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Research Article

AMPLITUDE ANALYSIS OF FUNCTIONALLY GRADED BEAMS UNDER LINEAR DECREASING AND EXPONENTIAL LOADS

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ABSTRACT:
The objective of this research is to study dynamic amplitude of functionally graded beams subjected to linear decreasing and exponential decay loads. The beams are assumed to be composed of ceramic and metal phases according to power law distribution. The Ritz method is utilized to solve free and forced vibration of the beams with various general boundary conditions. In case of dynamic analysis, the average acceleration method of Newmark is adopted to deal with the time dependent problem. Various effects of material composition, beam geometry and boundary condition, which have significant impact on beam analysis, are taken into account. Based on numerical results, it is found that the beam with high percentage of ceramic in material composition is very strong and has less dynamic deflection.

Keywords: *Functionally graded beam, Dynamic amplitude, Dynamic load, The Ritz method*

1. INTRODUCTION

One type of modern composite materials is functionally graded materials (FGMs) that have spatially varying properties. The material compositions are assumed to vary continuously from the top surface to the bottom one using mathematical functions such as polynomial or exponential functions. The idea and history of FGMs were introduced by Japanese scientists in the mid-1980s as ultra-high temperature-resistant materials for aerospace applications [1]. Nowadays, FGMs have been used in many applications, for example, in the application of rocket engine components, aerospace structures, turbine blades, etc. As described above, FGMs are important for engineering community, therefore, it is essential to understand their mechanical behavior.

According to open literature, there are a number of research studies on FGM behavior in different situations. For example, Aydogdu and Taskin [2] gave frequency results of simply supported FGM beams. Sina et al. [3] used traditional first order shear deformation theories to deal with vibration problem of functionally graded (FG) beams, using an analytical method. Simsek [4] used several beam theories to analyze vibration response of FG beams supported by different common boundary conditions. An improved third order shear deformation theory was employed by Wattanasakulpong et al. [5] to investigate thermal buckling and thermo-elastic vibration of FG beams with different immovable boundary conditions. Hein and Feklistova [6] applied Euler-Bernoulli theory and Haar matrices for analyzing vibration of non-uniform FG beams with various boundary conditions and cross-section ratios. By using Timoshenko beam theory, Xiang and Yang [7] presented the study of free and forced vibration of laminated FGM beams under heat conduction using the differential quadrature method (DQM).

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In recent investigation, many analytical and numerical techniques are adopted to find out mechanical solutions for FG beams. With the aid of dynamic stiffness method, FG beams and frameworks were considered to carry out their natural frequency in the study of Banerjee and Ananthapuvirajah [8]. In addition, Mahmoud [9] showed the general modeling for free vibration of axially FG cantilevers loaded at the tips with masses. By utilizing semi-analytical approach, the forced vibration of FG beams in form of linearly tapered beams resting on elastic foundation was investigated by neglecting the rotary inertia and shear deformation effects [10]. The development of sinusoidal beam theory was presented for stress and strain analysis of curved FG sandwich beams [11]. The beam is assumed to be composed of two functionally graded skins at the top and bottom and the core is made of homogenous material [12].

Based on above literature survey, it is found that there is no attempt on forced vibration analysis of FG beams. Therefore, in this investigation, the dynamic response of FG beams subjected to different dynamic loadings, which are linear decreasing and exponential decay loadings, are considered in this present study. The benefit of considering the linear and exponential decreasing loadings is that, in structural analysis, the models related to these types of loadings can be used to predict dynamic response of any structure under impact and blast loadings. To explain more about application, FG beams are structural members of whole structures in real application, for example, FG beams are stiffeners for reinforcing any panel in aerospace structures. When the panel subjected to the dynamic loading, the beams also have the same effect. The significant effects of material composition and beam geometry on the dynamic behavior of such beams are investigated for appropriate design.

2. FG BEAMS

A geometry of FG beam is shown in Fig. 1. The coordinate (x, z) is set at the middle and left end of the beam. The beam is made from the mixture of Alumina (Al_2O_3) ceramic phases and Aluminum (Al) metal in which their material properties are: Young's modulus (E_c = 380 GPa), density ($\rho_c = 3960$ kg/m^3) for Al_2O_3 and Young's modulus (E_m = 70 GPa), density ($\rho_c = 2702$ kg/m^3) for Al. The Poisson's ratio of the beam is $v = 0.3$.

