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Research Article

A THEORETICAL STUDY ON FUNDAMENTAL OF VEHICLE BEHAVIOR UNDER FORCE CONTROL

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ABSTRACT:
Drivers move steering wheel by controlling both angle and torque of steering wheel. The steering manner that drivers steer only by torque is called "force control." Since a dynamics of steering system is added to vehicle planar motion under force control, natural frequencies under force control differ from yaw natural frequency. To improve vehicle dynamic behavior, hence, not only yaw natural frequency, but also natural frequencies under force control are important. Therefore this paper aims to obtain symbolic natural frequencies under force control. Firstly, this paper obtains a characteristic equation under force control, which is expressed as a quartic equation of Laplace operator s. Secondly, Laplace operator s is replaced another variable to transform the quartic equation into a biquadratic equation. Solving the biquadratic equation, this paper obtains symbolic formulas of the frequencies. Considering these formulas, this paper proposes an improvement method of vehicle behavior under force control.

Keywords: *Force control, Natural frequency, Steering system, Vehicle dynamics, Automobile*

1. INTRODUCTION

Drivers control the steering wheel with both torque and angle [1]. The control method of steering wheel with only torque is defined as "force control", on the other side, that with only the angle is defined as "position control". For the transient response of position control, performance of handling is designed based on the formulas of yaw natural frequency and its damping ratio.

Therefore in order to design performance of handling under force control in the same way as performance design for position control transient response, we require formulas for natural frequencies and damping ratio under force control. However, since the characteristic equation under force control is a fourth-order equation for Laplace operator *s*, it is difficult to obtain exact formulas for the natural frequencies. However, the formulas have been obtained only for a special case such as neutral-steer vehicles [2]. However, all actual vehicles feature under-steer characteristics. Therefore in this paper, we obtain the formulas using 2-stage approximation for the natural frequencies and damping ratios under force control in vehicles with ordinary steering characteristics. Based on these formulas, moreover, we consider a method of configuring basic design parameters in order to improve transient response under force control.

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2. MODEL

2.1 Vehicle model

The model shown in Figure 1 describes planar motion with 2 degrees of freedom. Here, m is the vehicle mass, l_f is the distance from the center of gravity to the front wheels, l_r is the distance from the center of gravity to the rear wheels, I_z is the yaw moment of inertia, V is the vehicle velocity, r is the yaw angular velocity, β is the vehicle sideslip angle at the position of the center of gravity, β_f is the vehicle sideslip angle at the front wheels, β_r is the vehicle sideslip angle at the rear wheels, δ is the steering angle, K_f is the front wheel cornering power, K_r is the rear wheel cornering power, F_f is the front wheel lateral force, F_r is the rear wheel lateral force, and l is the wheelbase ($l = l_f + l_r$). Figure 2 shows the model of steering system. In this figure, ξ is the sum of the caster trail and pneumatic trail, I_h is the steering system moment of inertia, and T_h is the steering torque. In this model, steering system damping and friction are ignored, further the overall gear ratio is considered to be 1. The equations of motion for this vehicle model can be expressed as follows [3].

$$mV(r + \dot{\beta}) = F_f + F_r \tag{1}$$

$$I_z \dot{r} = l_f F_f - l_r F_r \tag{2}$$

$$I_h \ddot{\delta} = -\xi F_f + T_h \tag{3}$$

$$F_f = -2K_f(\beta_f - \delta) \tag{4}$$

$$F_r = -2K_r \beta_r \tag{5}$$

$$\beta_f = \beta + \frac{l_f}{V} r \tag{6}$$

$$\beta_r = \beta - \frac{l_r}{V} r \tag{7}$$

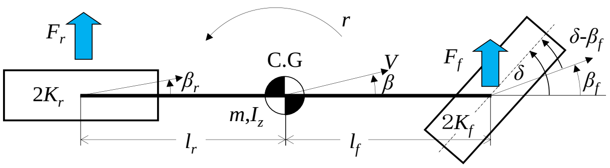


Fig. 1. Model of planar motion system.

