



Research Article

Discretisation for Optimal Control of Nonlinear System with Noises

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Abstract:

This paper presents an optimal control framework for nonlinear systems with noise using Hamiltonian-based discretisation to enhance the approximation of stochastic differential equations. It employs the symplectic Euler method and Newton's iteration, guided by Pontryagin's maximum principle to solve implicit equations, incorporating Gaussian white noise into the Hamiltonian formulation of state and costate dynamics. Numerical simulations on the van der Pol oscillator and Lorenz system demonstrate the proposed method's efficacy, achieving an average local truncation error reduction of 41% for the van der Pol oscillator and 1.1% for the Lorenz system, whilst reducing computational time by 27% on average for the Lorenz system, though often slower for the van der Pol oscillator. The method efficiently regulates noise, preserving dynamics, and minimising errors, contributing a robust numerical approach to nonlinear system control with advanced noise-handling capabilities.

Keywords: *Optimal control, Nonlinear system, Hamiltonian discretisation, Symplectic Euler method, Pontryagin's maximum principle*

1. Introduction

Optimal control involves determining a control input that minimises a given cost function whilst satisfying system's dynamics. It challenges in robotics, aerospace engineering, and marine vehicles, when the nonlinear system is subjected to noises. In those applications, noises perturbed by random processes are often unpredictable and complex [1-3]. Conventional methods, such as Kalman filter and its variants, may not sufficiently capture the complexity of the system with nonlinear dynamics and stochastic noise [4]. Their performance deteriorates in nonlinear systems due to their reliance on linearisation in both system dynamics and noise distribution. Thus, accurate modelling and controlling of a nonlinear system with noise are the key importance. Nonlinear dynamics represented by differential equations of nonlinear dynamics can cause unpredictable and complex behavior when perturbed by noise. These perturbations can disrupt stable control strategies, thus robust modelling techniques are required. One approach is presented by the systems governed by Hamiltonian dynamics [5]. The Hamiltonian systems are widely applied in various areas, for instance, mechanical, electrical, and quantum systems [6-8]. The approach is a powerful framework for optimal control by capturing the state and costate variables, preserving the structure of the system. This energy-based approach is particularly valuable for modelling nonlinear dynamic systems [9]. However, extending Hamiltonian control to noisy, nonlinear environments introduces significant challenges. The presence of stochastic elements significantly complicates the Hamiltonian structure. Stochastic terms not only add randomness to the state dynamics but also propagate through the costate equations, which result in coupled stochastic differential equations that are analytically intractable. Furthermore, noise modifies the deterministic Hamiltonian framework, necessitating the application of stochastic calculus [10] to account for the random fluctuations. This further complicates the derivation of explicit solutions.

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In practical applications, continuous-time Hamiltonian system with noise cannot be solved analytically. There is still a gap, the discretisation enables numerical simulation and allows the implementation of computational algorithms. Discretisation is a fundamental process for approximating continuous-time system dynamics with discrete-time equations. This important step is essential for digital simulation and implementation as they operate in discrete time. It allows for the explicit solution of both deterministic and stochastic terms. There are three common numerical methods namely Euler method [11-13], Runge-Kutta methods [14, 15], and backward differentiation formulas [16]. Runge-Kutta methods generally approximate numerical solution with higher accuracy and disturbance rejection. Nevertheless, this increased precision is obtained at the cost of high computation resources. Euler's scheme is typically defined by its simplicity and low computational cost. It offers a fast and direct approach for numerical integration due to its requirement of a single derivative evaluation per time step. The standard discretisation method may not fully preserve the structure. The symplectic Euler method (a variation of the Euler's scheme) is a discretisation technique that preserves the Hamiltonian system's properties, for example, energy conservation and symplectic structure [17]. Whilst effective for long-term stability, its application to nonlinear and noisy systems requires attention to noise propagation and the impact of discretisation on the system dynamics. This algorithm addresses a robust discrete-time representation that handles the noise whilst preserving the system [18, 19]. The symplectic Euler method leads to implicit equations. In addition, numerical simulations require discretisation to estimate solutions of stochastic differential equations. It ensures that stability and accuracy in numerical solutions preserve error from accumulating due to the presence of nonlinearities and noise interactions. Basically, Newton's method is used to solve these equations, but its convergence can be challenging in the presence of nonlinearity and noise [20]. The Hamiltonian system with noise increases computational cost and complexity, particularly discretisation strategies are modified to resolve stochastic dynamics. Iterative schemes are required to manage the interplay between nonlinearity and noise, imposing stringent demands on stability and convergence guarantees. In deterministic settings, Pontryagin's maximum principle provides a fundamental concept in optimal control theory that offers optimality criteria into state and costate equations [21, 22]. However, in a stochastic system, the adjoint equations must be modified to account for stochastic components. Solving these modified adjoint equations requires numerical techniques that guarantee stability and optimality in the control solution [23].

