A Tutorial Introduction to Linear Control Systems Theory

*P. Hetrakul**

Abstract: Fundamental concepts of linear control system theory are presented in an introductory manner. Only deterministic, finite-dimensional, linear, time-invariant dynamical systems are considered. Discussed are the concepts of systems and states, the state-space formulation of finite-dimensional, linear, time-invariant plants, various types of responses, controllability and observability, dynamical feedback and a description of linear regulator theory.

1. Introduction

Around the 1950's control theoretical study was based entirely on the classical theory which relied heavily on the input-output relation of the system or the transfer function to characterise the behaviour of the system. Various graphical techniques, notably root locus plots, Nyquist plots, etc. were developed as tools for the analysis and design of the system. Generally system design in classical control theory was based on trial-and-error procedures. This usually restricted the systems to deterministic, linear, time-invariant single input-single output systems and was extremely difficult to generalise to more complex systems having several inputs and outputs. Consequently this resulted in the accumulation of highly specialised and complicated procedures for dealing with multivariable systems.

This situation, along with the development of digital computers for large-scale calculation, motivated a completely new approach to the

* Lecturer, Faculty of Engineering, Khon Kaen University

description of dynamical systems. About a decade ago, researchers began to develop, instead of the transfer function description, a mathematical framework which exhibits the internal structure of the dynamical systems. This has led to the analysis and design of multivariate systems which are very general. Besides, this new approach has made possible the study of systems which are time-varying, nonlinear, stochastic and even distributed systems described by partial differential equations. At present a great deal of research activity is concentrated on these areas.

2. Systems and States

The concept of an abstract dynamical system is quite complex (1-8). For our purpose it will suffice to define an abstract dynamical system to be a process that can be described by a finite number of ordinary, linear, deterministic, time-invariant differential equations. That is to say the time evolution of the system is governed by this set of differential equations. The state of a dynamical system is defined as the minimum information about a system which is required to predict its future behaviour under any applying stimuli, or inputs. The state space then consists of all those values which the state may take on. This is best illustrated by the following example.

Example 1 The behaviour of the circuit shown in fig. 1 can be determined for all $t > o$ if the values of the inductor current i, and the capacitor voltage v, are both known at $t = o$ provided that the input voltage $u(t)$ is also known for $t > o$. Thus the two state variables may be taken as $i(t)$ and $v(t)$ and the differentail equations of the circuit are

$$\frac{1}{2}\frac{di(t)}{dt} = -v(t) + u(t) \qquad (1)$$

$$\frac{dv(t)}{dt} = i(t) - 3v(t) \qquad (2)$$

Defining $$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} i(t) \\ v(t) \end{bmatrix}$$

Then the state equation is

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \qquad (3)$$

The output $y(t)$ is the current through the resistor and may be expressed as

$$y(t) = v(t) = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad (4)$$

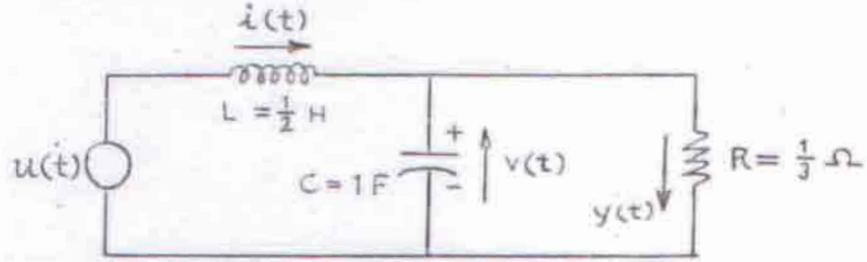

Fig. 1 Simple Linear Circuit

Equations (3) and (4) are examples of standard state variable form for linear differential systems. More generally the state $x = [x_1, x_2, x_3, ..., x_n]'$, the input $u = [u_1, u_2, ..., u_m]'$ and the output $y = [y_1, y_2, ..., y_p]'$ may be taken from real linear vector spaces of dimension n, m and p, respectively The dynamical behaviour of the system is then governed by an ordinary first order linear differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C'x(t) \end{aligned} \qquad (5)$$

where A, B, and C are matrices of dimension nxn, nxm, and nxp respectively.