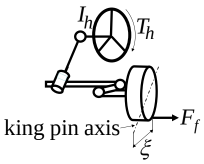


Fig. 2. Model of steering system.

2.2 Simplifying the notation

In order to eliminate m from the response parameters, the non-dimensional yaw radius of inertia k_N is defined by Eq. (8). In the case of passenger cars, $k_N^2 \approx 1$ [4].

$$k_N \equiv \sqrt{\frac{l_f l_r m}{I_z}} \tag{8}$$

Further, the front wheel normalized cornering power C_f is defined by Eq. (9), and the rear wheel normalized cornering power C_r is defined by Eq. (10).

$$C_f \equiv \frac{2K_f}{l_r m / l} \tag{9}$$

$$C_r \equiv \frac{2K_r}{l_f m / l} \tag{10}$$

In order to simplify the formulas for the natural frequencies and damping ratios to be obtained, we use the normal load distribution ratio of front axle as p. As a result, l_f and l_r are expressed by eqs. (11) and (12).

$$l_f = (1-p)l \tag{11}$$

$$l_r = pl \tag{12}$$

Using p, we perform the approximation shown in Appendix A-1.

2.3 Vehicle response under force control

Using eqs. (1) - (12), we obtain the transfer function of r against T_h as follows.

$$\frac{r(s)}{T_{hN}(s)} = \frac{C_f}{I_h k_N{}^2 l\xi V} \cdot \frac{sV + C_r}{s^4 + A_3 s^3 + A_2 s^2 + A_1 s + A_0}$$

where

$$A_3 \cong \frac{C_f}{k_N V} + \frac{C_r}{k_N V}$$

$$A_2 = -\frac{C_f}{k_N{}^2 l} + \frac{C_r}{k_N{}^2 l} + \frac{C_f C_r}{k_N{}^2 V^2} + \frac{C_f mp\xi}{I_h}$$

$$A_1 \cong \frac{C_f C_r mp\xi}{I_h k_N V}$$

$$A_0 = \frac{C_f C_r mp\xi}{I_h k_N{}^2 l} \tag{13}$$

Since s in the denominator in the above equation is a fourth-order equation, it is difficult to obtain a symbolic solution to the natural frequencies and damping ratios usually. However, only in the case of $C_f = C_r = C$ and $k_N = 1$, it is possible to factorize the right-side denominator in Eq. (13) as shown in Eq. (14) [2].

$$\frac{r(s)}{T_h(s)} = \frac{C}{I_h lV} \frac{1}{\left[s^2 + \frac{C}{V}s + \frac{1}{2}\frac{pmC\xi}{I_h}\left(1 + \sqrt{1 - 4\frac{I_h}{pml\xi}}\right)\right]} \frac{Vs + C}{\left[s^2 + \frac{C}{V}s + \frac{1}{2}\frac{pmC\xi}{I_h}\left(1 - \sqrt{1 - 4\frac{I_h}{pml\xi}}\right)\right]} \tag{14}$$

3. FIRST APPROXIMATION FORMULAS FOR NATURAL FREQUENCIES AND DAMPING RATIOS

3.1 Separation of the steering system and body system

The part remaining when the steering system is removed from the vehicle is called “body system” in this paper. As shown in Figure 3, we assume that F_f is the output from the steering system and the input to the body system. Under this assumption, this paper assumes that “the natural frequency of the steering system is sufficiently higher than the natural frequency of the body system”. (This assumption is here indicated as “Assumption 1”.)

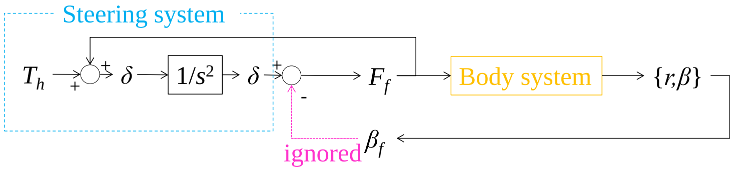


Fig. 3. Quasi border between steering system and body system.

