Novel Generic Analytical Model of Fractance Response

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ABSTRACT

Fractance is often cited in various engineering disciplines for example, analog circuits and systems, biomedical engineering, control engineering and electronic engineering etc. In this research, a novel generic analytical model of the voltage response of fractance in time domain has been proposed. This model composes of two parts for fractance with order ranges from 0 to 1 and larger than 1 respectively. With this model, the asymptotic and transient voltage responses of fractance can also be determined. Unlike the results of the previous works which are applicable only to fractance with certain orders under certain excitations, this model can be applied to fractance of arbitrary order under arbitrary type of periodic excitation. Moreover, it has been shown that the proposed model is in a realistic format. So, this model has been found to be beneficial to various fractance involved engineering disciplines stated above.

Keywords: Fractance, Fractional Order Calculus, Fourier Series, Generalized Trigonometric Functions

1. INTRODUCTION

Fractance or fractional impedance is the impedance which its order, α is not strictly integer but can be fractional [1-3] or arbitrary real value. It is often cited in various engineering disciplines for example, analog circuits and systems, biomedical engineering, control engineering [4] and electronic engineering [5] etc. In many circumstances, various characteristics of fractance are needed to be precisely determined [6-9]. Traditionally, they can be determined by using the numerical simulation in time domain based on the measured voltage data obtained from exciting the fractance with the predetermined current waveform [10]. This traditional methodology is cumbersome and yields non analytical results which are imprecise compared to the analytical ones. It can be seen that much of the effort can be reduced and the precise analytical result can be expected if the time domain responses of such fractance can be analytically obtained. By this motivation, the analytical expressions of these responses have been proposed in previous studies such as [3], [11] and [12] etc. Unfortunately, only fractance with certain under certain excitations have been concerned in these previous works which means that the results of these works are not generic and not versatile.

Hence, a novel versatile generic analytical model of the voltage response of fractance in time domain has been proposed in this research. The proposed model composes of two parts for fractance with $0 < \alpha \leq 1$ and $\alpha > 1$ respectively. By using this model, the asymptotic and transient voltage responses of fractance can also be determined. Unlike the results of the previous works which are applicable only to fractance with certain α under certain excitations, this model can be applied to fractance of arbitrary α under the periodic current excitation of arbitrary type. Such generality and versatility have been obtained from the usage of Fourier series [13] as the basis of the model. Moreover, it has been shown that the proposed model is in a realistic format.

The rest of the paper is organized as follows. The fractional derivative which is a key mathematical tool of this work will be briefly introduced in the subsequent section. The proposed model and its applications will be subsequently given in section 3 and 4. Finally, the discussion which shows that the proposed model is in a realistic format will be presented in section 5 and the conclusion will be drawn in section 6.

2. THE FRACTIONAL DERIVATIVE

Unlike the ordinary derivative, the order of the fractional derivative not strictly integer but can be fractional or arbitrary real value. There exist various mathematical definitions of fractional order derivative [14-16] etc., and the Riemann-Liouvielle definition has been chosen for this research as it is convenient and often cited in many previous works. Let $x(t)$ be an arbitrary function of time, t, its fractional derivative of arbitrary order denoted by γ i.e. $D^{\gamma}[x(t)]$ where $0 < \gamma \leq 1$ can be given in terms of the ordinary derivative and integral by using the Riemann-Liouvielle definition as follows

$$D^{\gamma}[x(t)] = \frac{d}{dt}[X(t,\gamma)] \tag{1}$$

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where

$$X(t,\gamma) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} x(\tau) d\tau \qquad (2)$$

and $\Gamma()$ stands for the Gamma function [13]. By using (1) and (2), $D^\gamma[x(t)]$ can be shortly given as follows

$$D^\gamma[x(t)] = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t (t-\tau)^{\gamma} x(\tau) d\tau \qquad (3)$$

As an illustration of fractional differentiation with Riemann-Liouvielle definition of fractional derivative, it can be seen that if $x(t)$ is a unit step function i.e. $u(t)$ then

$$D^\gamma[x(t)] = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \qquad (4)$$

where $t \geq 0$.

Moreover, if $x(t) = t^k$ where k can be any integer then

$$D^\gamma[x(t)] = \frac{\Gamma(k+1)}{\Gamma(k+1+\gamma)} t^{k-\gamma} \qquad (5)$$

Similarly to the ordinary derivative of x(t), the Laplace transform of $D^\gamma[x(t)]$ can be given by neglecting all initial values as

$$L[D^\gamma[x(t)]] = s^\gamma X(s) \qquad (6)$$

On the other hand, $D^\gamma[x(t)]$ with $\gamma > 1$ can be given according to the Riemann-Liouvielle definition as

$$D^\gamma[x(t)] = \frac{d^m}{dt^m}[X(m,t,\gamma)] \qquad (7)$$

Here, $X(m,t,\gamma)$ is given by

$$X(m,t,\gamma) = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-\tau)^{m-\gamma-1} x(\tau) d\tau \qquad (8)$$

where $m-1 < \gamma \leq m$ and $m \geq 2$. As a result, $D^\gamma[x(t)]$ can be now shortly given as follows

$$D^\gamma[x(t)] = \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\gamma-1} x(\tau) d\tau \qquad (9)$$

For an illustration, if $x(t) = u(t)$ then

$$D^\gamma[x(t)] = \frac{-t^\gamma}{\gamma\Gamma(m-\gamma+1)} \left[\prod_{n=0}^{m}(n-\gamma)\right] \qquad (10)$$

Moreover, if $x(t) = t^k$

$$D^\gamma[x(t)] = \frac{\Gamma(k+1)\left[\prod_{n=0}^{m}(n+k-\gamma)\right] t^{k-\gamma} u(t)}{(k-\gamma)\Gamma(m+1+k-\gamma)} \qquad (11)$$

Finally, the Laplace transform of $D^\gamma[x(t)]$ with $\gamma > 1$ can be given with all initial values being neglected by (6) as well as that of $D^\gamma[x(t)]$ with $0 < \gamma \leq 1$ even though their definitions and their resulting derivatives of the same functions may be different.

