Application of Modified Bessel Functions in Extended Surface Heat Transfer Problems

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Abstract

Modified Bessel functions are encountered in a range of engineering applications. This paper presents one such application in the analysis of extended surface heat transfer. Two classical problems of annular and triangular fins are selected for the purpose of this study. Differential equations describing the temperature distribution along the fins are formulated from the fundamentals of conduction and convection heat transfer, and the solutions of the equations are presented. Due to the tedious nature of the modified Bessel functions involved in the solutions, approximate relationships for the functions are recommended, and a simplified method for calculating fin efficiency is also discussed.

Introduction

A great variety of engineering problems are represented by modified Bessel's equation and other equations resembling Bessel's equation. These equations have solutions expressed in the form of Bessel functions. In heat transfer analysis of extended surface, the resulting differential equations are examples of such equations. In this paper, analyses will be made of annular and triangular fins in order to illustrate problem formulation and solutions. Since modified Bessel

Functions are cumbersome to evaluate, approximate polynomial relationships for the function will be presented and a simplified method for fin efficiency determination will also be discussed.

Annular fins

Consider an annular fin attached to a tube as shown in Figure 1. By considering one-dimensional, steady-state heat balance over a differential element, the equation describing the temperature distribution in the fin may be derived as follows.

The conduction heat transfer rate through the area at r may be written as

$$q_r = -[k 2\pi r t \frac{dt}{dr}]_r$$

Similarly the conduction heat transfer rate through the area at $r + \Delta r$ is given by

$$q_{r+dr} = -[k 2\pi r t \frac{dt}{dr}]_{r+\Delta r}$$

Heat is also convected away by the surrounding fluid at T_∞ and a convection heat transfer coefficient h. The convection loss is expressed as

$$dq = h(2)(2\pi r)\Delta r(T - T_\infty)$$

The energy balance of the element requires that

$$q_r = q_{r+\Delta r} + dq$$

Hence

$$-[k2\pi rt\frac{dt}{dr}]_r = -[k2\pi rt\frac{dt}{dr}]_{r+\Delta r} + h4\pi r\Delta r(T-T_\infty)$$

or

$$\frac{r\left.\frac{dt}{dr}\right|_{r+\Delta r} - r\left.\frac{dT}{dr}\right|_r}{\Delta r} - \frac{2h}{kt}r(T-T_\infty) = 0$$

As $\Delta r \rightarrow 0$

$$\frac{d}{dr}(r\frac{dT}{dr}) - \frac{2h}{kt}r(T-T_\infty) = 0$$

or

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} - \frac{2h}{kt}r(T-T_\infty) = 0$$

Multiplying by r and letting

$$\theta = T - T_\infty$$

$$m^2 = \frac{2h}{kt}$$

one obtains

$$r^2\frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - m^2r^2\theta = 0 \qquad (1)$$

This equation should be compared with a modified Bessel's equation of the form (1),

$$r^2\frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - (\lambda^2r^2 + v^2)y = 0 \qquad (2)$$

which has a complete solution

$$\theta = C_1 I_\nu(\lambda r) + C_2 K_\nu(\lambda r) \tag{3}$$

Obviously Eq.(1) is a special case of Eq.(2) with $\nu = 0$ and $\lambda = m$. Hence the solution of Eq.(1) is given by

$$\theta = C_1 I_0(mr) + C_2 K_0(mr) \tag{4}$$

The constants C_1 and C_2 may by evaluated by using the following boundary conditions

$$\theta = \theta_0 \quad \text{at} \quad r = r_0$$

$$\frac{d\theta}{dr} = 0 \quad \text{at} \quad r = r_e$$

At this point it is appropriate to introduce two relevant properties of modified Bessel functions.

$$\frac{d}{dx}[I_o(x)] = I_1(x)$$

$$\frac{d}{dx}[K_o(x)] = -K_1(x)$$

With the help of these two equations, the boundary conditions are used with Eq.(4) to evaluate

C_1 and C_2 which are substituted back into Eq.(4) to yield

$$\frac{\theta}{\theta_0} = \frac{K_1(mr_e)I_0(mr) + I_1(mr_e)K_o(mr)}{I_0(mr_0)K_1(mr_e) + I_1(mr_e)K_o(mr_0)} \tag{5}$$

The actual heat transfer from the fin may now be evaluated from the general relationship

$$q_0 = -2\pi k r_0 t\left[\frac{d\theta}{dr}\right]_{r=r_0} \tag{6}$$

The fin efficiency may also be evaluated by first determining the ideal heat flow from

$$q_i = 2\pi(r_e^2 - r_0^2)h\theta_0 \tag{7}$$

and the fin efficiency from

$$\eta = \frac{q_0}{q_i} \tag{8}$$

By calculating q_0 with the help of Eq.(5) and substituting q_0 and q_i into Eq.(8), one obtains

$$\eta = \frac{2r_o}{m(r_e^2 - r_o^2)} \cdot \frac{I_1(mr_e)K_1(mr_o) - K_1(mr_e)I_1(mr_o)}{I_o(mr_o)K_1(mr_e) + I_1(mr_e)K_o(mr_o)} \tag{9}$$

as required.