3.2 Body system natural frequency and damping ratio

This section obtains natural frequency for body system. At the body system natural frequency, the steering system behavior is quasi-static from assumption 1. Thus we can ignore the effect of I_h below this frequency. Therefore, Eq. (3) can be reduced below in this frequency.

$$F_f \approx \frac{T_h}{\xi} \tag{15}$$

Since this equation means that drivers input F_f to the vehicle as T_h, eq. (4) can be ignored. Therefore the equations of motion below the frequency consist of eqs. (15), (1), (2), (5) and (7). From these equations, when we obtain the following equation.

$$\frac{r(s)}{T_h(s)} \approx \frac{r(s)}{\xi F_f(s)} = \frac{1}{k_N{}^2 lmp\xi V} \cdot \frac{Vs + C_r}{s^2 + 2\zeta_B \omega_B s + \omega_B{}^2} \tag{16}$$

Here,

$$\omega_B = \sqrt{\frac{C_r}{k_N{}^2 l}} \tag{17}$$

$$\varsigma_B \omega_B \cong \frac{1}{2k_N} \frac{C_r}{V} \tag{18}$$

Here, ω_B is the body system first approximate natural frequency, and ζ_B is the first approximate damping ratio.

3.3 Steering system natural frequency and damping ratio

This section obtains the natural frequency of steering system. Based on assumption 1, it is almost impossible for the motion of the body system to follow the motion of the steering system at the steering system natural frequency. Therefore, we approximate as follows.

$$\beta_f \approx 0 \tag{19}$$

As a result, based on the above equation, eq. (3) and (4), the response of $\xi F_f(s)$ for $T_h(s)$ is expressed by the following.

$$\frac{\xi F_f(s)}{T_h(s)} = \frac{2\xi K_f(\beta_f(s) - \delta(s))}{T_h(s)} \approx 2\xi K_f \frac{\delta(s)}{T_h(s)} \tag{20}$$

Furthermore, we obtain $\delta(s)/T_h(s)$, which is the right hand of eq. (20). Using Eq. (19), Eq. (3) becomes the following.

$$I_h \ddot{\delta} \approx -2\xi K_f \delta + T_h \tag{21}$$

Reducing eq.(21), we obtain the following.

$$\frac{\delta(s)}{T_h(s)} \approx \frac{1}{I_h} \frac{1}{s^2 + \frac{2\xi K_f}{I_h}} \tag{22}$$

Using eq. (20) and Eq. (22), the transfer function of $F_f(s)$ against $T_h(s)$ is the following.

$$\frac{\xi F_f(s)}{T_h(s)} \approx \frac{\omega_S{}^2}{s^2 + \omega_S{}^2} \tag{23}$$

Here, ω_S is the first approximation of the steering system natural frequency, expressed as follows [3].

$$\omega_S{}^2 = \frac{2\xi K_f}{I_h} \tag{24}$$

Further we add the symbolic damping terms to Eq. (23), this term will be determined by the coefficient comparison method.

$$\frac{\xi F_f(s)}{T_h(s)} \approx \frac{\omega_S{}^2}{s^2 + 2\zeta_S \omega_S s + \omega_S{}^2} \tag{25}$$

Here, ζ_S stands for the steering system damping ratio. Further, from the product of Eq. (25) and Eq. (16), the first approximation for the transfer function of r against T_h is expressed as follows.

$$\begin{aligned} \frac{r(s)}{T_h(s)} &= \frac{\xi F_f(s)}{T_h(s)} \frac{r(s)}{\xi F_f(s)} \\ &\approx \frac{\omega_S{}^2}{k_N{}^2 lmp\xi V} \cdot \frac{1}{s^2 + 2\zeta_S \omega_S s + \omega_S{}^2} \cdot \frac{Vs + C_r}{s^2 + 2\zeta_B \omega_B s + \omega_B{}^2} \\ &= \frac{C_f}{I_h k_N{}^2 l\xi V} \cdot \frac{sV + C_r}{s^4 + D_3 s^3 + D_2 s^2 + D_1 s + D_0} \end{aligned} \tag{26}$$