3. THE PROPOSED MODEL

In this section, the proposed model will be presented by starting from its derivation. Before proceed further, the mathematical definitions of some useful special functions will be given. Firstly, let be arbitrary variable, the generalized sine function [17] of θ with order α i.e. $\sin_\alpha(\theta)$, can be given in term of a normal sine function as

$$\sin_\alpha(\theta) = \sum_{k=0}^{\infty}\left[\frac{\theta^{k-\alpha}\sin(0.5(k-\alpha)\pi)}{\Gamma(k-\alpha+1)}\right] \qquad (12)$$

If $\alpha = 1$ then

$$\sin_\alpha(\theta) = \sin_1(\theta) = \sin(\theta) \qquad (13)$$

Similarly to $sin_\alpha(\theta)$, the generalized cosine function of θ denoted by $cos_\alpha(\theta)$ can be mathematically defined as

$$\cos_\alpha(\theta) = \sum_{k=0}^{\infty}\left[\frac{\theta^{k-\alpha}\cos(0.5(k-\alpha)\pi)}{\Gamma(k-\alpha+1)}\right] \qquad (14)$$

Then

$$\cos_\alpha(\theta) = \cos_1(\theta) = \cos(\theta) \qquad (15)$$

if $\alpha = 1$. Moreover, let $E_\alpha^{j\theta}$ stands for the generalized exponential function [17] of $j\theta$ with order α, it can be seen that $E_\alpha^{j\theta}$ can be related to $\sin_\alpha(\theta)$ and $\cos_\alpha(\theta)$ by the following identity

$$E_\alpha^{j\theta} = \cos_\alpha(\theta) + j\sin_\alpha(\theta) \qquad (16)$$

So, this identity can be named the generalized Eulers formula. The generalized exponential function can be alternatively defined as a function of t and α in (17) and can be plotted against t and α as shown in Fig.1 [17].

$$E_\alpha^t = \sum_{k=0}^{\infty}\left[\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}\right] \qquad (17)$$

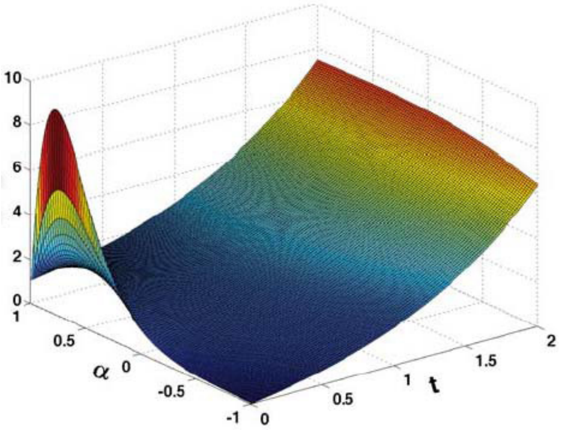

Fig.1: *E_α^t v.s. t and α [17].*

Finally, an often cited special function entitled the Mittag-Leffler function which is very important in the theory of fractional calculus as it is a generalization of many renowned functions such as exponential function and hyperbolic function etc., and always occurred in the solution of the fractional integrodifferential equation [18] will be now introduced. The Mittag-Leffler function of arbitrary variable z with a and b as parameters, $E_{a,b}(z)$ can be defined as [18]

$$E_{a,b}(z) = \sum_{k=0}^{\infty}\left[\frac{z^k}{\Gamma(ak+b)}\right] \tag{18}$$

At this point, it is ready to present the derivation of the proposed model. Mathematically, any periodic exciting current, $I(t)$ with arbitrary period, T can be expressed in term of its Fourier series expansion [13] as

$$I(t) = I_0 + \sum_{n=1}^{\infty}[I_{an}\cos(n\omega t) + I_{bn}\sin(n\omega t)] \tag{19}$$

where ω stands for the angular frequency and

$$I_0 = \frac{1}{T}\int_{-T/2}^{T/2} I(t)dt \tag{20}$$

$$I_{an} = \frac{1}{T}\int_{-T/2}^{T/2} I(t)\cos(n\omega t)dt \tag{21}$$

$$I_{bn} = \frac{1}{T}\int_{-T/2}^{T/2} I(t)\sin(n\omega t)dt \tag{22}$$

For any fractance with magnitude K and order α which its impedance function, $Z(s)$ can be given by

$$Z(s) = Ks^{\alpha} \tag{23}$$

where s denotes the complex frequency, its s-domain voltage response, $V(s)$ can be given in term of its s-domain exciting current, $I(s)$ as follows

$$V(s) = Ks^{\alpha}I(s) \tag{24}$$

and the time domain voltage response, $V(t)$ can be given by taking the inverse Laplace transform to both sides of (24) which yields

$$V(t) = KD^{\alpha}[I(t)] \tag{25}$$

By substituting (19) into (25), $V(t)$ can be found as

$$V(t)=K\left[D^{\alpha}[I_0]+\sum_{n=1}^{\infty}\left[I_{an}D^{\alpha}[\cos(n\omega t)]+I_{bn}D^{\alpha}[\sin(n\omega t)]\right]\right] \tag{26}$$