Triangular fins

Consider a metal fin of triangular shape attached to a plane wall to help convect heat from the latter. Assuming the dimensions and coordinates as shown in Figure 2, we wish to determine the temperature distribution along the fin if the wall temperature is T_O and if the fin is subject to convection condition specified by the surrounding fluid temperature T_∞ and the convection heat transfer coefficient h.

We shall base our analysis on a unit length of the fin and shall assume the fin is so thin that heat conduction is one-dimensional. Now consider the heat balance in the element of the fin between x and $x+\Delta x$. This element gains and loses heat by conduction through its right and left faces, respectively. The upper and lower surfaces lose heat by convection to the surrounding fluid.

The heat conducted in at $x+\Delta x$ is given by

$$q_{x+\Delta x} = [k\frac{(tx)}{l}\frac{dT}{dx}]_{x+\Delta x}$$

and the heat conducted out at x is expressed as

$$q_r = [\frac{tkx}{l}\frac{dT}{dx}]_x$$

The heat lost by the element to the surrounding fluid is given by

$$dq = \frac{2h(T-T_\infty)\Delta x}{\cos\alpha}$$

The energy balance requires that

$$q_{x+dx} = q_x + dq$$

Thus,

$$[\frac{tkx}{l}\frac{dT}{dx}]_{x+\Delta x} = [\frac{tkx}{l}\frac{dT}{dx}]_x + \frac{2h(T-T_\infty)\Delta x}{\cos\alpha}$$

After rearranging and dividing by Δx,

$$\frac{\left.\frac{xdT}{dx}\right|_{x+\Delta x} - \left.\frac{xdT}{dx}\right|_{x}}{\Delta x} - \frac{2hl(T-T_\infty)}{tk\cos\alpha} = 0$$

By letting $\theta = T - T_\infty$ and taking the limit as $\Delta x \to 0$,

$$\frac{d}{dx}\left(\frac{xd\theta}{dx}\right) - \frac{2hl\theta}{bk\cos\alpha} = 0$$

or

$$(x\theta') - m^2\theta = 0$$

where $m^2 = \dfrac{2hl}{tk\cos\alpha}$

Eq.(10) should be compared with the equation (1)

$$(x^r\theta')' + (ax^s + bx^{r-2})\theta = 0 \tag{11}$$

This equation, if $(1-r^2) \geq 4b$, a $\neq$ 0, and if either $r-2 < s$ or $b = 0$, has a complete solution

$$\theta = x^\alpha[C_1J_\nu(\lambda x^\gamma) + C_2Y_\nu(\lambda x^\gamma)] \tag{12}$$

where
$$\alpha = \frac{1-r}{2}$$
$$\gamma = \frac{2-r+s}{2}$$
$$\lambda = \frac{2\sqrt{|a|}}{2-r+s}$$
$$\nu = \frac{\sqrt{(1-r)^2-4b}}{2-r+s}$$

and if a < 0, J_ν and K_ν must be replaced by I_ν and K_ν respectively.

Comparison between Eq.(10) and Eq.(11) reveals that Eq.(10) is a special case of Eq.(11) with r = 1, a = m^2, s = 0 and b = 0. Hence $\alpha = 0$, $\gamma = \frac{1}{2}$, λ = 2 m and $\nu = 0$. By replacing J_ν and K_ν by I_ν and K_ν and using $\nu = 0$, the solution of Eq.(10) is therefore.

$$\theta = C_1 I_o(2m\sqrt{x}) + C_2 K_o(2m\sqrt{x}) \tag{13}$$

Since $K_o(2m\sqrt{x})$ is infinite when $x = 0$, C_2 must be zero, leaving

$$\theta = C_1 I_o(2m\sqrt{x})$$

By using the boundary condition $\theta = \theta_0$ at $x = l$,

$$\theta = C_1 I_o(2m\sqrt{l})$$

which gives

$$C_1 = \theta_0 / I_o(2m\sqrt{l})$$

Finally the temperature distribution along the triangular fin is expressed as

$$\frac{\theta}{\theta_0} = \frac{I_o(2m\sqrt{x})}{I_o(2m\sqrt{l})} \tag{14}$$

The heat flow rate from the fin is obtained by evaluating

$$q_o = k(t)\left.\frac{d\theta}{dx}\right|_{x=0} \tag{15}$$

By differentiating Eq.(14) and substituting the result in Eq.(15), the heat flow expression is obtained.