where

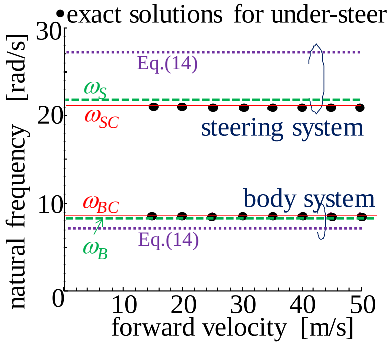
$$\begin{aligned} D_3 &= 2\zeta_S \omega_S + 2\zeta_B \omega_B \\ D_2 &= \omega_S{}^2 + \omega_B{}^2 + 4\zeta_S \omega_S \zeta_B \omega_B \\ D_1 &= 2\zeta_S \omega_S \omega_B{}^2 + 2\zeta_B \omega_B \omega_S{}^2 \\ D_0 &= \omega_S{}^2 \omega_B{}^2 \end{aligned}$$

From Eq.(26) and (13), we can obtain $D_3 \approx 2\zeta_S\omega_S + C_r/(2k_N V)$. Comparing this D_3 and A_3 in Eq. (13), we can obtain the following formula.

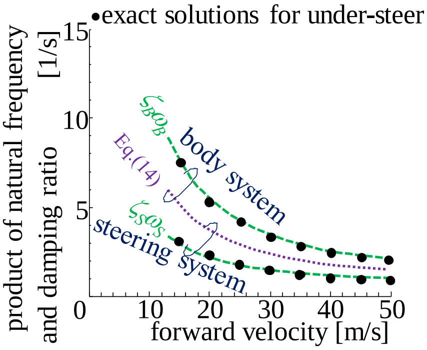
$$\varsigma_S \omega_S \cong \frac{C_f}{2k_N V} \tag{27}$$

3.4 Accuracy of approximation

Comparisons of ω_S, ω_B, and their exact solutions are shown in Figure 4(A). Further, those for $\zeta_S\omega_S$ and $\zeta_B\omega_B$ are shown in Figure 4(B). As shown fig. 4(A) and (B), the approximation error for ω_S, ω_B, $\zeta_S\omega_S$, and $\zeta_B\omega_B$ is much smaller than that of eq.(14). Therefore the author considers that these symbolic formulas are valid.



(A) Natural frequencies



(B) Damping ratios

Fig. 4. Approximation accuracy of natural frequencies (V= 24.5 [m/s], C_f =100[m/s^2], C_r =200[m/s^2], C =150[m/s^2], I_h=21.0[kg m^2], ξ=0.10[m], m=2000[kg], p=0.535, k_N^2=0.935 and l =3.00[m]).

4. Second Approximation Formulas for Natural Frequencies and Damping Ratios

4.1 Response parameters considering coupled motion (Second approximation)

Using the denominators of Eq. (16) and Eq. (25), eq. (13) is reduced into the following equation.

$$\frac{r(s)}{T_h(s)} \cong \frac{\omega_S^2}{k_N^2 lmp\xi V} \cdot \frac{Vs + C_r}{[(s^2 + 2\zeta_S\omega_S s + \omega_S^2)(s^2 + 2\zeta_B\omega_B s + \omega_B^2)] - (C_f / k_N^2 l)s^2 - 2\zeta_B\omega_B (C_f / k_N^2 l)s} \tag{28}$$

The approximation error for ω_S, $\zeta_S\omega_S$, ω_B, and $\zeta_B\omega_B$ is caused by the two terms outside the [ ] of the denominator in the right side of the above equation (hereafter referred to as the “coupling terms”). Therefore we expect that the approximation error will be further reduced when we correct ω_S or $\zeta_S\omega_S$, ω_B, and $\zeta_B\omega_B$ so that all the coupling terms are nearly 0. The equation with ω_S or $\zeta_S\omega_S$, ω_B, and $\zeta_B\omega_B$ corrected so that the coupled terms are almost 0 is expressed as follows.