For fractance with $0 < \alpha \leq 1$, the Riemann-Liouvielle definition of fractional derivative with order ranges from 0 to 1 which is given in short by (3) must be used for defining the fractional derivative operator, $D^{\alpha}[\ ]$. As a result, $V(t)$ can be given by

$$\begin{aligned} V(t) = {} & \frac{KI_0}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)d\tau \\ & + \sum_{n=1}^{\infty}\left[\frac{KI_{an}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (1-\tau)^{\alpha}\cos(n\omega\tau)d\tau\right] \\ & + \sum_{n=1}^{\infty}\left[\frac{KI_{an}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (1-\tau)^{\alpha}\sin(n\omega\tau)d\tau\right] \end{aligned} \tag{27}$$

Since it has been found that

$$\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}d\tau = t^{-\alpha} \tag{28}$$

$$\begin{aligned} & \frac{d}{dt}\int_0^t (t-\tau)\cos(n\omega\tau)d\tau \\ & = (n\omega)^{\alpha}\Gamma(1-\alpha)\left[\cos_{\alpha}(n\omega t)\cos\left(\frac{\alpha\pi}{2}\right)-\sin_{\alpha}(n\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right] \end{aligned} \tag{29}$$

$$\begin{aligned} & \frac{d}{dt}\int_0^t (t-\tau)\sin(n\omega\tau)d\tau \\ & = (n\omega)^{\alpha}\Gamma(1-\alpha)\left[\sin_{\alpha}(n\omega t)\cos\left(\frac{\alpha\pi}{2}\right)-\cos_{\alpha}(n\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right] \end{aligned} \tag{30}$$

, the analytical model of $V(t)$ of fractance with $0 < \alpha \leq 1$ can be obtained as follows

$$V(t) = K\Bigg[\frac{I_0 t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty}\Bigg[(n\omega)^{\alpha}\Bigg[I_{an}\left[\cos_\alpha(n\omega t)\cos\left(\frac{\alpha\pi}{2}\right) - \sin_\alpha(n\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right] + l_{bn}\left[\sin_\alpha(n\omega t)\cos\left(\frac{\alpha\pi}{2}\right) + \cos_\alpha(n\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right]\Bigg]\Bigg]\Bigg] \tag{31}$$

where I_0, I_{an} and Ibn can be given by (20)-(22), moreover, $\sin_\alpha(n\omega t)$ and $\cos_\alpha(n\omega t)$ denote the generalized sine and cosine function of $n\omega t$ with order α respectively.

Since it has been stated in [17] that the generalized trigonometric functions are asymptotically converged to normal ones, $\sin_\alpha(n\omega t)$ and $\cos_\alpha(n\omega t)$ asymptotically become $\sin(n\omega t)$ and $\cos(n\omega t)$ respectively i.e.

$$\lim_{t\to\infty}[\sin_\alpha(n\omega t)] = \lim_{t\to\infty}[\sin(n\omega t)] \tag{32}$$

$$\lim_{t\to\infty}[\cos_\alpha(n\omega t)] = \lim_{t\to\infty}[\cos(n\omega t)] \tag{33}$$

Moreover, it can also be seen that the first term of $V(t)$ is asymptotically very small i.e.

$$\lim_{t\to\infty}\left[\frac{KI_0 t^{-\alpha}}{\Gamma(1-\alpha)}\right] = 0 \tag{34}$$

As a result of (32)-(34), $V(t)$ at asymptotic, $V_{asymp}(t)$ can be given by

$$V_{asymp}(t) = K\sum_{n=1}^{\infty}\left[I_{an}\cos\left(n\omega t + \frac{\alpha\pi}{2}\right) + I_{bn}\sin\left(n\omega t + \frac{\alpha\pi}{2}\right)\right] \tag{35}$$

On the other hand, $V(t)$ at transient, $V_{trans}(t)$ can be found by using (31) and (35) as follows

$$V_{trans}(t) = K\Bigg[\frac{I_0 t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty}\Bigg[(n\omega)^{\alpha}\Bigg[I_{an}\Big[(\cos_\alpha(n\omega t) - \cos(n\omega t))\cos\left(\frac{\alpha\pi}{2}\right) - (\sin_\alpha(n\omega t) - \sin(n\omega t))\sin\left(\frac{\alpha\pi}{2}\right)\Big] + I_{bn}\Big[(\sin_\alpha(n\omega t) - \sin(n\omega t))\cos\left(\frac{\alpha\pi}{2}\right) + (\cos_\alpha(n\omega t) - \cos(n\omega t))\sin\left(\frac{\alpha\pi}{2}\right)\Big]\Bigg]\Bigg]\Bigg] \tag{36}$$

Moreover, it has been found that

$$\lim_{t\to\infty}[V_{trans}(t)] = 0 \tag{37}$$

Now, fractance with $\alpha > 1$ will be considered. So, the Riemann-Liouvielle definition of fractional derivative with order larger than 1 which is given in short by (9) must be utilized for defining $D^\alpha[\ ]$. So, (26) become

$$V(t) = \frac{KI_0}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}d\tau + \sum_{n=1}^{\infty}\left[\frac{KI_0}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}\cos(n\omega\tau)d\tau\right] + \sum_{n=1}^{\infty}\left[\frac{KI_0}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}\sin(n\omega\tau)d\tau\right] \tag{38}$$

which can be rewritten as

$$V(t) = V_{DC}(t) + V_{\sin}(t) + V_{\cos}(t) \tag{39}$$

where

$$V_{DC}(t) = \frac{KI_0}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}d\tau \tag{40}$$

$$V_{\sin}(t) = \sum_{n=1}^{\infty}\left[\frac{KI_0}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}\sin(n\omega\tau)d\tau\right] \tag{41}$$

$$V_{\cos}(t) = \sum_{n=1}^{\infty}\left[\frac{KI_0}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}\cos(n\omega\tau)d\tau\right] \tag{42}$$