$$q_o = ktm\sqrt{l}\theta_o \frac{I_1(2m\sqrt{l})}{I_o(2m\sqrt{l})}$$

$$q_o = l\sqrt{2kth/\cos\alpha}\,\theta_o \frac{I_1(2m\sqrt{l})}{I_o(2m\sqrt{l})} \tag{16}$$

The fin efficiency can be determined by Eq.(8). In this case the ideal heat flow is given by

$$q_i = 2(\frac{l}{\cos\alpha})h\,\theta_o$$

The fin efficiency is therefore

$$\eta = \sqrt{\frac{kt\cos\alpha}{2h}}\frac{I_1(2m\sqrt{l})}{I_o(2m\sqrt{l})} \tag{17}$$

as required.

Evaluation of modified Bessel functions

Equations (9) and (17) may be used to calculate the fin efficiency of annular fins and triangular fins, respectively. However, the determination of the fin efficiency requires the evaluations of modified Bessel functions, I_o , I_1, K_o and K_1. Mathematically these functions are given as.

$$I_o(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2}$$

$$I_1(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!}$$

$$K_o(x) = -\{\gamma + \ln(x/2)\}\ I_0 + \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2}\{1+1/2+1/3+...+1/k\}$$

$K_1(x)$ may be determined by substituting n = 1 in the general expression :

$$K_n(x) = \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1)^k(n-k-1)!}{k!}(2/x)^{n-2k}$$
$$+(-1)^{n+1}\sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!(n+k)!}\{\ln(x/2) - 1/2[\Psi(k+1) + \Psi(k+n=1)]\}$$

where,

$$\Psi(k+1) = (1+1/2+1/3+.....+1/k) - \gamma, \quad \Psi(1) = -\gamma$$
$$\Psi(k+n+1) = (1+1/2+1/3+.....+\frac{1}{K+n}) - \gamma$$
$$\gamma = 0.5772156649$$

While the expressions for I_o and I_1 take 14 terms to converge, the expressions for determining K_o and K_1 are extremely slow to converge. For the purpose of saving computer time and memory, it is better to represent these expressions by polynomial approximations. (2)

$$K_o(x) = -l_n(x/2)I_o(x) - 0.57721566 + 0.42278420(x/2)^2 + 0.23069756(x/2)^4$$
$$+0.03488590(x/2)^6 + 0.00262698(x/2)^8 + 000010750(x/2)^{10}$$
$$+0.00000740(x/2)^{12} + \varepsilon, |\varepsilon| < 1\times 10^{-8}, 0 < x \le 2$$

$$x^{1/2}e^{x}K_{o}(x) = 1.25331414 - 0.07832358(2/x) + 0.02189568(2/x)^2$$
$$- 0.1062446(2/x)^3 + 0.00587872(2/x)^4 - 0.00251540(2/x)^5$$
$$+ 0.00053280(2/x)^6 + \varepsilon, |\varepsilon| < 1.9 \times 10^{-7}, 2 \leq x < \infty$$

$$xK_1(x) =$$
$$x\ln(x/2)I_1(x) + 1 + 0.15443144(x/2)^2 - 0.67278579(x/2)^4$$
$$- 0.18156897(x/2)^6 - 0.01919402(x/2)^8 - 0.001104040(x/2)^{10}$$
$$- 0.001104040(x/2)^{10} - 0.00004686(x.2)^{12} + \varepsilon, |\varepsilon| < 8x10^{-9}, 0 < x \leq 2$$

$$x^{1/2}e^{x}K_1(x) =$$
$$1.25331414 + 0.25331414 + 0.23498619(2/x) - 0.03655620(2/x)^2$$
$$+ 0.01504268(2/x)^3 - 0.00780353(2/x)^4 + 0.00325614(2/x)^5$$
$$- 0.0068245(2/x)^6 + \varepsilon, |\varepsilon| < 2.2x10^{-7}, 2 \leq x < \infty$$

In practice, especially in computer simulation, it is necessary to evaluate the modified Bessel functions every time the change of flow conditions may dictate. This can be avoided if simple relationships describing fin efficiency are available. To cope with this problem, a simplified method based on polynomial regression of generated annular fin efficiency data was proposed by Charters and Theerakulpisut (3). The method enables simple calculation of fin efficiency of annular fins for any given geometry and flow conditions. With some modification, the method may also be extended to the case of triangular fins.

