FORMATION OF A SPARSE BUS IMPEDANCE MATRIX AND ITS APPLICATION TO SHORT CIRCUIT STUDY

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Introduction

Usually, the objective in the matrix analysis of network is to obtain the inverse of the matrix of coefficients of a system of simultaneous linear network equations. However, for large sparse systems such as occur in many network problems, the use of the inverse is very inefficient. In general; the matrix of the equations, formed from the given conditions of a network problem, is sparse whereas its inverseis full. However, in short circuit study, only a small number of elements of an inverted matrix are required. This paper will present a computational method for the formation of a sparse bus impedance matrix. An algorithm will be introduced which generates only the relevant terms of the bus impedance matrix needed for short circuit calculations. That is only the terms of the Z bus that are related to actual circuit elements will be generated. The new bus impedance matrix will not be as sparse as the bus admittance matrix but it will differ only by fill-in terms as a result of the factorization process. During the development of this method, a small sample system is provided to illustrate the computational procedure.

Problems formulation

An electrical network under short circuit conditions can be considered as a network supplied by several sources or generators with a single load to the system at the node subject to the short circuit. The normal customer load currents are usually ignored since they are small compare to the short circuit.An analysis of a network under short circuit conditions is required for the proper selection of circuit breakers and fuses in power systems. Usually, the short circuit study has been primarily concerned with three phase faults and single line-to-ground faults because they are commonly considered to be the severest and the most frequent types of system faults. Further attention is frequently focussed on total fault current and/or fault current in circuit connected to the faulted bus. The calculations for these types of faults are usually made assuming a 1 p.u. bus voltage prior to the fault as shown in Figure I. Expressions for the desired fault

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currents are given as follows:

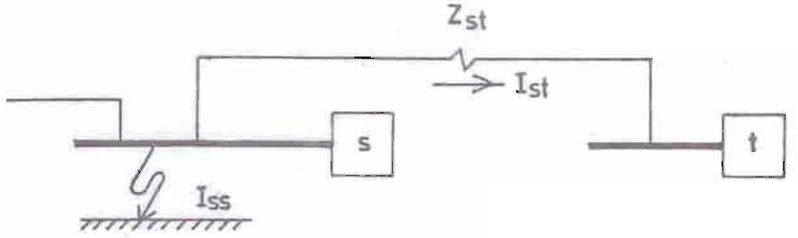

Fig.1 Short Circuit Currents for Bus Fault.

$$I_{ss} = 1/Z_{ss} \quad (1)$$

$$I_{st} = I_{ss}(Z_{ss} - Z_{st})/z_{st} \quad (2)$$

where I_{ss} is the short circuit current at the faulted bus.

I_{st} is the current at the connectiong bus due to the fault.

z_{st} is the impedance of the connecting line.

Z_{ss} is the driving point impedance at the faulted bus.

Z_{st} is the transfer impedance of the faulted bus to the adjacent bus.

Note that the current I_{st} is obtained with the assumption that the currents inject at any bus except at faulted bus are assumed to have values of zero during the fault. Also, the driving point and transfer impedance given in equations (1) and (2) are usually obtained by calculating all the terms of the bus impedance matrix Z.

The following method, a sparse bus impedance matrix method, is designed to calculate only the needed elements of the Z matrix. However, before this method can be discussed in detail; another method which is required for the sparse bus impedance matrix method will be presented. This method, the matrix triangular factorization, allows any non-singular symmetric matrix to be uniquely factorized into three matrices. Therefore, the Y matrix, which is easy to obtain in any power systems, can be factorized into the following three matrices.

$$Y = LDL^t \quad (3)$$

where L in a unit lower matrix.

D is a diagonal matrix.