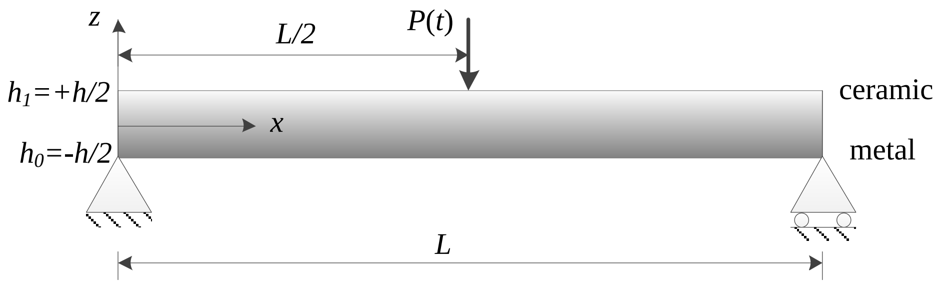

Fig. 1. A geometry of FG beam

It can be seen from Fig. 1 that the top surface is ceramic rich and bottom one is metal rich. The material property changes, Young's modulus (E) and material density (ρ), throughout the beam thickness (h) can be obtained from the power law equations as follows:

$$E = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^n + E_m \tag{1}$$

$$\rho = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^n + \rho_m \tag{2}$$

where n is the material volume fraction index or power law index, $0 \le n \le \infty$.

3. EQUATIONS OF MOTION

In this investigation, Timoshenko beam theory which takes into account shear deformation and rotational inertia is adopted to create the equations of motion. The displacement field of the theory is:

$$u(x,z,t)=\tilde{u}(x,t)+z\tilde{\psi}(x,t), \tag{3}$$

$$w(x,z,t)=\tilde{w}(x,t), \tag{4}$$

where $\tilde{u}$ and $\tilde{w}$ are axial and transverse displacements in the middle plane (z=0), respectively, $\tilde{\psi}$ is the rotation of the beam cross-section and t is time. The strain-displacement relations in terms of normal strain (ε_{xx}) and shear strain (γ_{xz}) are given by

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}=\frac{\partial \tilde{u}}{\partial x}+z\frac{\partial \tilde{\psi}}{\partial x}, \tag{4a}$$

$$\gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{\partial \tilde{w}}{\partial x}+\tilde{\psi}. \tag{4b}$$

The corresponding normal stress (σ_{xx}) and shear stress (τ_{xz}) can be obtained from the elastic constitutive law as

$$\sigma_{xx}=E(z)\varepsilon_{xx},\ \ \tau_{xz}=G(z)\gamma_{xz}=\frac{E(z)}{2(1+\nu)}\gamma_{xz}. \tag{5}$$

The strain energy (U_s) of the FG sandwich beams at any instant can be defined as

$$U_s=\frac{b}{2}\int_0^L\int_{-h/2}^{h/2}(\sigma_{xx}\varepsilon_{xx}+\tau_{xz}\gamma_{xz})dzdx. \tag{6}$$

Substituting Eqs. (4) and (5) into Eq. (6), one can obtain another form of the strain energy equation as

$$U_s=\frac{b}{2}\int_0^L\left[A_{11}\left(\frac{\partial \tilde{u}}{\partial x}\right)^2+2B_{11}\frac{\partial \tilde{u}}{\partial x}\frac{\partial \tilde{\psi}}{\partial x}+D_{11}\left(\frac{\partial \tilde{\psi}}{\partial x}\right)^2+A_{55}\left(\frac{\partial \tilde{w}}{\partial x}\right)^2+2A_{55}\tilde{\psi}\frac{\partial \tilde{w}}{\partial x}+A_{55}\tilde{\psi}\right]dx. \tag{7}$$

It is denoted that A_{11}, A_{55}, B_{11}, and D_{11} appearing in Eq. (7) are the extensional, shear, coupling and bending stiffness components which can be obtained from

$$[A_{11},B_{11},D_{11}]=\int_{-h/2}^{h/2}E(z)[1,z,z^2]dz,\ \text{and}\ \ A_{55}=\kappa\int_{-h/2}^{h/2}\frac{E(z)}{2(1+\nu)}dz \tag{8}$$

where κ = 5/6 is shear correction factor for layer of homogenous material.