In summary, this paper contributes a numerical framework for optimal control of nonlinear systems under noise, enhancing performance in stochastic dynamics. By discretising the Hamiltonian and incorporating noise into the formalism, it bridges gaps in solving stochastic differential equations. The symplectic Euler method preserves system dynamics, whilst Newton's iteration, guided by Pontryagin's maximum principle, efficiently solves implicit equations for optimal control. Numerical simulations of the van der Pol oscillator and Lorenz system demonstrate dynamic stability and error reduction.

This paper is structured as follows: Section 2 presents the discretisation of a continuous-time system. This is followed by a discretisation for the Hamiltonian nonlinear system with noise in Section 3. Simulation results for van der Pol and Lorenz oscillators are demonstrated in Section 4. Some conclusions are given in Section 5.

2. Discretisation of a continuous-time system

2.1 Basic concepts

Consider a continuous-time dynamical system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input and $f: \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^n$ is a nonlinear system function. The smooth nonlinear function f satisfies the Lipschitz condition. Therefore, a solution is guaranteed to exist and unique. The goals are to discretise the continuous-time system to continualise the resulting discrete system and to facilitate numerical computation for approximating continuous-time dynamics. These preserve both structural and accuracy properties.

2.2 Discretisation of the system

For a given a time step Δt , the discrete update is approximated as follows

$$\frac{x_{k+1} - x_k}{\Delta t} \approx f(x_k, u_k) \quad (2)$$

where $k = 0, 1, \dots, N$. For enhance stability and structure preservation, alternatives schemes, such as the following, include

- a) Implicit Euler method enhances stability in stiff nonlinear systems by solving implicit equation each time step, though it may not preserve symplectic structure as effectively as the symplectic Euler method, is given by

$$x_{k+1} = x_k + \Delta t f(x_k, u_k) \quad (3)$$

where $x_k \approx x(t_k)$ and $u_k \approx u(t_k)$

- b) Symplectic Euler method is a structure-preserving scheme, ensures long-time stability in nonlinear system by maintaining symplectic properties, as

$$x_{k+1} = x_k + \Delta t f(\tilde{x}_k, u_k) \quad (4)$$

where $\tilde{x}_k$ is chosen as either x_k explicit or x_{k+1} implicit

2.3 Continuous approximation of the discretised system

The continuous approximation procedure involves interpreting the discrete update x_k as a piecewise representation of a continuous trajectory $x(t)$ for time t . This is achieved through interpolation discrete point and reformulating the discrete update as a system of differential equations. Algorithm 2 outlines the steps to compute the continuous trajectory and the approximate differential equation from discrete updates.

2.3.1 Time interpolation

The state trajectory is interpolated between time steps to obtain a continuous approximation in $t \in [t_k, t_{k+1}]$ using piecewise-linear interpolation

$$x(t) \approx x_k + \frac{t - t_k}{\Delta t} (x_{k+1} - x_k) \quad (5)$$

substituting the discretised dynamics to obtain the corresponding continuous-time differential equation

$$x(t) \approx x_k + \frac{t - t_k}{\Delta t} \Delta t f(x_k, u_k) = x_k + (t - t_k) \Delta t f(x_k, u_k) \quad (6)$$

2.3.2 Continuous approximation of differential equations

Using linear interpolating, a finite difference approximation yields the discrete derivative

$$\dot{x}_k \approx \frac{x_{k+1} - x_k}{\Delta t} \quad (7)$$

A discrete differential equation

$$\dot{x}(t) \approx f(x_k, u_k), \quad t \in [t_k, t_{k+1}] \quad (8)$$

This equation approximates the original differential equation but introduces discretisation error. For implicit method, continuous approximation results in a modified differential equation that incorporates high-order system. As Δt approaches zero, the approximated dynamics converges to the original continuous differential equation.