$$\frac{r(s)}{T_h(s)} \cong \frac{\omega_S^2}{k_N^2 lmp\xi V} \cdot \frac{1}{[(1+\Delta_{S2})s^2 + 2(1+\Delta_{S1})\zeta_S\omega_S s + \omega_S^2]} \cdot \frac{Vs + C_r}{[(1+\Delta_{B2})s^2 + 2(1+\Delta_{B2})\zeta_B\omega_B s + \omega_B^2]} \tag{29}$$

The coefficient of Δ in the above equation is coupling terms. Since Eq. (29) and Eq. (28) are equal, when we compare the coefficients for each order of s in the denominators of Eq. (28) and Eq. (29), we obtain the following equations.

$$\begin{aligned}
&s^4 : \Delta_{S2} + \Delta_{B2} = 0 \\
&s^3 : 2\zeta_B\omega_B\Delta_{S2} + 2\zeta_S\omega_S\Delta_{S1} + 2\zeta_S\omega_S\Delta_{B2} + 2\zeta_B\omega_B\Delta_{B1} = 0 \\
&s^2 : \omega_B^2\Delta_{S2} + 2\zeta_S\omega_S \cdot 2\zeta_B\omega_B\Delta_{S1} + \omega_S^2\Delta_{B2} + 2\zeta_S\omega_S \cdot 2\zeta_B\omega_B\Delta_{B1} = -C_f / k_N^2 l \\
&s \ \ : 2\zeta_B\omega_B\omega_S^2\Delta_{B1} + 2\zeta_S\omega_S \cdot \omega_S^2\omega_B^2\Delta_{S1} = -2\zeta_B\omega_B C_f / k_N^2 l
\end{aligned} \tag{30}$$

Deriving the above equations, we assume that the absolute value of the products of the coupling terms are significantly smaller than 1 and can therefore be ignored. (This assumption is indicated as “assumption 2”.) Solving for four coupling terms in the above equations and ignoring the small terms produces the following formula.

$$\Delta_{S2} = \Delta_{S1} = -\Delta_{B2} = -\Delta_{B1} \approx \frac{C_f / k_N^2 l}{\omega_S^2} = \frac{I_h}{k_N^2 pml\xi} \tag{31}$$

Since the maximum value of $I_h/k_N^2 pml\xi$ measured in a passenger car is 0.2 [5], the maximum absolute value of the product of the coupling terms is 0.04. Therefore Assumption 2 is true. Substituting Eq. (31) for Eq. (29), we can obtain the transfer function of r(s) against $T_h(s)$ as follows.

$$\frac{r(s)}{T_h(s)} \approx \frac{1}{k_N{}^2 lmp\xi V} \cdot \frac{\omega_S{}^2}{\left(1+\frac{I_h}{k_N{}^2 pml\xi}\right)s^2 + 2\left(1+\frac{I_h}{k_N{}^2 pml\xi}\right)\varsigma_S \omega_S s + \omega_S{}^2} \cdot \frac{Vs + C_r}{\left(1-\frac{I_h}{k_N{}^2 pml\xi}\right)s^2 + 2\left(1-\frac{I_h}{k_N{}^2 pml\xi}\right)\varsigma_B \omega_B s + \omega_B{}^2} \tag{32}$$

Therefore, the second approximate steering system natural frequency ω_{SC}, its damping ratio ζ_{SC}, the second approximate body system natural frequency ω_{BC} and its damping ratio ζ_{BC} are described below.

$$\omega_{SC} = \frac{1}{\sqrt{1+\frac{I_h}{k_N{}^2 pml\xi}}}\omega_S \approx \sqrt{1-\frac{I_h}{k_N{}^2 pml\xi}}\sqrt{\frac{pmC_f\xi}{I_h}} \tag{33}$$

$$\omega_{BC} = \frac{1}{\sqrt{1-\frac{I_h}{k_N{}^2 pml\xi}}}\omega_B \approx \sqrt{1+\frac{I_h}{k_N{}^2 pml\xi}}\sqrt{\frac{C_r}{k_N{}^2 l}} \tag{34}$$

$$\omega_{SC}\zeta_{SC} = \omega_S\zeta_S \cong \frac{C_f}{2k_N V} \tag{35}$$

$$\omega_{BC}\zeta_{SC} = \omega_B\zeta_B \cong \frac{C_r}{2k_N V} \tag{36}$$

Hence, it is not necessary to correct for $\zeta_S\omega_S$ and $\zeta_B\omega_B$.