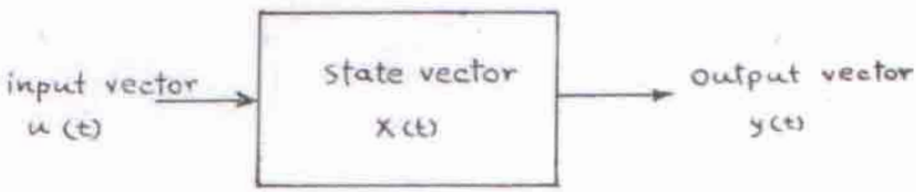

Fig. 2 Linear System

2.1 Similarity Transformation

From the definition of the state, it should be clear that state vectors may be chosen arbitrarily. In example 1, a state vector

$$x = \begin{bmatrix} i+v \\ i-v \end{bmatrix} \quad (6)$$

may just as adequately represent the system. Indeed, if **Q** is a nonsingular nxm matrix, then the system (5) may as well be described by a new state vector

$$\hat{x} = Qx \quad (7)$$

This change of **x** to $\hat{x}$ is known as a similarity transformation. Note that **x** may be recovered from $\hat{x}$ by

$$x = Q^{-1}\hat{x} \quad (8)$$

The differential equation of the system in terms of **x** is

$$\begin{aligned} \dot{\hat{x}}(t) &= QAQ^{-1}\hat{x}(t) + QBu(t) \\ y(t) &= C'Q^{-1}\hat{x}(t) \end{aligned} \quad (9)$$

Since nothing is lost through a similarity transformation, systems (5) and (9) are dynamical equivalent. Later on in the sequel, it will be shown that everything of importance (poles, transfer matrix, controllability and observability) is invariant under a similarity transformation. A similarity transformation may provide an added insight to the internal structure of the system and may often be used to provide a new system that is easy to handle computationally. For instance, if **A** has a linear independent set of eigenvectors and Q is chosen to be U^{-1}, U being a matrix of eigenvectors. Then the system of equation (9) takes on the diagonal form,

$$\dot{\hat{x}}(t) = \begin{bmatrix} \lambda_1 & 0 & ---- & 0 \\ 0 & \lambda_2 & 0 --- & 0 \\ | & & & \\ | & & & \\ 0 & ---- & 0 & \lambda_n \end{bmatrix} \hat{x} + U^{-1}Bu(t) \qquad (10)$$

$$y(t) = C'U\hat{x}(t)$$

3. Solution of Homogeneous Differential Equations: Natural Response

In this section solutions of the homogeneous differential equation

$$\dot{x}(t) = Ax(t) \qquad (14)$$

with initial state

$$x(o) = x_0 \qquad (14)$$

will be examined. To find the solution of (11), we take its Laplace transform

$$L\left\{\dot{x}(t)\right\} = sx(s) - x_0 = Ax(s) \qquad (13)$$

Rearranging equation (13) and taking the inverse transform yields

$$x(s) = (sI - A)^{-1} x_0$$

$$x(t) = L^{-1}\left\{(sI - A)^{-1}\right\}x_0 \qquad (14)$$

One method to solve for $x(t)$ is to expand $(sI - A)^{-1}$ in a matrix Binomial series, that is

$$(sI - A)^{-1} = s^{-1}I + s^{-2}A + s^{-3}A^2 + ... \qquad (15)$$

and taking the inverse transform term by term in this series yields

$$L^{-1}(sI - A)^{-1} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ...$$

$$= e^{At} \qquad (16)$$

Then the solution of the homogeneous differential equation (11) is given as

$$x(t) = e^{At}x_0 \qquad (17)$$

where the matrix e^{At} is known as a transition matrix.