Algorithm 1 Continuous approximation of discretised system

Initialisation:

input discrete state x_k , control u_k , dynamics $f(x_k, u_k)$,

set time step Δt , define time t_k

Interpolate trajectory (for $t \in [t_k, t_{k+1}]$)

Compute piecewise-linear interpolation:

$$x(t) = x_k + \frac{t - t_k}{\Delta t}(x_{k+1} - x_k)$$

Store $x(t)$ for $t \in [t_k, t_{k+1}]$

Substitute discrete dynamics for each time step k :

Compute discrete update

$$x_{k+1} - x_k \approx \Delta t f(x_k, u_k)$$

Substitute into interpolation

$$x(t) \approx x_k + (t - t_k)\Delta t f(x_k, u_k)$$

Derive differential equation

$$\dot{x}(t) \approx f(x_k, u_k), \quad t \in [t_k, t_{k+1}]$$

Output:

Return continuous trajectory: $x(t)$ for $t \in [0, T]$

Return approximate differential equation: $\dot{x}(t) \approx f(x_k, u_k)$

2.4 Evaluation

2.4.1 Error analysis

The continuous approximation of the discretised system matches the true dynamics with a local truncation error (LTE), yields

$$|x_{k+1}^{approx} - x_{k+1}^{exact}| = \mathcal{O}(\Delta t^q)$$

where q is the order of accuracy of the chosen discretisation scheme: for forward Euler $q = 1$, for backward Euler or symplectic Euler $q = 2$ and for high-order schemes (Runge-Kutta) $q = 3$.

2.4.2 Stability properties

Implicit schemes (backward Euler and symplectic Euler) enhance numerical stability for stiff system, ensuring that the continuous approximation remains well-behaved over long-time intervals. Whilst the discretisation scheme preserves geometric invariants, such as energy or symplectic structure, the continuous approximation inherits these properties ensuring consistent physical behaviour.

3. Discretisation for Hamiltonian nonlinear system with noise

3.1 Nonlinear Hamiltonian system

Consider a nonlinear Hamiltonian system with noise, where the goal is to minimise the following cost function

$$\min_u J = \int_0^T L(x(t), u(t)) dt + \Phi(x(T)), \quad (9)$$

subject to dynamics

$$\dot{x}(t) = f(x(t), u(t)) + g(x(t))w(t), \quad x(0) = x_0 \quad (10)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control vector, $w(t) \in \mathbb{R}^k$ is stochastic noise, $L(x, u)$ is the running cost that representing the instantaneous cost at time, and $\Phi(x(T))$ is the terminal cost. The goal is to determine the optimal control $u(t)$ that minimises the cost J .

The Hamiltonian system H is defined as

$$H(x, p, u, w) = p^T(f(x, u) + g(x)w) + L(x, u) \quad (11)$$

where $p \in \mathbb{R}^n$ is the costate vector associated with the state x . The stochastic dynamics extend to

a) state dynamics

$$\dot{x}(t) = \frac{\partial H}{\partial p} = f(x(t), u(t)) + g(x(t))w(t). \quad (12)$$

b) costate dynamics

$$\dot{p}(t) = -\frac{\partial H}{\partial p} \quad (13)$$

for noise, $w(t)$ affects both $f(x, u)$ and the cost $\int_0^T L(x, u) dt$ where requiring noise specific modelling.

3.2 Pontryagin maximum principle for systems with noise

The Pontryagin maximum principle ensures that the optimal control $u^*(t)$ maximises the Hamiltonian, yields

$$H(x^*, p^*, u^*, w) \geq H(x^*, p^*, u, w), \forall u \in U \quad (14)$$

The necessary conditions for optimal control include

a) state evolution

$$\dot{x}(t) = f(x, u) + g(x)w \quad (15)$$

b) costate evolution:

$$\dot{p}(t) = -\frac{\partial H}{\partial x} \quad (16)$$

c) optimal control

$$u^*(t) = \arg \max_{u \in U} H(x, p, u, w) \quad (17)$$

These equations are modified by noise, introducing stochastic effect and often necessitating numerical methods for evaluation.

3.4 Discretisation using the symplectic Euler method

The symplectic Euler method discretises the dynamics whilst preserving the Hamiltonian structure

a) discretised state dynamics

$$x_{k+1} = x_k + \Delta t f(x_{k+1}, u_k) + \Delta t g(x_{k+1})w_k \quad (18)$$

b) costate update

$$p_{k+1} = p_k - \Delta t \frac{\partial H}{\partial x}(x_{k+1}, p_{k+1}, u_k, w_k) \quad (19)$$

Thus, these equation preserves the symplectic structure but requires iterative methods to solve x_{k+1} and p_{k+1} .