4.2 Accuracy of approximation

Figure 4(A) also shows examples of approximation accuracy calculation for eq. (33) and (34). The approximation error with respect to the exact solutions for ω_{SC}, ω_{BC}, $\zeta_S\omega_S$, and $\zeta_B\omega_B$ is less than 1%. Moreover, Figure 5 shows the comparison of Eq. (13) and Eq. (32) on the time domain and frequency domain. Based on these results, it is believed that the accuracy of the second approximation is sufficient for practical purposes. In addition, figure 5 uses the non-dimensional steering torque T_{hN} which is defined as shown below in order to compare the steering torque and steering angle using the same dimensions [rad].

$$T_{hN} \equiv \frac{T_h}{2K_f\xi} \tag{37}$$

4.3 Valid range

Figure 6 shows the approximation accuracy for ζ_{SC} and ζ_{BC}. The accuracy is higher when the value of $I_h/k_N^2 pml\xi$ is lower; however it is believed that the approximation accuracy depends on C_f and C_r also. Therefore, in order to consider these at the same time, $I_h/k_N^2 pml\xi$ is converted to B, which is stability factor under force control [5]. B is expressed by the following formula.

$$B = \frac{\left(\frac{C_f}{C_f + C_r}\right)}{\left[\frac{(I_h/\xi)}{k_N{}^2 pml}\right]} \tag{38}$$

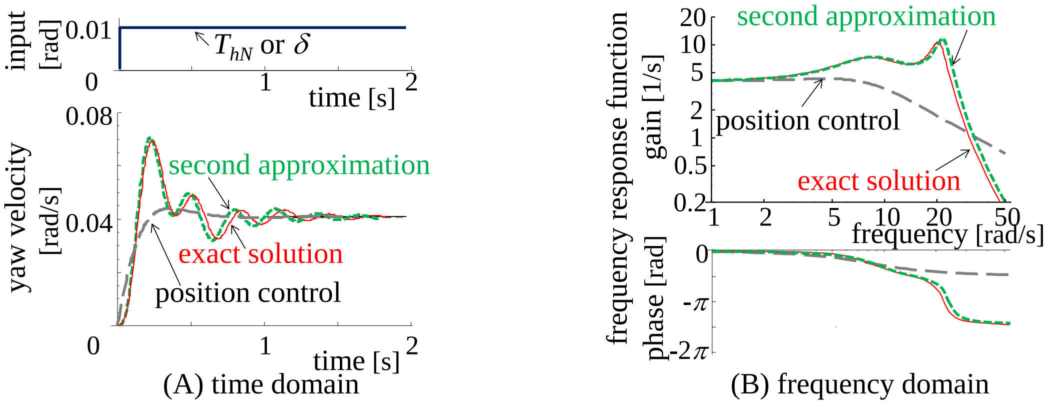


Fig. 5. Approximation accuracy of yaw velocity response ($V=(lC_r)^{1/2}$= 24.5 [m/s], C_f =100[m/s^2], C_r =200[m/s^2], I_h=21.0[kgm^2], ξ=0.10[m], m=2000[kg], p=0.535, $k_N^{\,2}$=0.935 and l=3.00[m]).

Figure 6 shows B as the horizontal axis Vernier scale. In both Figure 6(A) and Figure 6(B), the approximate validity range for eqs. (33) through (36) is $B \geq 2$. Further, the average value of B in actual vehicles is approximately 5 [5]. Since B = 4.76 in the calculation specifications for Figures 4 to 6, these figures are obtained assuming an average vehicle. Therefore eqs. (33) through (36) are valid for an ordinary passenger car.