For the kinetic energy of the beams, it is expressed as

$$\begin{aligned}U_k&=\frac{b}{2}\int_0^L\int_{-h/2}^{h/2}\rho(z)\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]dzdx\\&=\frac{b}{2}\int_0^L\left[I_0\left(\frac{\partial \tilde{u}}{\partial t}\right)^2+2I_1\frac{\partial \tilde{u}}{\partial t}\frac{\partial \tilde{\psi}}{\partial t}+I_2\left(\frac{\partial \tilde{\psi}}{\partial t}\right)^2+I_0\left(\frac{\partial \tilde{w}}{\partial t}\right)^2\right]dx\end{aligned} \tag{9}$$

where I_0, I_1 and I_2 are the inertia components which can be obtained from the following equation

$$[I_0, I_1, I_2] = \int_{-h/2}^{h/2} \rho(z)[1, z, z^2]dz. \tag{10}$$

In this investigation, the dynamic loads that are linear decreasing and exponential decay loads as shown in Fig. 2 are considered to excite FG beams.

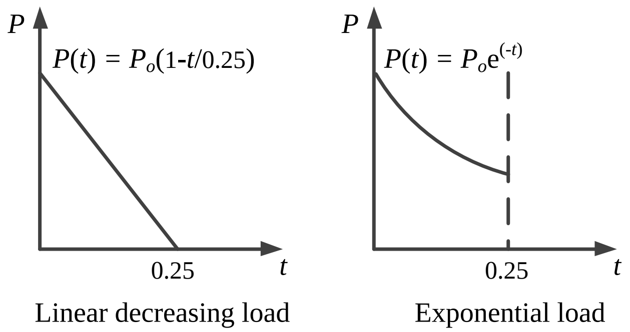

Fig. 2 Linear decreasing and exponential decay loads for exciting FG beams

Hence, the work done by the loads is $U_{ex} = P_0 w_0(x_p)$ where P_0 is the load magnitude and x_p indicates the position of the load on the top surface of the beams, in this case $x_p = L/2$.

From all energies described above, we can construct the total potential energy (Π) for the beam system as follows:

$$\Pi = U_s - U_k - U_{ex} \tag{11}$$

To solve the total potential energy in Eq. (11), the Ritz trial displacement functions in form of polynomial series are utilized for finding the solutions. The selected displacement functions must satisfy at least the essential or geometric boundary conditions. In order to consider FG beams with different boundary conditions such as clamped (C) and hinged (H) at any end of the beams, the trial displacement functions are

$$\begin{cases} u_0(x,t) = \sum_{j=1}^{N} A_j(t)\Delta_{1j}(x) \\ w_0(x,t) = \sum_{j=1}^{N} B_j(t)\Delta_{2j}(x) \\ \psi(x,t) = \sum_{j=1}^{N} C_j(t)\Delta_{3j}(x) \end{cases} \tag{12}$$

in which $\Delta_{1j}(x), \Delta_{2j}(x)$ *and* $\Delta_{3j}(x)$ are polynomial-series shape functions that are dependent on boundary conditions of the beams. The shape functions of the beams with different boundary conditions (B.C.) are shown in Table 1. For example, the beams which are hinged at both ends are defined as H-H beams.