After performing the integration and derivative in an iterative manner, $V_{DC}(t)$ can be found as

$$V_{DC}(t) = \frac{-KI_0}{\alpha\Gamma(m-\alpha+1)}\left[\prod_{n=0}^{m}(n-\alpha)\right]t^{-\alpha} \tag{43}$$

For determining $V_{\sin}(t)$, we simply use the following identity [13]

$$\sin(\theta) = \sum_{p=0}^{\infty}\left[\frac{(-1)^p(\theta)^{2p+1}}{\Gamma(2p+2)}\right] \tag{44}$$

So, $V_{\sin}(t)$ can be given by

$$V_{\sin}(t) = \sum_{n=1}^{\infty}\left[\frac{KI_{bn}}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1} \sum_{p=0}^{\infty}\frac{(-1)^p(n\omega t)^{2p+1}}{\Gamma(2p+2)}d\tau\right] \quad (45)$$

After integration and differentiation in an iterative manner, $V_{\sin}(t)$ can be found as

$$V_{\sin}(t) = KI_{bn}\sum_{k=0}^{\infty}\left[\frac{\omega t^{m+1-\alpha}(-(\omega t)^2)^k}{\Gamma(2k+m+2-\alpha)}\right] \sum_{p=0}^{\infty}\left[\prod_{n=1}^{m}(n+2p+1-\alpha)\right] \quad (46)$$

and can be rearranged as follows

$$V_{\sin}(t) = KI_{bn}\omega t^{n+1-\alpha}\sum_{k=0}^{\infty}\left[\frac{((-\omega t)^2)^k}{\Gamma(2k+m+2-\alpha)}\right] \sum_{p=0}^{\infty}\left[\prod_{n=1}^{m}(n+2p+1-\alpha)\right] \quad (47)$$

Since it can be found that

$$\sum_{k=0}^{\infty}\left[\frac{((-\omega t)^2)^k}{\Gamma(2k+m+2-\alpha)}\right] = E_{2,m+2-\alpha}(-(\omega t)^2) \quad (48)$$

which is the Mittag-Leffler function with $a = 2$, $b = m+2-\alpha$ and $x = -(\omega t)^2$, $V_{\sin}t$ can be given in term of the Mittag-Leffler function as follows

$$V_{sin}(t) = KI_{bn}\omega t^{m+1-\alpha}E_{2,m+2-\alpha}(-(\omega t)^2) \sum_{p=0}^{\infty}\left[\prod_{n=1}^{m}(n+2p+1-\alpha)\right] \quad (49)$$

For determining $V_{\cos}(t)$, we simply use the following identity [13]

$$\cos(\theta) = \sin\left(\frac{\pi}{2}-\theta\right) \quad (50)$$

So, $V_{cos}(t)$ can be determined by using (49) and (50) as

$$V_{\cos}(t) = KI_{an}\omega^{\alpha-m}\left(\frac{\pi}{2}-\omega t\right)^{m+1-\alpha}E_{2,m+2-\alpha}\left(-\left(\frac{\pi}{2}-\omega t\right)^2\right) \times\sum_{p=0}^{\infty}\left[\prod_{n=1}^{m}(n+2p+1-\alpha)\right] \quad (51)$$

Finally, the analytical model of $V(t)$ of fractance with $\alpha > 1$ can be obtained as given by (52) where I_0, I_{an} and I_{bn} can also be given by (20)-(22).

By using the proposed model i.e. (31) and (52) with I_0, I_{an} and I_{bn} as given by (20)-(22) for fractance with $0 < \alpha \leq 1$ and $\alpha > 1$ respectively, $V(t)$ of fractance of arbitrary α due to arbitrary periodic $I(t)$ can be obtained as will be shown in the subsequent section.

$$V(t) = K\left[\frac{-I_0 t^{\alpha}}{\alpha\Gamma(m-\alpha+1)}\left[\prod_{n=0}^{m}(n-\alpha)\right] + \left[I_{an}\omega^{\alpha-m}\left(\frac{\pi}{2}\right)^{m+1-\alpha}E_{2,m+2-\alpha}\left(-\left(\frac{\pi}{2}-\omega t\right)^2\right) + I_{bn}\omega t^{m+1-\alpha}E_{2,m+2-\alpha}(-(\omega t)^2)\right]\sum_{p=0}^{\infty}\left[\prod_{n=1}^{m}(n+2p+1-\alpha)\right]\right] \quad (52)$$

It can be seen that the applicability to fractance of arbitrary due to arbitrary periodic I(t) is the superiority of the proposed model over the results of the previous works which are applicable to only to fractance with certain values of excited by certain excitations, for example, the results of [3] are applicable only to fractance with $\alpha = 0.5$ and the main result of [11] given by

$$V(t) = KI_0\omega^{\alpha}\left[\sin_{\alpha}(\omega t)\cos\left(\frac{\alpha\pi}{2}\right) + \cos_{\alpha}(\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right] \quad (53)$$

is applicable only to fractance with $0 < \alpha \leq 1$ excited by $I(t) = I_0\sin(\omega t)$. So, this model which is obviously more generic and versatile as it can be applied to fractance with arbitrary α excited by arbitrary periodic $I(t)$, has been found to be beneficial to the analysis and design of any fractance involved systems.