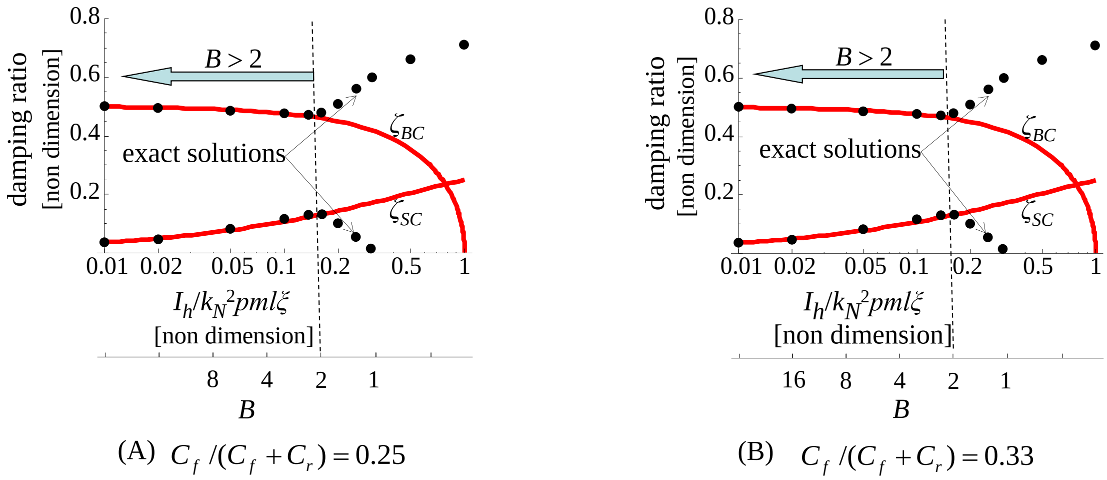


Fig. 6. Validity range of the formulas (V=24.5 [m/s], C_f=100[m/s^2], C_r=200[m/s^2], I_h=21.0[kgm^2], ξ=0.10[m], m=2000[kg], p=0.535, $k_N^{\,2}$=0.935 and l =3.00[m]).

5. Methods of Handling Performance Design for Improving Response

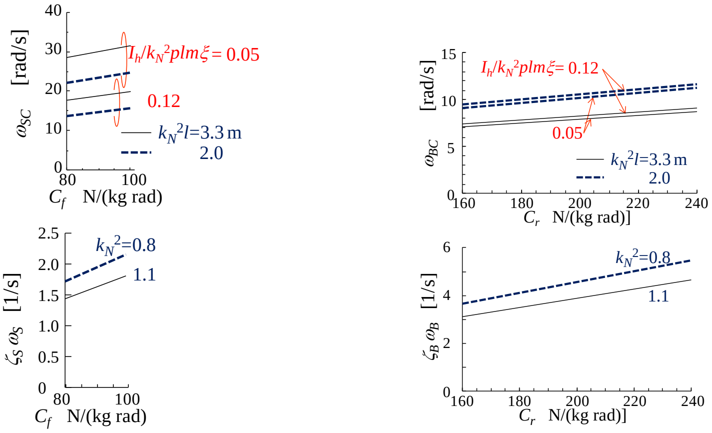
5.1 Response parameters considering coupled motion (Second approximation)

Figure 7 shows the results of a parameter study for ω_{SC}, $\zeta_S\omega_S$, ω_{BC}, and $\zeta_B\omega_B$. It is believed that the parameters which have a degree of design freedom at the steering stability development stage are C_f, C_r, and $I_h/k_N^{\,2}pml\xi$. Based on Figure 7, the basic design policy for these 3 parameters is the following.

1) Set a larger C_f and smaller $I_h/k_N^{\,2}pml\xi$ in order to increase both ω_{SC} and $\zeta_S\omega_S$.

2) Set a larger C_r in order to increase ω_{BC} and $\zeta_B\omega_B$.