Table 1: Polynomial shape functions of FG beams

B.C.	$\boldsymbol{\Delta_{1j}(x)}$	$\boldsymbol{\Delta_{2j}(x)}$	$\boldsymbol{\Delta_{3j}(x)}$
C-C	$\left(\frac{x}{L}\right)^j\left(1-\frac{x}{L}\right)$	$\left(\frac{x}{L}\right)^j\left(1-\frac{x}{L}\right)$	$\left(\frac{x}{L}\right)^j\left(1-\frac{x}{L}\right)$
C-H	$\left(\frac{x}{L}\right)^j\left(1-\frac{x}{L}\right)$	$\left(\frac{x}{L}\right)^j\left(1-\frac{x}{L}\right)$	$\left(\frac{x}{L}\right)^j$
H-H	$\left(\frac{x}{L}\right)^j\left(1-\frac{x}{L}\right)$	$\left(\frac{x}{L}\right)^j\left(1-\frac{x}{L}\right)$	$\left(\frac{x}{L}\right)^{j-1}$

These shape functions are expanded to suitable number of polynomial terms (N) which can find from convergence study. Inserting the trial displacement functions written above into the total potential energy of Eq. (11) and then following the Lagrange equation method

$$\frac{\partial \Pi}{\partial q_j} - \frac{d}{dt}\frac{\partial \Pi}{\partial \dot{q}_j} = 0 \tag{13}$$

with q_j representing the time-dependent unknown parameters ($A_j(t)$, $B_j(t)$, $C_j(t)$), one can obtain the following equation of motion

$$\begin{bmatrix} K^{11} & K^{12} & K^{13} \\ K^{21} & K^{22} & K^{23} \\ K^{31} & K^{32} & K^{33} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} + \begin{bmatrix} M^{11} & M^{12} & M^{13} \\ M^{21} & M^{22} & M^{23} \\ M^{31} & M^{32} & M^{33} \end{bmatrix} \begin{bmatrix} \ddot{A} \\ \ddot{B} \\ \ddot{C} \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}. \tag{14}$$

It is noted that the over-dot stands for the derivative with respective to time, K and M are the stiffness and mass matrices, respectively, in which their size is ($3N \times 3N$). Additionally, F is the vector of dynamic force. It is noted that matrices K and M can be used to find out natural frequency of FG beams using procedure described in Ref. [12]. The matrix form in Eq. (14) can be solved for time-dependent problem by using the average acceleration method of Newmark, see Ref. [12].

4. Numerical results

The numerical technique of Ritz method is utilized to solve free and forced vibration of FG beams in this investigation. As a result, by using this method, it is important to do a convergence study first in order to check the suitable number of terms in polynomial-series shape functions as presented in Table 2. It can be clearly seen that using only N-terms =10 is enough to obtain the correct results in which the first mode frequency (ω_1) of C-C beam is validated by that of Ref. [4].

Table 2: Convergence study and validation for free vibration of FG beams (L/h = 20, n = 0.5)

			n = 0.5	
B.C.	***N***	ω_1	ω_2	ω_3
H-H	4	4.7894	23.1976	59.2622
	6	4.7879	18.4590	41.4229
	8	4.7879	18.3835	40.7646
	9	4.7879	18.3831	40.7645
	10	4.7879	18.3831	40.7549
C-H	4	7.2719	23.4254	77.8225
	6	7.2648	23.1015	48.3163
	8	7.2648	23.0956	47.1299
	9	7.2648	23.0956	47.1037
	10	7.2648	23.0956	47.1033
C-C	4	10.4521	28.7040	109.1131
	6	10.4229	28.1816	55.9458
	8	10.4228	28.1733	53.9190
	9	10.4228	28.1733	53.8644
	10	10.4228	28.1733	53.8640
	Ref. [4]	10.4263	-	-

To study the effect of beam thickness ratio (L/h), Fig. 3 presents the fundamental frequency of FG beams hinged at both ends (H-H) with different values of the ratio. In this figure, the relationship between the frequency and material volume fraction index (n) is also presented. Increasing n leads to the decrease of the frequency for every beam.