4. ITS APPLICATION

In this section, the applications of the proposed model to both single fractance and fractional order circuit i.e. the circuit which which the ordinary impedances have been replaced by fractances, will be presented respectively. For demonstrating the application to a single fractance, analytically obtaining $V(t)$ of fractance with arbitrary α due to arbitrary periodic $I(t)$ will be presented where both fractance with $0 < \alpha \leq 1$ and with $\alpha > 1$ will be considered. Firstly, let fractance with $0 < \alpha \leq 1$ be excited by a rectified sinusoidal current with magnitude $I_M, T = 2\pi$ and zero phase shift which can be given by

$$I(t) = |I_M\sin(t)| \quad (54)$$

With this $I(t)$, $V(t)$ for fractance with arbitrary K and α where$0 < \alpha \leq 1$ can be obtained by applying part of the proposed model for fractance with $0 < \alpha \leq 1$ as follows

$$V(t) = \frac{4KI_M}{\pi}\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty}\left[\frac{n^{\alpha}}{1-n^2}\left[\cos_{\alpha}(nt)\cos\left(\frac{\alpha\pi}{2}\right) - \sin_{\alpha}(nt)\sin\left(\frac{\alpha\pi}{2}\right)\right]\right]\right] \tag{55}$$

It can be seen that this $V(t)$ can be alternatively given for $0 \leq t \leq 2\pi$ as.

$$V(t) = \left|KI_M\left[\cos_{\alpha}(t)\sin\left(\frac{\alpha\pi}{2}\right) + \sin_{\alpha}(t)\cos\left(\frac{\alpha\pi}{2}\right)\right]\right| \tag{56}$$

which is a periodic function with $T = 2\pi$.

At asymptotic, this $V(t)$ become $V_{asymp}(t)$ which can be given by

$$V_{asymp}(t) = \frac{4KI_M}{\pi}\sum_{n=1}^{\infty}\left[\frac{n^{\alpha}}{1-n^2}\cos\left(nt + \frac{\alpha\pi}{2}\right)\right] \tag{57}$$

and can be plotted against t and α by assuming that $I_M = 1$ A. and $K = 1$ as depicted in Fig. 2.

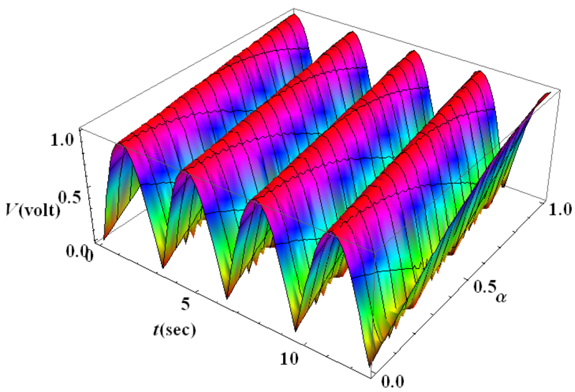

Fig.2: *$V_{asymp}(t)$ due to $I(t) = |I_M \sin(t)|$ of fractance when I_M = 1A. and $K = 1$.*

Furthermore, the corresponding $V_{trans}(t)$ of this $V(t)$ can be analytically given by using (56) and (57) as

$$V_{trans}(t) = \frac{4KI_M}{\pi}\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty}\left[\frac{n^{\alpha}}{1-n^2}\left[(\cos_{\alpha}(nt) - \cos(nt))\cos\left(\frac{\alpha\pi}{2}\right) - \sin_{\alpha}(nt) - \sin(nt)\sin\left(\frac{\alpha\pi}{2}\right)\right]\right]\right] \tag{58}$$

As another example, let the fractance with $0 < \alpha \leq 1$ be excited by a sawtooth current with magnitude I_M and $T = 2\pi$ which can be analytically given for $0 \leq t \leq 2\pi$ as

$$I(t) = I_M t \tag{59}$$

where

$$I(t + 2m\pi) = I(t) \tag{60}$$

for $\{m\} = \{1, 2, 3, \ldots\}$.

By using part of the proposed model for fractance with $0 < \alpha \leq 1$, $V(t)$ of fractance with arbitrary K and α can be obtained as follows

$$V(t) = 2KI_M\left[\frac{\pi t^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{n=1}^{\infty}\left[n^{\alpha-1}\left[\sin_{\alpha}(nt)\cos\left(\frac{\alpha\pi}{2}\right) + \cos_{\alpha}(nt)\sin\left(\frac{\alpha\pi}{2}\right)\right]\right]\right] \tag{61}$$

where $V_{trans}(t)$ and $V_{asymp}(t)$ for this case can be simply found respectively as

$$V_{trans}(t) = 2KI_M\left[\frac{\pi t^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{n=1}^{\infty}\left[n^{\alpha-1}\left[(\sin_{\alpha}(nt) - \sin(nt))\cos\left(\frac{\alpha\pi}{2}\right) + (\cos_{\alpha}(nt) - \cos(nt))\sin\left(\frac{\alpha\pi}{2}\right)\right]\right]\right] \tag{62}$$

$$V_{asymp}(t) = -2KI_M\sum_{n=1}^{\infty}\left[n^{\alpha-1}\sin\left(nt + \frac{\alpha\pi}{2}\right)\right] \tag{63}$$

Now, consider a half order inductor which is a practical fractance device. Let the magnitude of such half order inductor be K, its $Z(s)$ can be given by

$$Z(s) = s^{0.5}K \tag{64}$$

By using the continue fraction expansion, $s^{0.5}$ can be realized in term of a rational integer order polynomial as follows [19]

$$s^{0.5} = \frac{11s^5 + 165s^4 + 462s^3 + 330s^2 + 55s + 1}{s^5 + 55s^4 + 330s^3 + 462s^2 + 165s + 11} \tag{65}$$

So, based on the assumption that $K = 1$, this half order inductor can be constructed as a passive ladder network composed of ordinary resistors and inductors as depicted in Fig. 3.