3.5 Solving implicit equations using Newton's method

To solve the implicit equations in the discretised dynamics

a) let's define the residual for the costate update as

$$R(x_{k+1}) = x_{k+1} - x_k + \Delta t (f(x_{k+1}, u_k) + g(x_{k+1})w_k) \quad (20)$$

b) linearise using Taylor series

$$R(x_{k+1}^{(n)}) \approx R(x_{k+1}^{(n-1)}) + \frac{\partial R}{\partial x_{k+1}} \delta x_{k+1} \approx 0 \quad (21)$$

where $\delta x_{k+1} = x_{k+1}^{(n)} - x_{k+1}^{(n-1)}$

c) update each Newton's iteration

$$x_{k+1}^{(n)} = x_{k+1}^{(n-1)} + \left(\frac{\partial R}{\partial x_{k+1}} \right)^{-1} R(x_{k+1}^{(n-1)}) \quad (22)$$

where the Jacobian $\frac{\partial R}{\partial x_{k+1}}$ accounts for nonlinear dynamics and noise.

3.6 Optimal control update

The optimal control satisfies as follows

$$\frac{\partial H}{\partial u} = 0 \quad (23)$$

At each time step, compute the optimal control by minimising the Hamiltonian

$$u_k^* = \arg \max_{u \in \mathcal{U}} (p_k^T f(x_k, u) + L(x_k, u)) \quad (24)$$

3.7 Algorithm

Algorithm 2 defines the numerical simulations of the proposed framework. Let's set $x_0(t)$ as initial guess. At the k th iteration, the current approximate solution is denoted by x_k and x_{k+1} is the new approximate solution.

4. Simulation Results

In this section, the proposed discretisation method is applied to the van der Pol oscillator and Lorenz system with noise. A simulation is conducted to assess the performance of the proposed method. The van der Pol system is a second-order nonlinear differential equation often used to model oscillatory behavior [24], whilst the Lorenz system is a differential equation used in meteorology [25]. The simulations were calculated using Matlab/Simulink in a computer, whose processor is Intel Core i3-8130U CPU @2.21 GHz.

4.1 Van de Pol oscillator

The van de Pol oscillator with noise is described by the second-order differential equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} - x = u + \sigma \dot{w}(t) \quad (25)$$

Algorithm 2 Numerical iteration for optimal control

Initialisation:

set $x_0(t)$, initial costate $p_N = \frac{\partial \Phi}{\partial x} \Big|_{x_N}$, initial control $u_k^{(0)}$,

choose Δt , ϵ and maximum Newton iterations M

Iterative updates (for $k = 0, 1, \dots, N-1$): repeat until convergence $|u_{k+1}^{(n+1)} - u_{k+1}^{(n)}| < \epsilon$

Forward state propagation:

Simulate the state trajectory

$$x_{k+1} = x_k + \Delta t (f(x_{k+1}, u_k^{(n)}) + g(x_k)w_k)$$

Solve implicitly for x_{k+1} using Newton's method

Backward costate propagation:

Propagate the costate backward in time

$$p_{k+1} = p_k + \Delta t \frac{\partial H}{\partial x}(x_{k+1}, p_{k+1}, u_k^{(n)}, w_k)$$

Use the discretised Hamiltonian $H(x, p, u, w)$ for computation

Control update:

Update the control each time step by maximising the Hamiltonian

Solve the optimisation problem numerically if no closed-form solution exists

Check convergence

Compute the norm of the control difference

$$\text{error} = \|u_{k+1}^{(n+1)} - u_{k+1}^{(n)}\|$$

If error $< \epsilon$, terminate iterations

Output:

Return trajectories $x_{k=0}^N, p_{k=0}^N$ and $u_{k=0}^{N-1}$

rewritten as a first-order system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1 + u + \sigma \dot{w}(t)\end{aligned}\tag{26}$$

where x_1 is the displacement, x_2 is the velocity, μ is nonlinearity parameter, σ is noise intensity and $\dot{w}(t)$ is derivative of noise.

The Hamiltonian for this system is

$$H = p_1 x_2 + p_2 [\mu(1 - x_1^2)x_2 - x_1 + u + \sigma \dot{w}(t)] - \frac{1}{2}u^2\tag{27}$$

To determine optimal control, the Pontryagin maximum principle is applied. The costate dynamics are derived from partial derivatives of the Hamiltonian with respect to the state variables

$$\begin{aligned}\dot{p}_1 &= 2\mu p_2 x_1 x_2 + p_2, \\ \dot{p}_2 &= -p_1 + p_2 \mu(1 - x_1^2)\end{aligned}\tag{28}$$

Differentiating the Hamiltonian with respect to the control effort u and setting the derivative to zero, yields

$$u = -p_2\tag{29}$$

Thus, the optimal control u minimises the Hamiltonian, balancing control effort and system dynamics