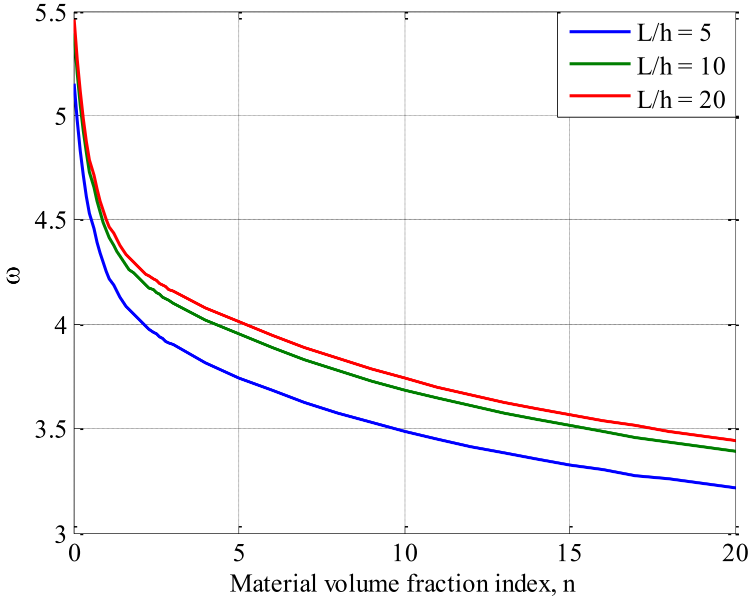

Fig. 3 Effects of thickness ratio and material volume fraction index on fundamental frequency of FG beams

The results in Table 2 and Fig. 3 are the results of free vibration analysis of FG beams. Next, we move to analyze forced vibration of FG beams under the dynamic loads. Now, we begin our forced vibration analysis with FG beams under the action of linear decreasing load in Figs. 4-6. The dynamic deflections presented in forced vibration are normalized by dividing them with static deflection ($w_s = P_0L^3/48E_mI$) of simply supported metal beam where I is the moment of inertia of area of the beam cross-section. According to force schematic in Fig. 2, it can be observed that the normalized dynamic deflections are reduced linearly from initial time (t = 0) to the setting time of t = 0.25 s before vibrating continuously to the final time of t = 0.5 s.

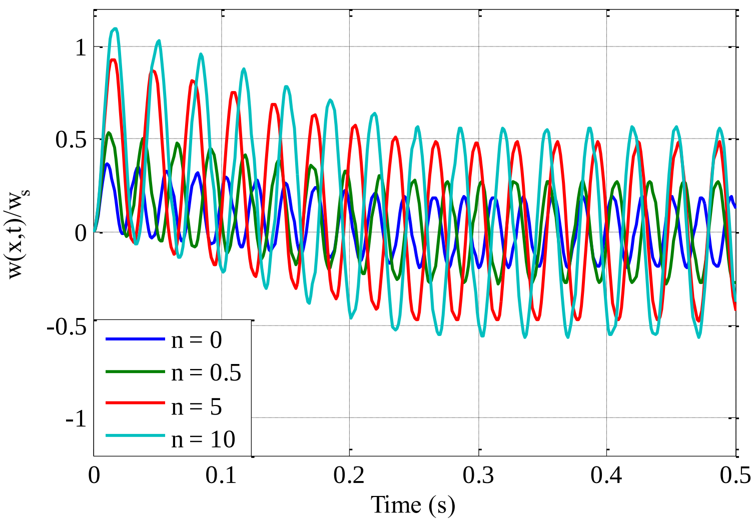

Fig. 4 Dynamic deflection of FG beams under linear decreasing load: Effect of n index

The effect of material index (n) is considered in Fig. 4. The deflection is increased as the increase of n. It is noted that the percentage of ceramic in FG beam is reduced continuously when the n index increase. As a result, the beam with n = 10 has lower percentage of ceramic than other beams in Fig. 4.