L^t is a transpose of L.

that is:

$$
L = \begin{bmatrix} 1 & 0 & 0 - - - - 0 \\ l_{21} & 1 & 0 - - - - 0 \\ l_{31} & l_{32} & 1 - - - - 0 \\ & & \\ l_{n1} & l_{n2} & l_{n3} - - - 1 \end{bmatrix}
$$

and;

$$
D = \begin{bmatrix} d_{11} & 0 & 0 - - - - 0 \\ & d_{22} & 0 - - - - 0 \\ 0 & 0 & d_{33} - - - 0 \\ \vdots & \vdots & \ddots \\ 0 & 0 & 0 - - - - d_{nn} \end{bmatrix}
$$

Each term of the factorized matrices can be successively determined as follows;

diagonal terms: $$d_{ii} = y_{ii} - \sum_{k<i} l_{ik}^2 d_{kk} \qquad (4)$$

for $i = 1,---,n$

triangular terms: $$l_{ij} = (y_{ij} - \sum_{k<j} l_{ik}\, l_{jk} d_{kk})/d_{jj} \qquad (5)$$

for $i = 2,---,n$; $j = 1,---,i-1$

It is recommend that the above expressions are most convenient to compute by column, that is to compute the elements along the 1^{st} column first, then the elements along the 2^{nd} column and so on, as will be illustrated in an example later on.

Since the Y matrix of typical systems is very sparse, a proper reordering of the buses in the Y matrix is very important to reduce the number of new non-zero terms produced in the factorization process. Suboptimal ordering, which is one of the optimal ordering schemes, stated that at each stage of elimination, select the row which has the fewest off-diagonal elements; if more than one row has this minimum number, pick any one of them. This scheme will also be illustrated in the example later on.

In general, the sparse Z matrix method will generate only relevant terms of the Z matrix for usual short circuit study. That is, it is sufficient enough to know only those terms of the Z matrix, which correspond to node and branch elements including fill-in terms as a result of the factorization process, in order to calculate the short circuit currents at the faulted bus and in the lines connected to the faulted bus. The equation for this method is derived as follows. From the definition of the Z matrix.

$$YZ = I \qquad (6)$$

where I is an identity matrix. Recall that $Y = LDL^t$, so from equation (3) we have.

$$LDL^tZ = I$$

or:

$$L^tZ = D^{-1}L^{-1} \qquad (7)$$

Add $(Z-L^tZ)$ into both sides of equation (7)

$$Z - L^tZ + L^tZ = D^{-1}L^{-1} + Z - L^tZ$$

or:

$$Z = D^{-1}L^{-1} + (I-L^t)Z \qquad (8)$$

Now we define matrix W by:

$$W = D^{-1}L^{-1} \qquad (9)$$

and another matrix by:

$$T = (I-L^t) \qquad (10)$$

Substitute the two matrices W and T into equation (8) will result in the following equation.

$$Z = W + TZ \qquad (11)$$

Note that W is a linear triangular matrix with its diagonal terms w_{ii} equal to $1/d_{ii}$ for i=1,----,n; and T is strictly an upper triangular matrix, that is all the diagonal and lower terms of matrix T have values of zero. It also has been mentioned that the Y matrix is symmetric. Thus, the Z matrix will be symmetric, that is $z_{ij} = z_{ji}$ for all i and j. Therefore, it can be shown that all the terms of the Z matrix are successively obtained as follows.

$$z_{nn} = w_{nn}$$

$$z_{n-1,n} = t_{n-1,n}z_{nn}$$

$$z_{n-1,n-1} = w_{n-1,n-1} + t_{n-1,n}z_{n,n-1}$$

$$z_{n-2,n-1} = t_{n-2,n}z_{n,n-1}$$

$$z_{12} = t_{1n}z_{n2} + t_{1,n-1,2} + \text{-----} + t_{13}z_{32}$$

$$z_{11} = w_{11} + t_{1n}z_{n1} + \text{---} + t_{14}z_{41} + t_{13}z_{31} + t_{12}z_{21}$$

The relation noted by expressions above indicates that the terms of the Z natrix can be calculated by backward substitution. It should also be noted that according to equation (11) and above expressions, no lower terms of matrix W arerequired for the calculation of the elements in the Z matrix. Thus, the off-diagonal elements of the W matrix are not required nor obtained.