Clearly equation (16) is quite impractical to implement as it involves summation of infinite series. Another method for obtaining e^{At} is as follows. First we invert $(sI-A)^{-1}$ to obtain

$$(sI-A)^{-1} = \frac{Q(s)}{P(s)} = \frac{Q_0 + Q_1 s + \dots\dots\dots + Q_{n-1} s^{n-1}}{P_0 + P_1 s + \dots\dots\dots + P_{n-1} s^{n-1} + s^n} \qquad (18)$$

where (P) is the characteristic polynomial of $(sI-A)$ defined by

$$P(s) = \det(sI-A)$$

The matrix coefficients Q_i and the scalar coefficients P_i can be generated from Faddeev's algorithm [17]

$$Q_{n-k} = Q_{n-k+1} A + d_{n-k+1} I \qquad (19)$$

and $$d_{n-k} = -\frac{1}{k} \operatorname{tr}(Q_{n-k} A) \qquad (20)$$

$$k = 1, \dots n \qquad \text{and } Q_o = O,\ Q_{n-1} = I$$

To be specific, suppose

$$P(s) = (s-\lambda_1)^m (s-\lambda_{m+1}) \dots (s-\lambda_n) \qquad (21)$$

That is, the first eigenvalue of A is repeated m times and the others are distinct. Equation (18) may then be expanded in a partial fraction expression as

$$(sI-A)^{-1} = \frac{Q(s)}{P(s)} = \frac{R_1}{(s-\lambda_1)} + \frac{R_2}{(s-\lambda_1)^2} + \frac{R_3}{(s-\lambda_1)^3} + \dots\dots$$
$$+ \frac{R_m}{(s-\lambda_1)^m} + \frac{R_{m+1}}{(s-\lambda_{m+1})} + \dots + \frac{R_n}{(s-\lambda_n)} \qquad (22)$$

where R_1, R_2, ...Rm are matrices of residues.

Taking the inverse Laplace transform of each term, we find that

$$L^{-1}\left\{(sI-A)^{-1}\right\} = e^{At} = R_1 e^{\lambda_1 t} + R_2 t e^{\lambda_2 t} + \dots + R_m t^{m-1} e^{\lambda_m t}$$
$$+ R_{m+1} e^{\lambda_{m+1} t} + \dots R_n e^{\lambda_n t} \qquad (23)$$

or, if the eigenvalues are all distinct

$$e^{At} = \sum_{i=1}^{n} R_i e^{\lambda_i t} \qquad (24)$$

equations (23) and (24) are known as the spectral factorization of A and explicitly display the stability properties of the homogeneous differential equation (11).

Theorem 3.1 Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A

a) All solutions of the homogeneous differential equation

$$\dot{x}(t) = Ax(t) \qquad (26)$$

approach zero as $t \rightarrow \propto$ (that is, the system is asymptotically stable) if and only if

$$Re(\lambda_i) < 0 \qquad i = 1,2,\ldots\ldots,n, \qquad (26)$$

where $Re(\lambda_i)$ denotes the real part of λ_i

b) All solutions of equation (25) remain bounded as $t \rightarrow \propto$ (that is, the system is stable) if and only if

$$Re(\lambda_i) \leq 0 \qquad i = 1,2,\ldots,n \qquad (27)$$

and the residue matrices of order greater than one are all zero for any repeated eigenvalues with zero real part.

Another useful expression for the transition matrix is [1]

$$e^{At} = \sum_{i=1}^{n} \propto_i (t) A^i \qquad (28)$$

where $\propto_i(t)$ are scalar time functions and A^0 is defined to be I.

4. Solution of Nonhomogeneous Differential Equations:

Forced Response

We now consider the system of the previous section with an input and output defined as

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \; x(o) = x_0$$
$$y(t) = C'x(t) \tag{29}$$

The solution of equation (19) may again be found by taking Laplace transform

$$L\left\{\dot{x}(t)\right\} = sx(s) - x_0 = Ax(s) + Bu(s) \tag{30}$$

$$x(t) = L^{-1}\left\{sI-A\right)^{-1}\right\} x_0 + L^{-1}\left\{(sI-A)^{-1} Bu(s)\right\} \tag{31}$$

The first term on the right is the natural response and the second term is the forced response due to the input $u(t)$. This second term is a convolutian in time domain of $L^{-1}\left\{(sI-A)^{-1}B\right\}$ and $L^{-1}\left\{u(s)\right\}$.