Figure 1 and 2 present the simulation results of the van der Pol oscillator with Gaussian white noise. The phase portraits (see Fig. 1(a)) highlight the influence of nonlinearity ($\mu = [0.1, 0.5, 1, 2]$) and noise intensity ($\sigma = 0.5$) on trajectory behavior. For low $\mu = 0.1$ and sampling time $T = 0.1$ seconds, the trajectories exhibit nearly circular limit cycles with minor perturbations. Whilst at higher $\mu = 2$, the trajectories become highly irregular, reflecting the

interplay between nonlinearity and noise-induced deviations. The state trajectories x_1 and x_2 demonstrate in Fig. 1(b) for $\mu = 0.5$. The fluctuations in x_1 and x_2 are moderate but grow in amplitude as μ increases, showing how the oscillator becomes increasingly sensitive to stochastic inputs (see Fig. 2(a)). The costate trajectories p_1 and p_2 reflect the response to both the control input and noise. Their oscillatory behavior intensifies with noise, particularly in regions of high system nonlinearity, where the Pontryagin principle works to minimise the Hamiltonian whilst ensuring stability. The control input u exhibits smooth oscillations between -0.2 and 0.2 over time as shown in Fig. 2(a), with its magnitude significantly increasing with higher μ and noise intensity. As shown in Fig. 2(b), the trajectory error analysis shows that the deviation between noisy and deterministic trajectories grows with both time and μ , indicating increased system sensitivity. The cost function evaluation indicates that at $\mu = 0.5$ results in greater cumulative costs due to an increase in energy control and state error penalties. The sensitivity analysis confirms that the system becomes more susceptible to parameter changes as μ rises. The symplectic Euler method ensures numerical stability and preserves Hamiltonian structure, demonstrating its robustness in capturing the dynamics of stochastic nonlinear systems effectively.

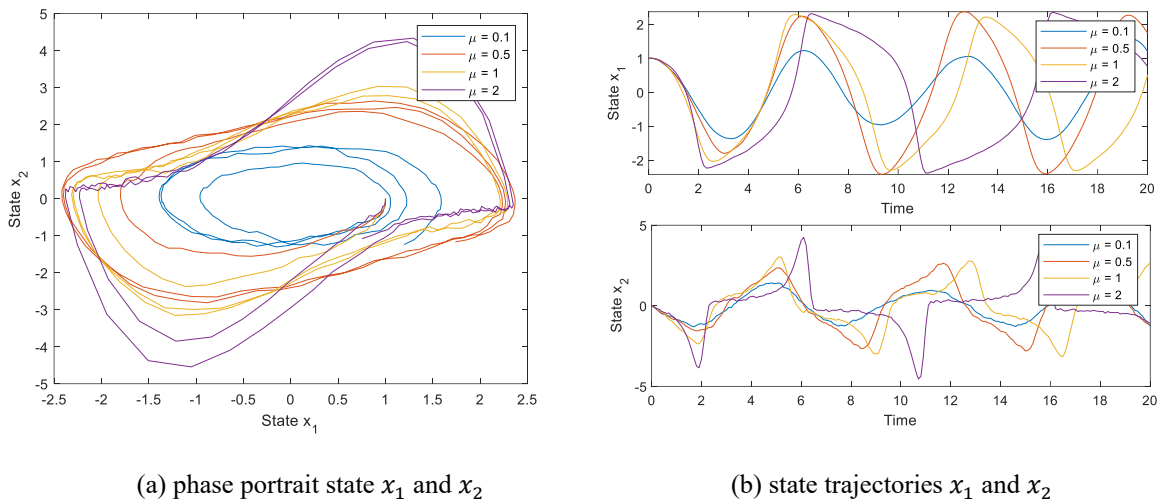


Fig. 1. Trajectories of the state x_1 and x_2 .