In general the sparse Z matrix method is determined by the following expressions.

off-diagonal terms: $$z_{pq} = \sum_{k > p} t_{pk}z_{kq} \quad \text{for } p < q \qquad (12)$$

diagonal terms: $$z_{pp} = w_{pp} + \sum_{k > p} t_{pk}z_{kp} \qquad (13)$$

Note that the summation for the above expressions is taken for only such k that $t_{pk} \neq 0$ and $k > p$. Moreover, the calculation process which is given by both equations is restricted to the backward substitution as previously shown. Thus, the terms which are generated by the sparse Z matrix method can fill the data requirement for the Z matrix needed for the usual short circuit calculation.

A small DC netword is provided in Figure 2 to illustrate the method already mentioned.

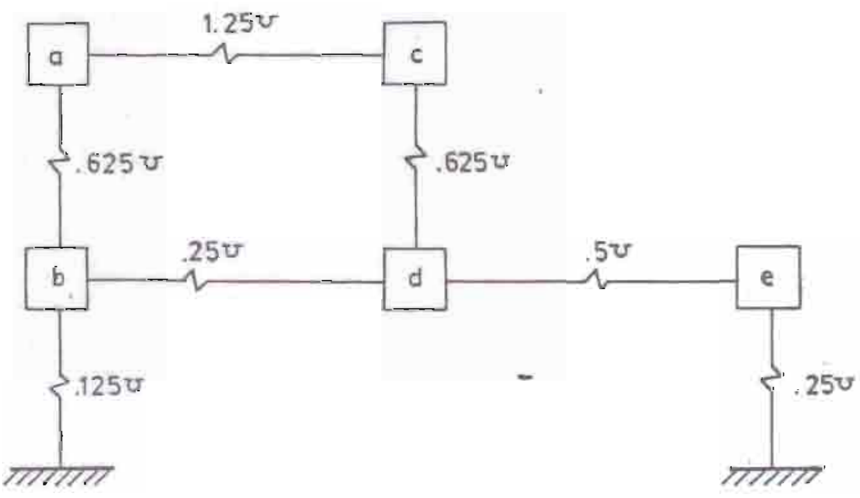

Fig.2 Example of DC Network.

In Figure 2, the series admittance of branches represent reactances of transmission lines or transformers, whereas the shunt admittance at nodes represent equivalent reactances of generators connected to the nodes. It is assumed that real numbers are used in the example. The Y matrix is then constructed as follows.

Y =	a	b	c	d	e
a	1.875	-.625	-1.25	0	0
b	-.625	1.0	0	-.250	0
c	-1.25	0	1.875	-.625	0
d	0	-.250	-.625	1.375	-.5
e	0	0	0	-.500	.75

The new ordering of the Y matrix is obtained using the Suboptimal Ordering scheme as follows.

Y =	(1) e	(2) a	(3) b	(4) c	(5) d
(1) e	.75	0	0	0	-.500
(2) a	0	1.875	-.625	-1.25	0
(3) b	0	-.625	1.0	0	-.250
(4) c	0	-1.25	0	1.875	-.625
(5) d	-.5	0	-.250	-.625	1.375

Row e was selected first because it has 3-zero off-diagonal values. After the selection, its row and column have been crossed-out. Thus, leaving only 4 rows and 4 columns (a,b,c and d), and this can be seen immediately that each row of the remaining matrix has only one-zero off-diagonal value. Rows a,b,c and d are then selected arbitrary to fill the new Y matrix.

Next step is to obtain the factorized matrices using the matrix triangular factorization method. These matrices are shown and computed as follows (the elements of these matrices are computed by column as previously mentioned).