When setting these parameters, it is necessary to take care that the stability factor under position control does not become too small.



(A) Front axle (B) Rear axle

Fig. 7. Influences of design parameters on response parameters (V=24.5m/s).

Table 1 shows a specific example of this basic design policy. When the input shown in Figure 8(A) is applied to a vehicle with the specifications in Table 1, the response is shown in Figure 8(B). As shown in Figure 8(C), the response with the improved specifications matches the steady-state response beginning from the second cycle following the first steering (Figure 8(D)). It was possible to improve the force control transient response characteristics as a result of the basic design policy 1) to 2).

Table 1: Vehicle parameters to improve vehicle transient behavior under force control

	unit	origin	improved
C_f	m/s^2	80	160
C_r		160	240
$I_h/k_N^2 plm\xi$	non dimension	0.10	0.09
m	kg	2000	2000
l	m	3	3
k_N	non dimension	0.91	0.91
ξ	m	0.1	0.1
p	non dimension	0.54	0.54

(A) Input torque (B) Lateral acceleration

(C) Response of original vehicle (D) Response of improved vehicle

Fig. 8. Improvement of vehicle transient behavior under force control.

5.2 Design of handling performance

Stability and control consists of multiple performance items. On the other hand, their design parameters are also related to other performances (e.g. ride, noise and vibration). Thus, their design parameters cannot be determined by numerical optimization for a certain metrics of stability and control. Hence, this optimization of the design parameters will be done by insights of the designer. To obtain the insights, symbolic formulas consisting of the design parameters for each performance item should be useful. As one of these formulas, the author believes equations (33) - (36) having the accuracy described in the former section is useful.

6. Conclusions

This paper describes a study aimed at improving the force control transient response in a vehicle with ordinary steering characteristics. The results are summarized below.

1) Formulas were obtained for the natural frequencies and damping ratios by separating the steering system and body system of a vehicle, and finding the characteristic equation for each.

2) The error between these approximation formulas and the exact solution is less than 1%. These approximation formulas are valid for ordinary passenger cars, however they are not valid for cases when the force control stability index B is less than 2.

3) Considering the formulas for natural frequencies and damping ratios, a basic configuration was derived.

Nomenclature

B	stability factor under force control, non-dimension
C	coefficient of cornering stiffness, m/s^2
F	cornering force
$_f$	front axle
I_h	moment of inertia of steering wheel, kg m^2
I_z	yaw moment of inertia, kg m^2
K	cornering stiffness, N/rad
k_N	ratio of radius of yaw moment of inertia
l	wheelbase, m
l_f	length between front axle and C.G., m
l_r	length between rear axle and C.G., m
m	vehicle mass, kg
p	ratio of normal load distribution (l_r/l)
r	yaw velocity, rad/s
$_r$	rear axle
T_{h}	steering torque, N m
T_{hN}	ratio of steering torque
V	vehicle velocity, m/s
β	attitude angle of body, rad
δ	angle of steering wheel, rad
ω_B	natural frequency of body system (first order approximation), rad/s
ω_{BC}	natural frequency of body system (second order approximation), rad/s
ω_S	natural frequency of steering system (first order approximation), rad/s
ω_{SC}	natural frequency of steering system (second order approximation), rad/s
ξ	length of trail, m
ζ_B	damping ratio of body system (first order approximation)
ζ_S	damping ratio of steering system (first order approximation)

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APPENDIX A-1: FORMULA SIMPLIFICATION

When we consider p = 1/2, then the following relationship is satisfied.

$$1 - p + pk_N^2 \cong \frac{1 + k_N^2}{2} \quad \text{(A1)}$$

Further, because $k_N^2 \approx 1$[4], for the terms which include k_N, the formula (A1) which allows first order Taylor expansion for k_N at around $k_N = 1$ produces the following relationship.

$$1 + k_N^2 \cong 2k_N \quad \text{(A2)}$$

Therefore, the following approximation is possible based on Formula (A1) and Formula (A2).

$$1 - p + pk_N^2 \cong k_N \quad \text{(A3)}$$