By letting $\alpha = 0.5$ and $K = 1$ in (61) which can be obtained by using part of the proposed model for fractance with $0 < \alpha \leq 1$, $V(t)$ of this unity magnitude half order inductor excited by a sawtooth current defined by (59) and (60) can be given as

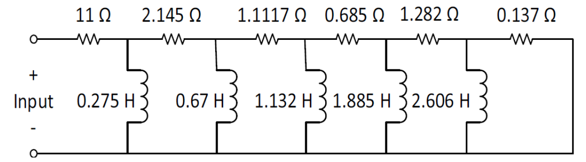

Fig.3: *Passive network realization of a half order inductor with unity magnitude.*

$$V(t) = 2I_M\left[0.56419\pi t^{-0.5} - \sum_{n=1}^{\infty}\left[(2n)^{0.5}\left[\sin_{0.5}(nt) + \cos_{0.5}(nt)\right]\right]\right] \quad (66)$$

and can be given for $0 \leq t \leq 2\pi$ by

$$V(t) = 1.1284\ I_M\sqrt{t} \quad (67)$$

where for $\{m\} = \{1, 2, 3, \ldots\}$,

$$V(t + 2m\pi) = V(t) \quad (68)$$

It can be seen that this $V(t)$ can be plotted against t for $0 \leq t \leq 2\pi = 6.286$ sec., under the assumption that $I_M = 1$ A. as shown in Fig.4. A half order differentiation of the sawtooth function can be observed in this figure and an ordinary differentiation is expected if the device of our interested become an ordinary inductor which has $\alpha = 1$. So, we now let $\alpha = 1$ in order to see that whether the proposed model satisfies this expectation. With $\alpha = 1$ and $K = 1$, $V(t)$ due to a sawtooth current defined by (59) and (60) can be as follows

$$V(t) = I_M \quad (69)$$

and can be plotted against t under the similar assumption to that of Fig.4 as shown in Fig.5 where an ordinary differentiation of the sawtooth function can be seen as expected.

In order to demonstrate the application of the proposed model to fractance with $\alpha > 1$, let fractance with arbitrary K and $\alpha = 1.5$ be considered. By simply letting $\alpha = 1.5$ and $m = 2$, $V(t)$ of this fractance can be determined by using part of the proposed model for fractance with $\alpha > 1$ as

$$V(t) = K\left[-0.282095 I_0 t^{-1.5} + \left[I_{an}\omega^{-0.5}\left(\frac{\pi}{2} - \omega t\right)^{1.5} E_{2,2.5}\left(-\left(\frac{\pi}{2} - \omega t\right)^2\right) + I_{bn}\omega t^{1.5} E_{2,2.5}(-(\omega t)^2)\right]\sum_{p=0}^{\infty}[(2p+0.5)(2p+1.5)]\right] \quad (70)$$

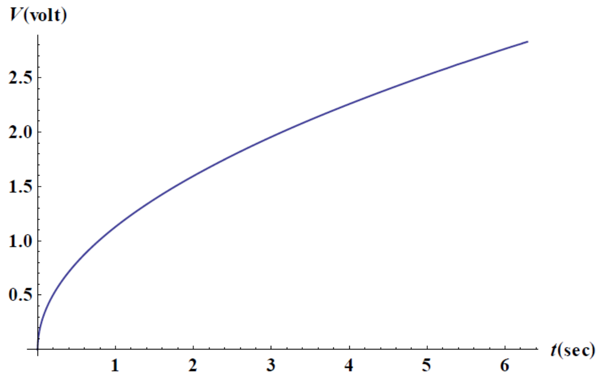

Fig.4: *$V(t)$ of a half order inductor with unity magnitude excited by a sawtooth current waveform with $T = 2\pi$.*

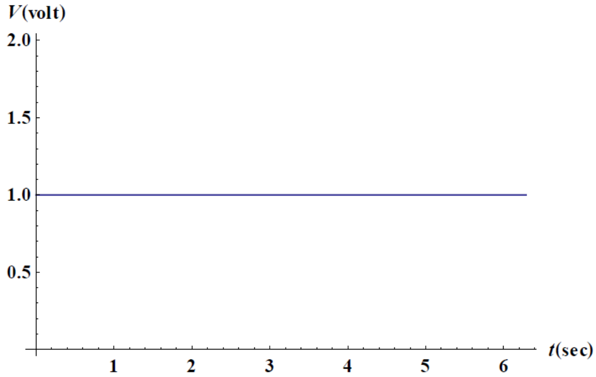

Fig.5: *$V(t)$ of an ordinary inductor with unity magnitude excited by a sawtooth current waveform with $T = 2\pi$.*