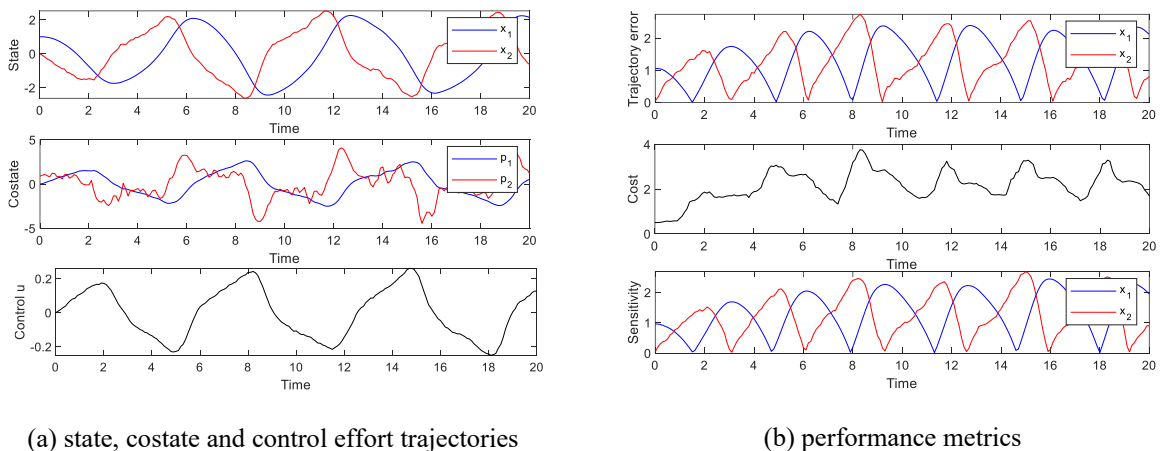


Fig. 2. Dynamics plot for $\mu = 0.5$.

Table 1 summarises the local truncation error (LTE) and computational time of the proposed method and the conventional Kalman filter for the van der Pol oscillator under various simulation conditions. At $\mu = 0.5$, the proposed method for the van der Pol oscillator outperforms the Kalman filter, achieving a lower LTE of 0.029 versus 0.051 (43% improvement) and a faster computational time of 2.911 seconds compared to 4.627 seconds (37% reduction). This efficiency, an outlier among higher times at other μ values, likely due to balanced nonlinearity reducing the

number of Newton iterations in the symplectic Euler method. The proposed method consistently delivers superior accuracy across all μ , but its generally higher computational cost highlights a trade-off. This makes it ideal for accuracy-critical applications with potential for real-time use at optimal conditions like $\mu = 0.5$.

Table 1: Errors and computation time of the van der Pol oscillator.

Case	Proposed method		Conventional method	
	LTE	Computation time (s)	LTE	Computation time (s)
$\mu = 0.1$	0.012	12.852	0.023	8.205
$\mu = 0.5$	0.029	2.911	0.051	4.627
$\mu = 1.0$	0.047	32.984	0.078	16.912
$\mu = 2.0$	0.083	42.702	0.124	38.270

4.2 Lorenz system

The Lorenz system is an example of a nonlinear dynamical system exhibiting chaotic behavior. The system is represented by three couples, the first order differential equations are

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1) + u_1 + \sigma_1 w_1(t) \\ \dot{x}_2 &= rx_1 - x_2 + x_1 x_3 + u_2 + \sigma_2 w_2(t) \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 + u_3 + \sigma_3 w_3(t)\end{aligned}\quad (30)$$

where x_1, x_2, x_3 are the state of the system, $\sigma_1 w_1(t), \sigma_2 w_2(t), \sigma_3 w_3(t)$ are independent Gaussian noise term modelled as Wiener processes representing system uncertainty, and σ is Prandtl number, β is parameter related to the geometry, and r is the reduced Rayleigh number.

Then, the Hamiltonian for the Lorenz system with noise is

$$H = p_1(\sigma(x_2 - x_1) + u_1) + p_2(x_1(r - x_3) - x_2 + u_2) + p_3(x_1 x_2 - \beta x_3 + u_3) - \frac{1}{2} u^T R u \quad (31)$$

where $p = [p_1, p_2, p_3]^T$ is costate and R is positive definite weighting matrix for the control effort. The Pontryagin maximum principle is employed to determine the optimal control, yielding the following partial derivatives

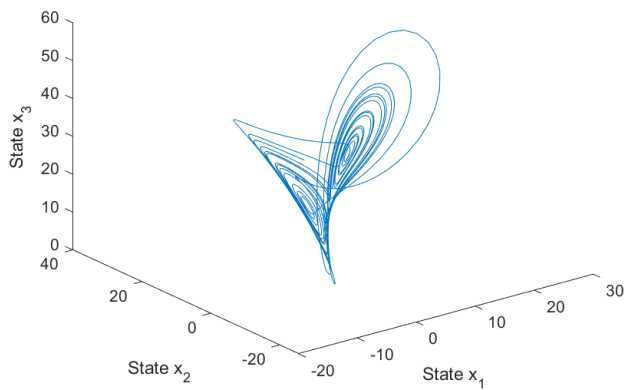
$$\begin{aligned}\dot{p}_1 &= -p_2(r - x_3) - p_3 x_2, \\ \dot{p}_2 &= \sigma p_1 - p_3 x_1, \\ \dot{p}_3 &= p_2 x_1 - \beta p_3\end{aligned}\quad (32)$$