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{32}$$

This equation is often known as the <u>variation of constants formula.</u>

The output y is given by

$$y(t) = C'e^{At}x_0 + \int_0^t C'e^{A(t-\tau)}Bu(\tau)d\tau \tag{33}$$

If the natural response is ignored $(x_0=0)$ the input/output relation of the system is

$$y(t) = \int_0^t C'e^{A(t-\tau)}Bu(\tau)d\tau \tag{34}$$

or in the transform domain

$$y(s) = \left[C'(sI-A)^{-1}B\right]u(s) \tag{35}$$

Thus we see that

$$H(t) = C'e^{At}B \tag{36}$$

is the impulse response matrix of the system of equation (29) and its transform

$$H(s) = C'(sI-A)^{-1}B \tag{37}$$

is the transfer function matrix from input to output.

For the system of equation (9) the transfer function matrix can readily be written as

$$\begin{aligned}\hat{H}(s) &= C'Q^{-1}\left(sI-QAQ^{-1}\right)^{-1}QB \\ &= C'\left[Q^{-1}\left(sI-QAQ^{-1}\right)^{-1}Q\right]B \\ &= C'\left(sI-A\right)^{-1}B\end{aligned} \tag{38}$$

then the transfer function matrix is invariant under a similarity transformation. Also, since the poles of the transfer function or the eigenvalues of A are found from the determinant of $(sI-A)$, then poles are also invariant.*

Theorem 4.1 The systems of equations (20) and (9) have the same transfer function and the same poles.

The ideas of sections 3 and 4 are illustrated by the following example.

Example 2 For the system of fig. 1, the system equation were found in example 1 as

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = c'x(t)$$

where $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $c' = \begin{bmatrix} 0 & 3 \end{bmatrix}$

To determine the response of this system, we first compute

$$(sI-A)^{-1} = \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{-2}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

add $e^{At} = L^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix} 2e^{-t}-e^{-2t} & -2e^{-t}+2e^{-2t} \\ e^{-t}-e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$

Thus the transfer function for this single input-single output system is

$$h(s) = c'(sI-A)^{-1}b = \frac{6}{(s+1)(s+2)}$$

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*This is because

$$\begin{aligned}\det\left(sI-QAQ^{-1}\right) &= \det\left(Q\left(sI-A\right)Q^{-1}\right) \\ &= \det\left(sI-A\right)\end{aligned}$$

and the impulse response is

$$h(t) = c'e^{At}b = 6e^{-t} - 6e^{-2t}$$

we note that the system has eigenvalues −1 and −2 and is asymptotically stable.

The forced response for an initial capacitor voltage and inductor current of −1 and a unit step input is

$$\begin{aligned} y(t) &= c'e^{At}x_0 + \int_0^t c'e^{A(t-\tau)}bu(\tau)d\tau \\ &= \begin{bmatrix} 0 & 3 \end{bmatrix} e^{At} \begin{bmatrix} -1 \\ -1 \end{bmatrix} + 6\int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right)d\tau \\ &= -3e^{-2t} + 3 + 3e^{-2t} - 6e^{-t} = 3 - 6e^{-t} \end{aligned}$$

Suppose a new state $x(t)$ is defined as

$$\hat{x}(t) = Qx(t) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} x(t)$$

Then the new differential equations are

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{b}u(t)$$
$$\hat{y}(t) = \hat{c}'\hat{x}(t)$$

where

$$\hat{A} = QAQ^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\hat{c}' = c'Q^{-1} = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \end{bmatrix}$$