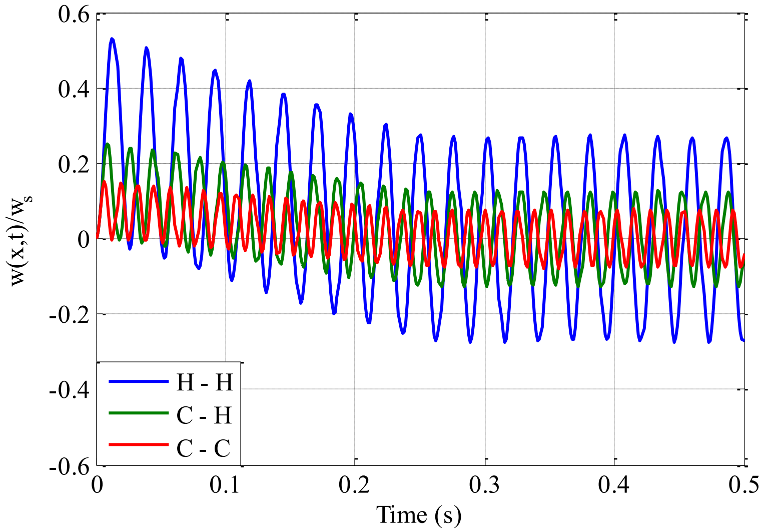

Fig. 5 Dynamic deflection of FG beams under linear decreasing load: Effect of boundary condition

The beams with different boundary conditions are investigated in Fig. 5. As can be observed, the beam with clamped at both end (C-C) has lower dynamic deflection than that of other beams throughout the time.

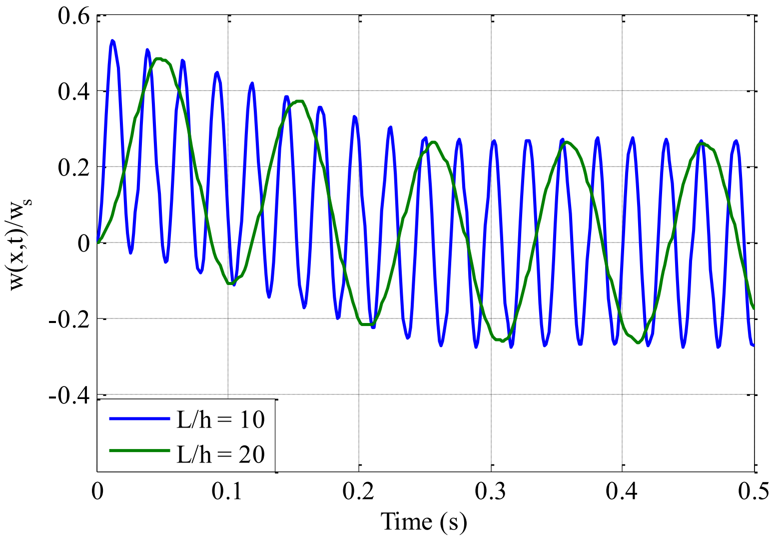

Fig. 6 Dynamic deflection of FG beams under linear decreasing load: Effect of beam thickness ratio

In Fig. 6, the beams with two different beam thickness ratios are considered. The beam with high thickness ratio is known as the thin beam. In general, thin beams have lower natural frequency than that of thick beam. Consequently, the beam with L/h = 10 vibrates more than the beam with L/h = 20 within the same period.

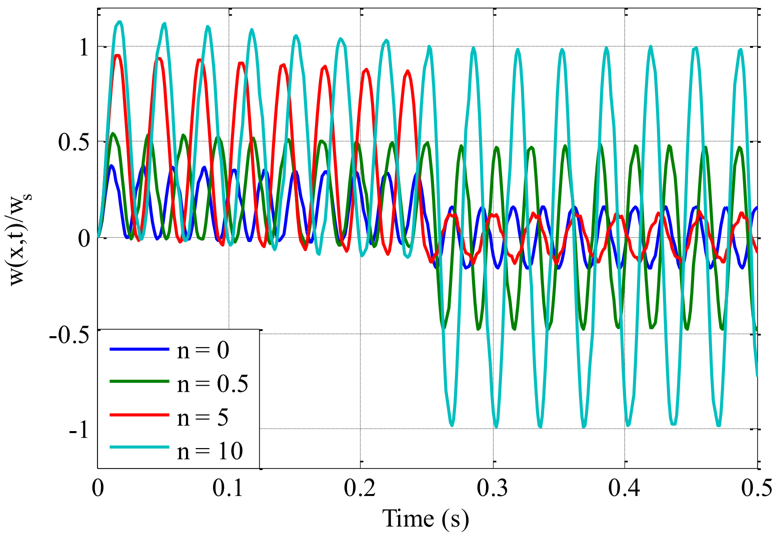

Fig. 7 Dynamic deflection of FG beams under exponential decay load: Effect of *n* index

After forced vibration of FG beams under linear decreasing load, we move to investigate the dynamic response of the beams under exponential decay load, which can be seen in Figs. 7-9. Similarly, in this case, we also consider the significant effects of material volume fraction index (n), boundary condition and the beam thickness ratio which have dramatic impact of the vibration response of FG beams subjected to exponential load. As can be seen, the deflection reduces exponentially in the first period of t = 0-0.25 s before vibrates freely until the time t = 0.5 s.