The Hamiltonian is differentiated with respect to u , thus the optimal control input is

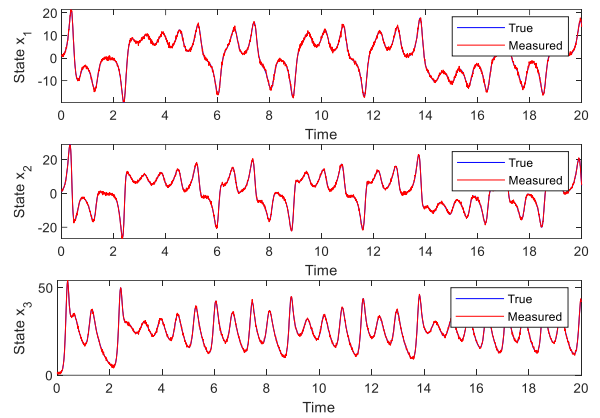
$$u(t) = -R^{-1}p \quad (33)$$

where R is a weighting matrix for penalising control effort

The simulations of the Lorenz system with noise have been carried out with zero initial conditions, taking sampling time $T = 0.1$ seconds, Prandtl number $\sigma = 10$, geometric factor $\beta = 8/3$, and reduced Rayleigh number $r = [10, 17, 20, 27]$. As shown in Fig. 3(a), the phase portrait shows the Lorenz system's characteristic butterfly attractor, which remains chaotic even with Gaussian white noise (Wiener process). The trajectory exhibits noise-induced deviations whilst still respecting the system's chaotic nature. The state variables (x_1, x_2, x_3) with noise introducing fluctuations as shown in Fig. 3(b), show the response of the proposed method still tracking the true value of the Lorenz system. State trajectories and errors (see Fig. 4), show the deviation from the desired path, grows with noise. Sensitivity analysis shows that the system is highly responsive to noise, with small changes in initial conditions leading to significant differences in behavior as shown in Fig. 5(a). Control effort becomes highly irregular in the presence of noise, as the system deviates more from the desired path as shown in Fig. 5(b)). The control strategy reduces this sensitivity, but the chaotic nature of the Lorenz system means the system remains highly sensitive to perturbations.

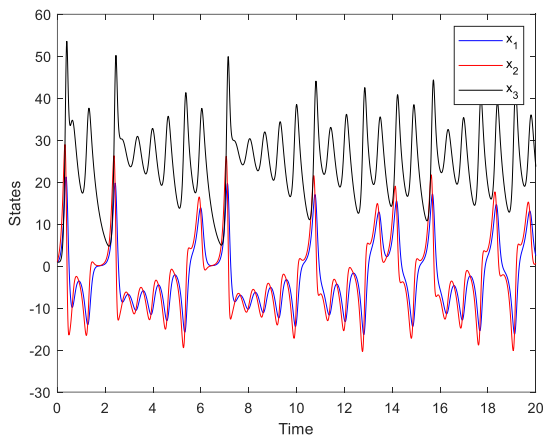


(a) phase portrait of state x_1, x_2 and x_3

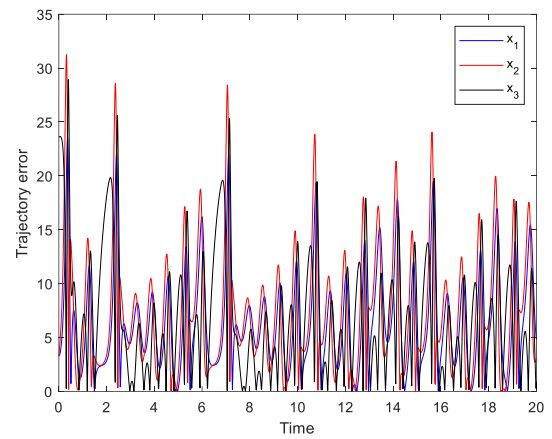


(b) state trajectory x_1, x_2 and x_3 with noise

Fig. 3. Trajectories of state x_1, x_2 and x_3 .