Discussion and conclusion

Application of Bessel functions to the problem of extended surface heat transfer was presented and discussed with reference to two classical annular and triangular fin problems. Relationships for calculating fin efficiency which is an important parameter in the analysis of finned heat exchanger performance are also presented. The fin efficiency equations are cumbersome to evaluate since the calculation involves evaluation of modified Bessel functions which are slow to coverage. Approximate relationships for the modified Bessel function are therefore recommended. A simplified method by which fin efficiency may be calculated is also suggeted in order to expedite the calculation for any given fin geometry and flow conditions.

Nomenclature

h	=	convection heat transfer coefficient
I_o	=	modified Bessel function of the first kind of order zero
I_1	=	modified Bessel function of the first kind of order one
k	=	thermal conductivity of fin material
K_0	=	modified Bessel function of the second kind of order zero
K_1	=	modified Bessel function of the second kind of order one
l	=	length of fin
q_0	=	heat transfer rate evaluated at the base of the fin
q_i	=	ideal heat transfer rate
r	=	radial distance from the centre of the tube
r_0	=	inner radius of the fin
t	=	fin thickness
r_e	=	outer radius of the fin
γ	=	Euler-Macheroni constant
η	=	fin efficiency
θ	=	temperature excess

References

1. C.R.Wylie, ***Advanced Engineering Mathematics.*** McGraw-Hill Kogakusha, 1975.

2. N.M.Mclachlan, ***Bessel Functions for Engineers.*** Oxford Press (1955).

3. W.W.S. Charters and S. Theerakulpisut, ***Efficiency Equations for Constant Thickness Annular Fins***, Int. Comm. Heat mass Transfer, Vol. 16, pp 547-558 (1989).

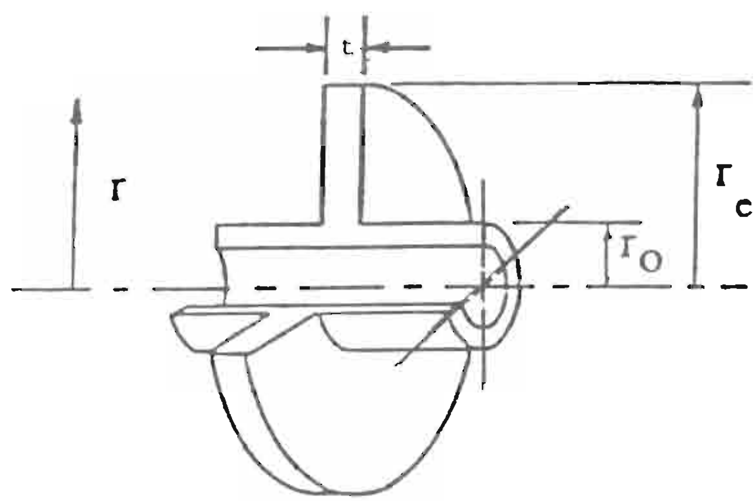

Figure 1 Annular fin

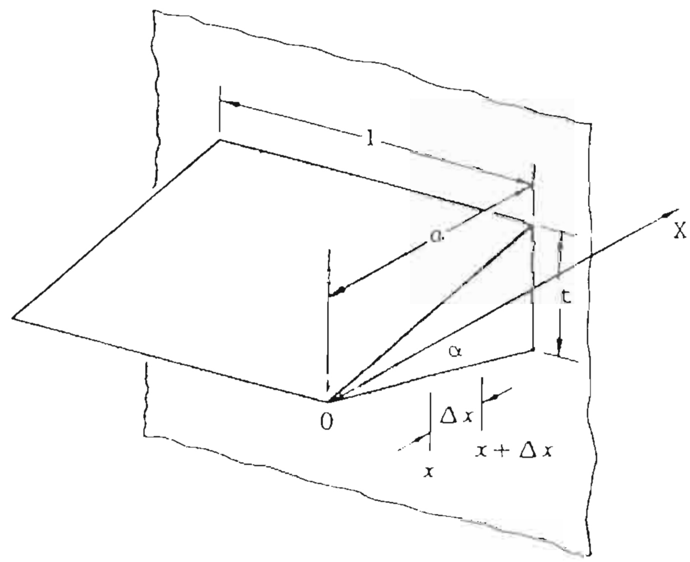

Figure 2 Triangular fin.