As another example, let us consider the fractance with arbitrary K and $\alpha = 2.8$. By letting $\alpha = 2.8$ and $m = 3$, $V(t)$ of this fractance can also be determined by applying part of the proposed model for fractance with $\alpha > 1$ as follows

$$V(t) = K\left[0.313668 I_0 t^{-2.8} + \left[I_{an}\omega^{-0.2}\left(\frac{\pi}{2} - \omega t\right)^{1.2} E_{2,2.2}\left(-\left(\frac{\pi}{2} - \omega t\right)^2\right) + I_{bn}\omega t^{1.2} E_{2,2.2}(-(\omega t)^2)\right]\sum_{p=0}^{\infty}[(2p-0.8)(2p+0.2)(2p+1.2)]\right] \quad (71)$$

For demonstrating the application of the proposed model to the fractional order circuit, a fractional order filter depicted in Fig. 6 where R is an ordinary resistor, F is fractance, $I(t)$ is the current flowing through R and F and $V_{in}(t)$ is an arbitrary periodic input voltage, has been chosen as a candidate fractional order circuit. Here, we will show that the proposed model can be applied for determining the output voltage, $V_{out}(t)$ of this fractional order filter.

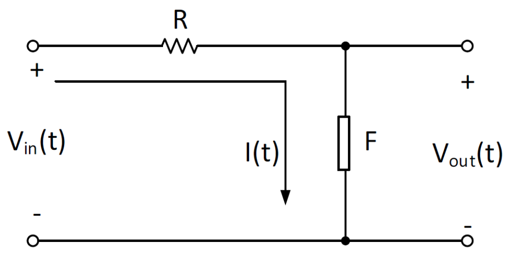

Fig. 6: *Fractional order filter.*

Firstly, the time domain expression of $I(t)$ must be determined. Since $Z(s)$ of F can be be given by (23), the s-domain equivalences of $I(t)$, $I(s)$ can be given in term of the s-domain equivalence of $V_{in}(t)$, $V_{in}(s)$ as

$$I(s) = \frac{V_{in}(s)}{R + Ks^{\alpha}} \tag{72}$$

By taking the inverse Laplace transformation and applying the convolution theorem [13], $I(t)$ can be obtained as a function of $V_{in}(t)$ as follows

$$I(t)=\frac{1}{K}\int_0^t V_{in}(\tau)(1-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{R}{K}(t-\tau)^{\alpha}\right)d\tau \tag{73}$$

Since $I(t)$ is the excitation of F, $V_{out}(t)$ can be immediately determined by using the proposed model and such $I(t)$. This is because the voltage response of F serves as $V_{out}(t)$. If the order of the filter lies between 0 and 1, $V_{out}(t)$ can be given by using part of the proposed model for fractance with $0 < \alpha \leq 1$. On the other hand, $V_{out}(t)$ of the filter with order larger than 1 can be determined by using part of the proposed model for fractance with $\alpha > 1$.

5. DISCUSSION

In this section, some worthy of mentioned issues on the alternative generic analytical expressions of the voltage responses of fractance which can be obtained based on complex exponential Fourier series expansion of $I(t)$, will be discussed. It will also be shown that the proposed model is more preferable than these complex exponential Fourier series based expressions since the format of the proposed model is realistic. Before proceed further, it should be mentioned here that $I(t)$ can be alternatively expressed in term of its complex exponential Fourier series expansion [13] as

$$I(t) = \sum_{n=-\infty}^{\infty} \left[I_{cn}\exp(jn\omega t)\right] \tag{74}$$

where

$$I_{cn} = \frac{1}{T}\int_{T/2}^{T/2} I(t)\exp(-jn\omega t)dt \tag{75}$$

As a result, $V(t)$ can be found as

$$V(t) = K\sum_{n=-\infty}^{\infty}\left[I_{cn}D^{\alpha}\left[\exp(jn\omega t)\right]\right] \tag{76}$$

Via the ordinary Eulers formula [13] given by

$$\exp(jn\omega t) = \cos(n\omega t) + j\sin(n\omega t) \tag{77}$$

, $V(t)$ become

$$V(t)=K\sum_{n=-\infty}^{\infty}\left[I_{cn}\left[D^{\alpha}[\cos(n\omega t)]+jD^{\alpha}[\sin(n\omega t)]\right]\right] \tag{78}$$

With the usage of the Riemann-Liouvielle definition of fractional derivative with order ranges from 0 to 1 for defining $D^{\alpha}[\,]$, (78) can be rewritten as

$$V(t)=K\sum_{n=-\infty}^{\infty}\left[I_{cn}\left[\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\gamma}\cos(n\omega t)d\tau + \frac{j}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\gamma}\sin(n\omega t)d\tau\right]\right] \tag{79}$$

After the integration and differentiation, we obtained

$$V(t) =K\sum_{n=-\infty}^{\infty}\left[\left[(n\omega)^{\alpha}I_{cn}\left[\cos_{\alpha}(n\omega t)\cos\left(\frac{\alpha\pi}{2}\right) - \sin_{\alpha}(n\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right] + j\left[\sin_{\alpha}(n\omega t)\cos\left(\frac{\alpha\pi}{2}\right) + \cos_{\alpha}(n\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right]\right]\right] \tag{80}$$

which can be thought of as a complex exponential Fourier series based expression of $V(t)$ of fractance with $0 < \alpha \leq 1$. At asymptotic, this complex exponential Fourier series based $V(t)$ become its asymptotic counterpart i.e. $V_{asymp}(t)$ which can be approximately given by

$$V_{asymp}(t) =K\sum_{n=-\infty}^{\infty}\left[(n\omega)^{\alpha}I_{cn}\left[\cos\left(n\omega t + \frac{\alpha\pi}{2}\right) + j\sin\left(n\omega t + \frac{\alpha\pi}{2}\right)\right]\right] \tag{81}$$