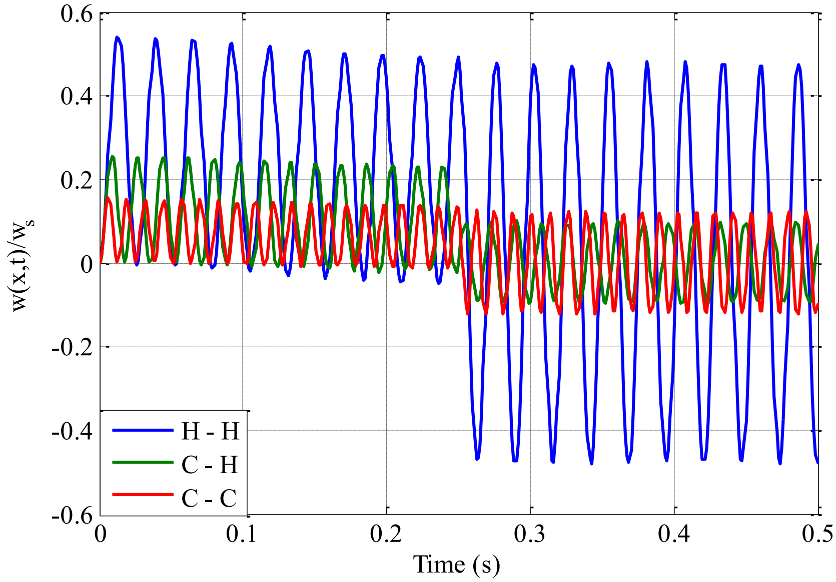

Fig. 8 Dynamic deflection of FG beams under exponential decay load: Effect of boundary condition

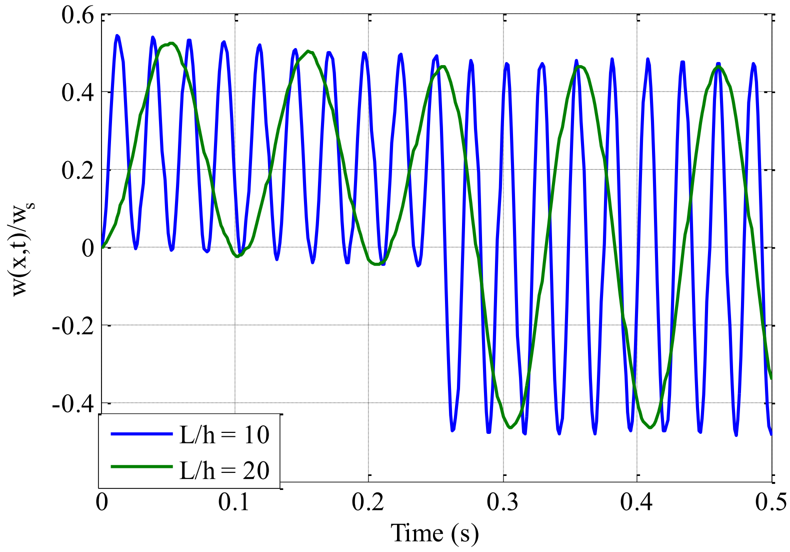

Fig. 9 Dynamic deflection of FG beams under exponential decay load: Effect of beam thickness ratio

5. Conclusion

The dynamic behavior of FG beams subjected to linear decreasing and exponential loads is considered in this research. By using the Ritz method cooperated with the time integration technique of Newmark, the dynamic deflections of such beams can be obtained for various boundary conditions. In case of free vibration, the natural frequencies of FG beams are reduced as the increase of the material volume fraction index (n). For dynamic analysis, the beams having clamped at both ends have less deflections than those of other beams throughout time history. In addition, increasing n also yields the increase of the dynamic deflections. The mathematical model in this research can be extended to deal with other kinds of dynamic analysis such as the problems of moving forces and moving masses on FG beams.