D =

	1	2	3	4	5
1	0.750	0	0	0	0
2	0	1.875	0	0	0
3	0	0	0.792	0	0
4	0	0	0	0.822	0
5	0	0	0	0	0.267

and;

L =

	1	2	3	4	5
1	1.000	0	0	0	0
2	0	1.000	0	0	0
3	0	-.333	1.000	0	0
4	0	-.667	-.526	1.000	0
5	-.667	0	-.316	-.920	1.000

At this point, the matrices W and T can be obtained

W =

	1	2	3	4	5
1	1.333	0	0	0	0
2	*	0.533	0	0	0
3	*	*	1.263	0	0
4	*	*	*	1.216	0
5	*	*	*	*	3.750

and;

T =

	1	2	3	4	5
1	0	0	0	0	0.667
2	0	0	0.333	0.667	0
3	0	0	0	0.526	0.316
4	0	0	0	0	0.920
5	0	0	0	0	0

where an asterisk indicates a term that is not required in the iteration process.

Finally the sparse Z matrix is obtained:

$$Z = \begin{array}{cc} & \\ & \\ (1) & e \\ (2) & a \\ (3) & b \\ (4) & c \\ (5) & d \end{array} \begin{array}{ccccc} (1) & (2) & (3) & (4) & (5) \\ e & a & b & c & d \\ \end{array}$$

		(1) e	(2) a	(3) b	(4) c	(5) d
(1)	e	3.00				2.50
(2)	a		4.44	3.60	4.06	
(3)	b			4.00	3.40	3.00
(4)	c				4.39	3.45
(5)	d					3.75

Note that Z_{34} is a term in the Z matrix which is associated with a fill-in term created in the factorization of the Y matrix. Furthermore, all the remaining off-diagonal elements of Z corresponding to zero elements in the T matrix are computed.

The algorithm that has been presented so far has dealt with a symmetric matrix. However, in a fault study, we must sometimes deal with an asymmetric matrix. This happened in the bus admittance and/or bus impedance matrices when phase-shifting transformer is connected in the systems. Application of the sparse Z bus method to a system having an asymmetric Y matrix is introduced in this section.

It has been previously noted that any non-singular symmetric matrix can be factorized into three matrices. This statement applies to an asymmetric matrix as well. Therefore, a non-singular asymmetric matrix A can be uniquely factorized into the following form.

$$A = LDU \qquad (14)$$

where L is a unit lower triangular matrix.

D is a diagonal matrix.

U is a unit upper triangular matrix.

Each term of the factorized matrices can be successively found as follows:

diagonal term: $$d_{ii} = y_{ii} - \sum_{k<i} r_{ik} u_{ki} d_{kk} \qquad (15)$$

$$\text{for } i = 1,----,n$$

upper triangular term: $u_{ij} = (y_{ij} - \sum_{k<i} l_{ik}u_{kj}d_{kk})/d_{ii}$ (16)

for $i = 1,---,n-1$; $j = 2,---,i+1$

lower triangular term: $l_{ij} = (y_{ij} - \sum_{k<j} l_{ik}u_{kj}d_{kk})/d_{jj}$ (17)

for $i = 2,---,n$; $j = 1,---,i-1$

Then, denoting X as the inverse of A, we have the following expressions.

$$AX = I \quad (18)$$

and:

$$XA = I \quad (19)$$

By substitution equation into expression (18) we have:

$$LDUX = I \quad (20)$$

Let, $$D^{-1}L^{-1} = W \quad (21)$$

and $$I-U = T \quad (22)$$

Then, by substitution equations (21) and (22) in the same procedure as in symmetric matrix, equation (20) becomes:

$$X = W + TX \quad (23)$$

where W is a lower triangular matrix with the elements of D^{-1} as its diagonal terms.

T is a strictly upper triangular matrix.