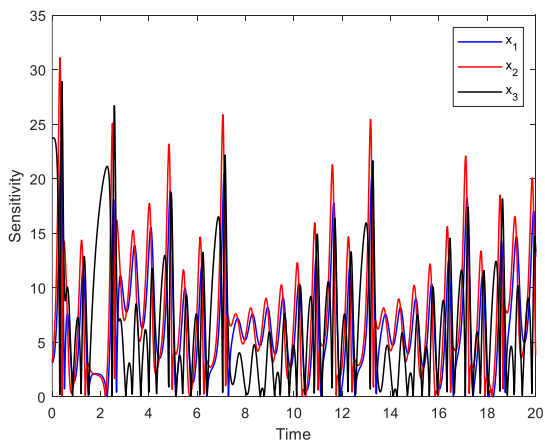


(a) state trajectory x_1, x_2 and x_3

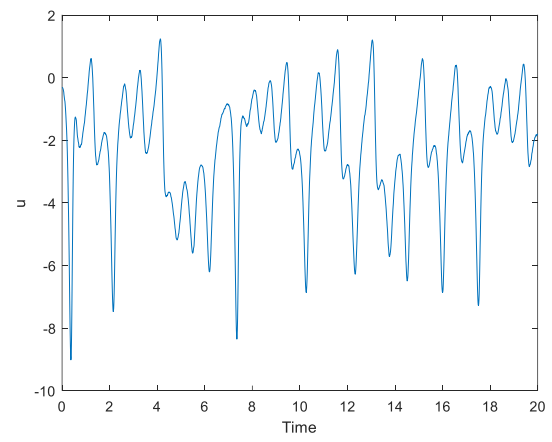


(b) state trajectory error

Fig. 4. Dynamics plots



(a) state trajectory sensitivity



(b) control effort u

Fig. 5. Performance metrics

Table 2 summarises the LTE and computational times of the proposed method and the conventional method for the Lorenz system under various simulation conditions. The proposed method consistently achieves lower LTE, with an average reduction of 1.1% (e.g., 2.7% at $r = 10$), due to its enhanced state estimation capabilities. As Rayleigh number r increases, the proposed method shows a slightly increase in LTE (from 0.252 to 0.628), suggesting enhanced chaos behavior in the system. Additionally, it is faster by an average of 27% (e.g., 30% at $r = 10$), outperforming the conventional method across all r values. Overall, the proposed method performs effectively in nonlinear systems with noise, achieving a balance between accuracy and computational efficiency.

Table 2: Errors and computation time of the Lorenz system.

Case	Proposed method		Conventional method	
	LTE	Computation time (s)	LTE	Computation time (s)
$r = 10$	0.252	28.412	0.259	40.352
$r = 17$	0.385	37.643	0.389	43.801
$r = 20$	0.461	47.342	0.463	76.047
$r = 27$	0.628	56.771	0.630	77.355

In summary, the proposed method offers significant value in engineering application where precise control of complex, noisy system is critical. Applied to the van der Pol and Lorenz systems with Gaussian white noise, simulations show the proposed method reduces LTE by 41% on average for the Van der Pol oscillator and 1.1% for the Lorenz system, whilst being 27% faster on average for the Lorenz system, though often slower for the Van der Pol oscillator except at $\mu = 0.5$ (37% faster). The following examples outline potential directions for applying the proposed method in real-world applications. In robotics, autonomous systems rely on nonlinear control systems to navigate dynamic environments with disturbances. The van der Pol oscillator, used in the study, models oscillatory behavior similar to robotic actuators or vehicle suspension systems. Reduced control effort and cost (battery usage), as the method optimises control inputs whilst handling noise-induced deviation. Moreover, an autonomous drone navigating turbulent air, the method could stabilise flight by computing optimal rotor adjustments, preserving dynamic stability and minimising power consumption despite wind gusts modeled as Gaussian noise. Another application in aerospace and satellite control, a satellite maintaining a geostationary orbit, the method could optimise reaction wheel adjustments to counteract noise from solar radiation pressure, ensuring precise alignment with ground stations. The Lorenz system, tested in the study, models chaotic behavior relevant to attitude control or orbital dynamics. Extended mission lifespans by optimising control effort under noisy conditions.

5. Conclusion

The study integrates Hamiltonian-based discretisation with the symplectic Euler method and Newton's iteration, guided by optimal control scheme for nonlinear systems affected by noises. The method preserves the system's dynamics and optimal control, though noise significantly affects state trajectories, control effort, and cost function. The proposed method was applied to van der Pol and Lorenz systems with Gaussian white noise. The numerical simulations demonstrate enhanced performances in error reduction and computation efficiency. The proposed method is currently limited to modelling system with Gaussian noise, requiring further research to address non-Gaussian noise types, such as impulsive or heavy-tailed distributions. Despite this limitation, it provides robust performance for digital system analysis applications, such as digital control and signal processing.

Nomenclature

f	nonlinear system function
J	cost function
H	Hamiltonian system
p	costate vector
$L(x, u)$	running cost
u	control vector
$w(t)$	noise
x	state vector
$\Phi(x(T))$	terminal cost
Δt	time step

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