These equations have the uncoupled or diagonal form as shown in fig. 3

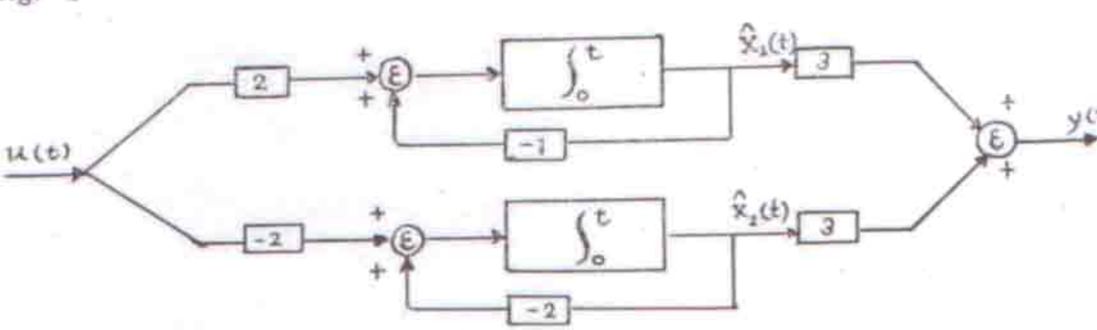

Fig. 3 Diagonal Form for Circuit of Fig. 1

And the transition matrix for this new system reduces to

$$e^{At} = \begin{vmatrix} e^{-t} & o \\ o & e^{-2t} \end{vmatrix}$$

5. Controllability and Observability

In the previous sections we introduced the concept of a linear dynamical system (fig. 2) with input, output and state vectors governed by differential equations. We also discussed the general form of the response of the system for the initial state x_0 and control $u(t)$. Now we consider the fundamental problems of controlling the state of such a system with the control $u(t)$ and observing (or determining) its state from the output $y(t)$ The following definition gives a precise meaning to controlling the state.

Definition 5.1 A particular state $\bar{x}$ of the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = C'x(t) \qquad (39)$$

is controllable if for every $T > o$ there exists a control $u(t)$, $o \leq t \leq T$, that drives the system from the initial state $x(o) = \bar{x}$ to $x(T) = O$

The entire system of equation (28) is completely controllable if every state is controllable.

Conditions for a state to be controllable is given by the following theorem.

Theorem 5.1 A state $\bar{x}$ of the system of equation (39) is controllable if and only if $\bar{x}$ belongs to the range space $R(P)$** where P is the nxmm controllability matrix

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** If $\bar{x}$ belongs to $R(P)$, then there exists a vector y such that $\bar{x} = Py$

$$P = \left[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \ldots\ldots \,\vdots\, A^{n-1}B\right] \tag{40}$$

Proof Suppose first that $\bar{x}$ is controllable, then from definition 5.1, there exists a control $u(.)$ such that

$$O = x(T) = e^{AT}\bar{x} + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau \tag{41}$$

multiplying equation (30) by e^{-AT} and using equation (28) yields

$$\begin{aligned}\bar{x} &= -\int_0^T e^{-A\tau}Bu(\tau)d\tau \\ &= -\int_0^T \sum_{i=1}^{n} \alpha_i(-\tau)A^{i-1}Bu(\tau)d\tau \\ &= \sum_{i=1}^{n} A^{i-1}B\int_0^T -\alpha_i(-\tau)u(\tau)d\tau\end{aligned} \tag{42}$$

defining the m-vector

$$b_i = \int_0^T -\alpha_i(-\tau)d\tau \qquad i = 1, 2, \ldots\ldots, n \tag{43}$$

an the mn-vector

$$b' = \left[b_1 \,\vdots\, b_2 \,\vdots\, \ldots\ldots \,\vdots\, b_n\right] \tag{44}$$

equation (31) may be written as

$$\bar{x} = \left[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \ldots\ldots \,\vdots\, A^{n-1}B\right]b = Pb \tag{45}$$

If $\bar{x}$ is controllable, then the equation has a solution and it is clear that $\bar{x}$ must belong to the range space $R(P)^{**}$. Conversely, if $\bar{x}$ belong to $R(P)$ then equation (45) has a solution b. Because of the properties of $\alpha_i(.)$, $i = 1, 2, \ldots.\, n$, a control $u(.)$ may always be found to generte b via equations (43) and (44) and $\bar{x}$ is controllable.