Similarly, by substitution equation (14) into expression (19), we have.

$$XLDU = I \quad (24)$$

Let, $$U^{-1}D^{-1} = V \quad (25)$$

and, $$I - U = S \quad (26)$$

Rearrange equation (19) using equations (25) and (26), the following relation is obtained:

$$X = V + XS \quad (27)$$

where V is an upper triangular matrix with the elements of D^{-1} as its diagonal terms.

S is a strictly lower triangular matrix.

It should also be noted that the off-diagonal elements of W and V are not required in the iteration process. Furthermore, backward substitution is restricted to both expressions above (23 and 27).

For an example, a (3x3) asymmetric matrix is selected. Then, using equations (23) and (27), the matrices are expressed as follows,

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} w_{11} & 0 & 0 \\ * & w_{22} & 0 \\ * & * & w_{33} \end{bmatrix} + \begin{bmatrix} 0 & t_{12} & t_{13} \\ 0 & 0 & t_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

and:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} v_{11} & * & * \\ 0 & v_{22} & * \\ 0 & 0 & v_{33} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ s_{21} & 0 & 0 \\ s_{31} & x_{32} & 0 \end{bmatrix}$$

where $w_{ii} = v_{ii} = d/d_{ii}$ for all i

$t_{ij} = -u_{ij}$ for i<j

$s_{ij} = -l_{ij}$ for i<j

The upper triangular terms of x can be determined by the first expression as can the lower triangular terms of x by the second expression; whereas the diagonal terms can be obtained by either expression. Algebraically, both equations or expressions above can be expressed as follows:

$$x_{33} = w_{33} = v_{33}$$

$$x_{23} = t_{23}x_{33}$$

$$x_{32} = x_{33}s_{32}$$

$$x_{22} = w_{22} + t_{23}x_{32} = v_{22} + x_{23}s_{32}$$

$$x_{13} = t_{13}x_{33} + t_{12}x_{23}$$

$$x_{31} = x_{33}s_{31} + x_{32}\bar{s}_{21}$$

$$x_{12} = t_{13}x_{32} + t_{12}x_{22}$$

$$x_{21} = x_{23}s_{31} + x_{22}s_{21}$$

$$x_{11} = w_{11} + t_{13}x_{31} + t_{12}x_{21}$$
$$= v_{11} + x_{13}s_{31} + x_{12}s_{21}$$

A simple example is provided to illustrate the above method.

$$\text{For} \quad A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

the factorized matrices are obtained using equations (15), (16) and (17).

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} ; \; D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \; U = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

hence;

$$S = I-L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} ; \quad T = I-U = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

and,

$$W = \begin{bmatrix} 1 & 0 & 0 \\ * & -1 & 0 \\ * & * & 1 \end{bmatrix} ; \quad V = \begin{bmatrix} 1 & * & * \\ 0 & -1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

where an asterisk indicates a term that is not required in the calculation of the matrix X process.

Applying these matrices into the two-matrices previously discussed.

$$\begin{bmatrix} x_{11} & * & x_{13} \\ * & x_{22} & x_{23} \\ * & * & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ * & -1 & 0 \\ * & * & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & * & x_{13} \\ * & x_{22} & x_{23} \\ x_{31} & * & x_{33} \end{bmatrix}$$

$$\begin{bmatrix} * & * & * \\ * & * & * \\ x_{31} & * & * \end{bmatrix} = \begin{bmatrix} 1 & * & * \\ 0 & -1 & * \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} x_{11} & * & x_{13} \\ * & x_{22} & x_{23} \\ x_{31} & * & x_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

This implies:

$$\begin{aligned} x_{33} &= w_{33} && = 1 \\ x_{23} &= t_{23}x_{33} && = 3 \\ x_{22} &= w_{22} + t_{23}x_{32} && \\ &= v_{22} + x_{23}s_{32} && = -1 \\ x_{13} &= t_{13}x_{33} && = 2 \\ x_{31} &= x_{33}s_{31} + x_{32}s_{21} && = -1 \\ x_{11} &= w_{11} + t_{13}x_{31} + t_{12}x_{21} && = -1 \end{aligned}$$

Note that before the above procedure is taken, a proper reordering for the columns and rows of the given matrix should also be obtained in order to reduce the total number of new non-zero elements in L and U.

