Consider

$$\Phi=\int_0^L h_1(s)v(s)ds+\int_0^u h_2(s)v(s)ds$$

$$=\overline{k}(\underline{\beta})\left(\frac{\min p_2}{\Gamma(\alpha)}\int_0^L (L-s)^{\alpha-1}s^{\underline{\beta}-4}ds+\frac{\min p_3}{\Gamma(\alpha)}\int_0^u (u-s)^{\alpha-1}s^{\underline{\beta}-4}ds\right.$$

$$\left.+\frac{\min p_4}{\Gamma(\alpha-1)}\int_0^u (u-s)^{\alpha-2}s^{\underline{\beta}-4}ds\right)$$

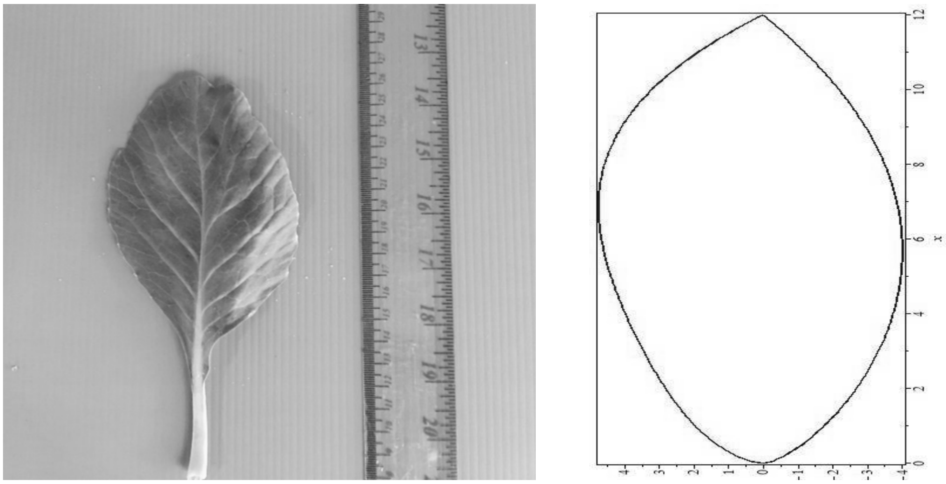
By using properties of beta function, we obtain (5). Similarly for Ψ, (6) is obtained. Recall theorem 3.1.1, we get $(1-g\Phi)>0$ and $(k-\Psi)>0$. By choosing $g<\Phi^{-1}$ and $k>\Psi$ then previous inequalities holds. Therefore the positive solution exists.

From the parameters of the Chinese kale and rewrite solution in form of (3), we get $p_1(x)\in[0,4.47]$, $p_2(x)\in[-13.53,0]$, $p_3(x)\in[0,1]$ and $p_4(x)\in[-2.04,4.03]$ yield $\Phi=80.597$ and $\Psi=-169.94$. If we choose $g=1/81<1/\Theta$ and $k=1>\Psi$ then there exists at least one solution on $[0,L]$. Alternatively, we may use solution is the form of (4), we obtain the solution of the upper shape and the lower shape in form of

$$\overline{y}(x)=\overline{c}_1x^{\alpha-1}+\overline{c}_2x^{\alpha-2}+\overline{c}_3x^{\alpha-3}+\overline{k}(\overline{\beta})x^{\alpha+\overline{\beta}-4}/\Gamma(\alpha+\overline{\beta}-3)\text{, and}$$

$$\underline{y}(x)=\underline{c}_1x^{\alpha-1}+\underline{c}_2x^{\alpha-2}+\underline{c}_3x^{\alpha-3}+\underline{k}(\underline{\beta})x^{\alpha+\underline{\beta}-4}/\Gamma(\alpha+\underline{\beta}-3)\text{,}$$

respectively. Its simulation is presented as the figure 3(b). However, the leaf area is equal to $\int_0^L\left(\overline{y}(s)-\underline{y}(s)\right)ds=73.847$, which difference from estimated by image J software $|73.847-72.587|=1.26$.



(a) Chinese kale (b) simulation of the Chinese kale

Figure 3 The Leaf of Chinese kale and its simulation

4. Conclusions

Asymmetric leaf shapes can be represented by the solution of fractional differential inclusion with four conditions of three point boundary value problems. For suitable fractional order and characteristic parameters yield realistic simulation of the leaf shapes, also is its area. Furthermore, we investigate theoretical approach for checking positive solution of leaf shape function without determining the solution. It is very useful if we want to ensure quantity of its area without simulation.

Comparing figure 1 and figure 2, this methodology obviously yields realistic shape and area due to the fact that leaves shape are not symmetry. Either twice of upper integration or lower integration in the classical method [8] maybe given more error if the curves and the widths of upper shape and lower shape are very different. However this method may use complexity computation to obtain the closed form of the solution. Determining the suitable of the right hand side function and fractional order are the problem challenge for other plants. In addition to the well-known method to measure leaf are such as digital image [11], Regression [12] etc., this methodology is a new alternative method for simulation and measure area of the leaves. We can investigate the innovation of application for simulating and computing area from the closed form of leaf shape functions in the future.

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