Q.E.D.

Another proof may be found in references [1,2,6-9] along with the generalization to time varying systems.

The condition of complete controllability of a system follows easily from Theorem 3 and the proof is left for interested readers.

Corollary 5.1 The system of equation (28) is completely controllable if and only if the matrix P has full rank (i.e rank n). For a single input system the condition reduces to P being nonsingular.

To illustrate the above concepts, let us consider the following example.

Example 3 Consider the system

$$\dot{x}(t) = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u(t), \; x(o) = \bar{x}$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)$$

This system is not completely controllable, as (46)

$$P = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$$

has rank = 1. The controllable states are those which belongs to the range space of P. Here only those controllable states $\left\{ \bar{x} : \bar{x} \propto \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ may be driven to O.

To verify this, we compute

$$(sI - A)^{-1} = \begin{bmatrix} s-2 & -2 \\ o & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-2} & \frac{2}{(s-1)(s-2)} \\ o & \frac{1}{(s-1)} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{2t} & -2e^{2t} - e^{t} \\ o & e^{t} \end{bmatrix}$$

Then, from equation (42)

$$\bar{x} = - \int_o^T \begin{bmatrix} e^{-2t} & 2e^{-2t} - e^{-t} \\ o & e^{-t} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} u(t)\, dt$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \int_o^T e^{-t} \; u(t)\, dt$$

Regardless of what u(.) is chosen, only those states which have this form may be driven to O.

Now we turn to the question of whether a particular state is observable at the output of the system. For the system of equation (39) if the input u(.) is known, then the forced response at the output due to u(.)

$$\int_o^t C' \, e^{A(t-\tau)} Bu(\tau) d\tau$$

is a known quantity independently of the initial state. Thus it suffices to consider instead a system with zero input

$$\dot{x}(t) = Ax(t)$$
$$y(t) = C'x(t)$$

Definition 5.2 A particular state $\bar{x}$ of the system of equation (47) is unobservable if for every $T > 0$ the initial condition $x(0) = \bar{x}$ produces an output

$$y(t) = 0 \qquad 0 < t < T$$

The entire system is said to be completely observable if no nonzero state is unobservable.

Conditions for observability are as follows.

Theorem 5.2 A state $\bar{x}$ of the system of equation (47) is unobservable* if and only if it belongs to the null space $N(M)$ where M is the pn × p observability matrix

$$M = \begin{bmatrix} C' \\ C'A \\ C'A^2 \\ \vdots \\ C'A^{n-1} \end{bmatrix}$$

Proof For an initial state $x(0) = \bar{x}$, the output of the system is

$$y(t) = C'e^{At}\bar{x}$$

substituting for e^{At} by equation (28) yields

$$y(t) = \sum_{i-1}^{n} \alpha_i(t)\, C'A^{i-1}\bar{x}$$

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* A state is said to be observable if it is orthogonal to the null space $N(M)$. Thus a state may have both observable and unobservable components.

$$= \left[\alpha_1(t) I \;\vdots\; \alpha_2(t) I \;\vdots\; \ldots\; \alpha_n(t) I \right] \begin{bmatrix} C' \\ C'A \\ C'A^2 \\ \vdots \\ C'A^{n-1} \end{bmatrix} \bar{x}$$

$$= \left[\alpha_1(t) I \;\vdots\; \alpha_2(t) I \;\vdots\; \ldots\; \alpha_n(t) I \right] M\bar{x}$$

If $\bar{x}$ belongs to $N(M)$ (that is $M\bar{x} = 0$), then $y(t) = 0$ and $\bar{x}$ is unobservable. Conversely if $\bar{x}$ is unobservable, by the definition $y(t) = 0$, $o < t < T$. Then $M\bar{x} = 0$, so $\bar{x}$ belongs to $N(M)$.

Another proof and generalization to time varying system may be found in references [1,2,7–10] The condition for complete obserability follows easily and the proof is left for interested readers.