Other applications.

The sparse Z matrix method that has been presented in this paper is applicable not only to computing currents in lines connected to the faulted bus; it can in fact be applied to the calculation of those Z elements required for the calculation of the fault current in any line. As an example, considers the system shown in Figure 3. Expressions for fault currents given as shown.

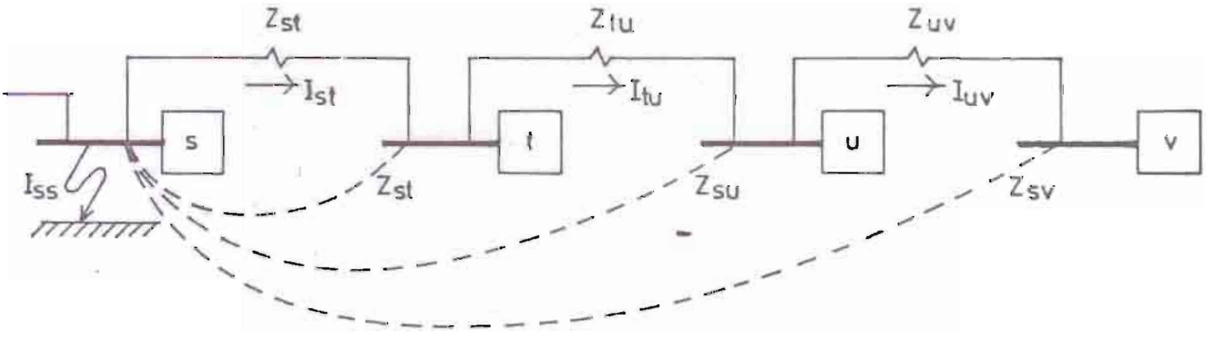

Fig.3 The Short Circuit Currents of Lines Two or More Nodes Away from the Faulted Bus.

$$I_{ss} = 1/Z_{ss}$$

$$I_{st} = I_{ss}(Z_{ss} - Z_{st})/z_{st}$$

$$I_{tu} = I_{ss}(Z_{st} - Z_{su})/z_{tu}$$

$$I_{uv} = I_{ss}(Z_{su} - Z_{sv})/z_{uv}$$

where I_{tu} is the current between the first and second adjacent buses from the faulted bus.

I_{uv} is the current between the second and third adjacent buses from the faulted bus.

Z_{su} is the transfer impedance of the faulted bus to the second adjacent bus.

Z_{sv} is the transfer impedance of the faulted bus to the third adjacent bus.

z_{tu} is the self impedance of the connecting line between between bus t and bus u.

z_{uv} is the self impedance of the connecting line between bus u and bus v.

I_{ss}, I_{st}, z_{st}, Z_{ss} and Z_{st} have already been mentioned in page 2.

From the above expressions one can immediately see that Z_{st}, Z_{su} and Z_{sv} are in fact off-diagonal elements in the Z matrix. Thus, one can obtain these elements of the Z matrix using the sparse Z matrix method. The general formula to obtain these Z elements for the system above is as follows:

$$z_{sr} = \sum_{k>s} t_{sk} z_{kr} \qquad s < r \tag{30}$$

This approach is applicable to the calculation of Z elements for all nodes s and r. However, as previously encountered, the calculation given in the equation above must also be solved backwardly with respect to node s at which the fault is applied.

Conclusion

A new algorithm for short circuit studies, designated as the sparse Z matrix method, has been presented. The procedure is quite superior to any other conventional methods because it provides a method for computing only those elements of the Z matrix actually needed in short circuit calculations, thereby reducing required computer time and storage in the computational process. Also, this method seems to be applicable to system contingency studies and sentivity analysis for system stability problems.

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