By using (77), $V_{asymp}(t)$ can be alternatively given as

$$V_{asymp}(t) = K \sum_{n=-\infty}^{\infty} \left[(n\omega)^{\alpha} I_{cn} \exp\left[j \left(n\omega t + \frac{\alpha\pi}{2} \right) \right] \right] \quad (82)$$

On the other hand, the transient counterpart of this complex exponential Fourier series based $V(t)$ i.e. $V_{trans}(t)$ can be given as

$$V(t) = K \sum_{n=-\infty}^{\infty} \left[(n\omega)^{\alpha} I_{cn} \left[\left[\cos_{\alpha}(n\omega t) \cos\left(\frac{\alpha\pi}{2}\right) - \sin_{\alpha}(n\omega t) \sin\left(\frac{\alpha\pi}{2}\right) - \cos\left(n\omega t + \frac{\alpha\pi}{2}\right) \right] + j \left[\sin_{\alpha}(n\omega t) \cos\left(\frac{\alpha\pi}{2}\right) + \cos_{\alpha}(n\omega t) \sin\left(\frac{\alpha\pi}{2}\right) - \sin\left(n\omega t + \frac{\alpha\pi}{2}\right) \right] \right] \right] \quad (83)$$

Moreover, after some rearrangement, (80) become

$$V(t) = K \sum_{n=-\infty}^{\infty} \left[(n\omega t)^{\alpha} I_{cn} \left[(j\omega t)^{-\alpha} (\cos_{\alpha}(n\omega t) + j \sin_{\alpha}(n\omega t)) \exp\left(j \frac{\alpha\pi}{2} \right) \right] \right] \quad (84)$$

By using the generalized Eulers formula, the complex exponential Fourier series based $V(t)$ can be now given by

$$V(t) = !K \sum_{n=-\infty}^{\infty} \left[(n\omega t)^{\alpha} I_{cn} \left[E_{\alpha}^{jn\omega t} \exp\left(j \frac{\alpha\pi}{2} \right) \right] \right] \quad (85)$$

which is in term of the generalized exponential function. If we use the following identity

$$(jn\omega t)^{-\alpha} E_{1,1-\alpha}(jn\omega t) = \cos_{\alpha}(n\omega t) + j \sin_{\alpha}(n\omega t) \quad (86)$$

instead of the generalized Eulers formula, the complex exponential Fourier series based $V(t)$ can be now given in term of the Mittag-Leffler function as follows

$$V(t) = K \sum_{n=-\infty}^{\infty} \left[(n\omega)^{\alpha} \left[(j\omega t)^{-\alpha} E_{1,1-\alpha}(jn\omega t) \exp\left(j \frac{\alpha\pi}{2} \right) \right] \right] \quad (87)$$

On the other hand, for fractance with $\alpha > 1$, the Riemann-Liouvielle definition of the fractional derivative with order larger than 1 must be used for defining $D^{\alpha}[\,]$ in (78). As a result, (78) is now become

$$V(t) = K \sum_{n=-\infty}^{\infty} \left[I_{cn} \left[\frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} \cos(n\omega\tau) d\tau + \frac{j}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} \sin(n\omega\tau) d\tau \right] \right] \quad (88)$$

and can be rearranged as

$$V(t) = \sum_{n=-\infty}^{\infty} \left[\frac{KI_{cn}}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} \cos(n\omega\tau) d\tau \right] + \sum_{n=-\infty}^{\infty} \left[\frac{jKI_{cn}}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} \sin(n\omega\tau) d\tau \right] \quad (89)$$

After the integration and differentiation in an iterative manner, the complex exponential Fourier series based $V(t)$ of fractance with $\alpha > 1$ can be finally obtained as in (90).

At this point, it can be seen that even though these complex exponential Fourier series based expressions are also the generic analytical expressions of $V(t)$ of fractance as well as the proposed model, they are not realistic. This is because they contain j which is an imaginary value. So, the proposed model has been found to be more preferable as it is realistic. This is because it composes of real valued functions and parameters only.

$$V(t) = KI_{cn} \left[\omega^{\alpha-m} \left(\frac{\pi}{2} - \omega t \right)^{m+1-\alpha} E_{2,m+2-\alpha}\left(\left(-\frac{\pi}{2} - \omega t \right)^2 \right) + j\omega t^{m+1-\alpha} E_{2,m+2-\alpha}(-(\omega t)^2) \right] \sum_{p=0}^{\infty} \left[\prod_{n=1}^{\infty} (n+2p+1-\alpha) \right] \quad (90)$$

6. CONCLUSION

A novel generic analytical model of $V(t)$ of fractance with arbitrary α excited by arbitrary periodic $I(t)$ has been proposed in this work. The Fourier series has been adopted as the modelling basis. The proposed model composes of two parts for fractance with $0 < \alpha \leq 1$ and that with $\alpha > 1$ respectively. By using this model, both $V_{asymp}(t)$ and $V_{trans}(t)$ of fractance can be simply determined. Moreover, it has been shown that this model which is in a realistic format is applicable to both single fractance and fractional order circuit. Hence, it has been found to be beneficial to various fractance involved engineering disciplines e.g. analog circuits and systems, biomedical engineering, control engineering and electronic engineering etc.

7. ACKNOWLEDGEMENT

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