Nomenclature

A_{11}, A_{55}	Extensional stiffness, N.m/m^2
B_{11}	Coupling stiffness, N.m^2/m^2
b	Width of beam, m
D_{11}	Bending stiffness, N.m^3/m^2
E	Young's modulus, N/m^2
F	Vector of dynamic force, N
G	Shear modulus, N/m^2
h	Beam Thickness, m
I	Moment of inertia, m^4
K	Stiffness matrix, F/m
κ	Shear correction factor
M	Mass Matrix, kg
N	Number of terms in polynomial-series shape functions
n	Material volume fraction index
P	Force, N
q_j	Time-dependent unknown parameters
t	time, s.
U_{ex}	Work done by Force, N.m
U_k	kinetic energy, N.m
U_s	Strain energy, N.m
W	Deflection, m
u, v, w	Components of displacement, m

$\tilde{u}, \tilde{v}, \tilde{w}$	Components of displacement in middle plane (z = 0), m
x,y,z	Rectangular coordinates
ω	Fundamental frequency, radian/s
ρ	Density, kg/m^3
$\tilde{\psi}$	Rotation of beam cross-section, radian
ε	Strain, m/m
γ	Shear strain, radian
ν	Poisson's ratio
Π	Total potential energy, N.m
Δ	Polynomial-series shape functions

REFERENCES

[1] Miyamoto, Y., Kaysser, W.A., Rabin, B.H., Kawasaki, A. and Ford, R.G. Functionally graded materials: design, processing and application, 1999, Kluwer Academic Publishers, London.

[2] Aydogdu, M. and Taskin, V. Free vibration analysis of functionally graded beams with simply supported edges, Materials and Design, Vol. 28, 2007, pp. 1651-1656.

[3] Sina, S.A., Navazi, H.M. and Haddadpour, H. An analytical method for free vibration analysis of functionally graded beams, Materials and Design, Vol. 30, 2009, pp. 741-747.

[4] Simsek, M. Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories, Nuclear Engineering and Design, Vol. 240, 2010, pp. 697-705.

[5] Wattanasakulpong, N., Prusty, B.G. and Kelly, D.W. Thermal buckling and elastic vibration of third-order shear deformable functionally graded beams, International Journal of Mechanical Sciences, Vol. 53, 2011, pp. 734-743.

[6] Hein, H. and Feklistova, L. Free vibrations of non-uniform and axially functionally graded beams using Haar wavelets, Engineering Structures, Vol. 33, 2011, pp. 3696-3701.

[7] Xiang, H.J. and Yang, J. Free and forced vibration of a laminated FGM Timoshenko beam of variable thickness under heat conduction, Composites Part B: Engineering, Vol. 39, 2008, pp. 292-303.

[8] Banerjee, J.R. and Ananthapuvirajah, A. Free vibration of functionally graded beams and frameworks using the dynamic stiffness method, Journal of Sound and Vibration, Vol. 422, 2018, pp. 34-47.

[9] Mahmoud, M.A. Natural frequency of axially functionally graded, tapered cantilever beams with tip masses, Engineering Structures, Vol. 187, 2019, pp. 34-42.

[10] Lohar, H., Mitra, A. and Sahoo, S. Large amplitude forced vibration analysis of an axially functionally graded tapered beam resting on elastic foundation, Materialstoday: Proceedings, Vol. 5, 2018, pp. 5303-5312.

[11] Sayyad, A.S. and Ghugal, Y.M. A sinusoidal beam theory for functionally graded sandwich curved beams, Composite Structures, Vol. 226, 2019, 111246.

[12] Songsuwan, W., Pimsarn, M. and Wattanasakulpong, N. Dynamic responses of functionally graded sandwich beams resting on elastic foundation under harmonic moving loads, International Journal of Structural Stability and Dynamics, Vol. 18, 2018, 1850112.