Corollary 5.2 The system of equation (47) is completely observable if and only if the observability matrix

$$M = \begin{bmatrix} C' \\ C'A \\ C'A^2 \\ \vdots \\ C'A^{n-1} \end{bmatrix}$$

has nullity 0 (or rank n). For a single output system this reduces to M being nonsingular.

Let us consider the following example

Example 4 Consider the system.

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 2 \end{bmatrix} u(t), \; x(o) = \bar{x}$$

$$y(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} x(t)$$

Since $M = \begin{bmatrix} C' \\ C'A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ the system is not completely observable. The unobservable states are those belonging to the null space of M and are proportional to $[\,0 \;\; 1\,]'$. To verify this, we compute.

$$\left(sI-A\right)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)(s-2)} & \frac{1}{(s-2)} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ e^{2t}-e^{t} & e^{2t} \end{bmatrix}$$

Them for initial state $x\left(o\right)$

$$y\left(t\right) = c'e^{At}x\left(o\right) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e^{t} & o \\ e^{2t}-e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} x_1 & (o) \\ x_2 & (o) \end{bmatrix}$$

$$= x_1\left(o\right) e^{t}$$

and cleary any states of the form $\left\{ \bar{x} ; \bar{x} \propto \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ produce zero output and thus are unobservable.

The resemblance between the conditions for controllability and observability should be apparent. In fact, they are dual concepts, as explained in the following theorem.

Theorem 5.3 Controllability and observability are dual in the sense that the system

$$\dot{x} = Ax + Bu$$

$$y = Cx \qquad (51)$$

is completely controllable (observable) if and only if the system

$$\dot{\hat{x}} = A'\hat{x} + C'\hat{u} \qquad (52)$$

$$\hat{y} = B'\hat{x}$$

is completely observable (controllable)

Proof The statement of this theorem is easily proved by noting that the controllability (or observability) matrix for one system becomes the observability (or controllability) matrix for the other system.

Finally we note that both controllability are invariant under similarity transformation.

Theorem 5.4 Suppose two systems

$$\dot{x} = Ax + Bu \qquad (53)$$

$$y = Cx$$

and

$$\dot{\hat{x}} = QAQ^{-1}\hat{x} + QBu \qquad (54)$$

$$y = CQ^{-1}\hat{x}$$

are related by the similarity transformation

$$\hat{x} = Q\,x \qquad (55)$$

Then $\bar{x}$ is controllable (unobservable) if and only if $Q\bar{x}$ is controllable (unobservable)

Proof The controllability matrices for systems of equation (53) and (54) are

$$P = \left[\, B \,\middle|\, AB \,\middle|\, A^2B \,........ A^{n-1}B \,\right]$$

and

$$\hat{P} = \left[\, QB \,\middle|\, Q\,AQ^{-1}QB \,\middle|\, ...(QAQ^{-1})^{n-1}QB \,\right]$$

$$= QP$$

Thus $\bar{x}$ belonging to $R(P)$ implies $Py = \bar{x}$ for some y and hence $QPy = \hat{P}y = Q\bar{x}$. Conversely $Q\bar{x}$ belonging to $R(P)$ implies $\hat{P}y = QPy = Q\bar{x}$ for some y and $Py = Q^{-1}\hat{P}y = \bar{x}$.

The observability matrices are related by

$$\hat{M} = MQ^{-1}$$

and the proof for unobservable states follows analogously.

6. Feedback Control

The fundamental problem of control systems theory is to determine a control which is to be applied at the input of the system to produce. some desired output. By far the most common form of control is that of a feedback control law where some operation is performed at the output of the system and the result is applied to the input. In the case of linear systems, a linear function of the output is often added to the input to form a proportional feedback control.

6.1 Proportional Feedback

In terms of a linear system

$$\dot{x} = Ax + Bu \qquad (56)$$
$$y = C'x$$

Proportional feedback from output to input takes the form

$$u = Ky + u_{ext} \qquad (57)$$

where u_{ext} is the external input and the resulting closed loop system is

$$\dot{x} = Ax + B\left(Ky + u_{ext}\right) = \left(A + BKC'\right)x + Bu_{ext}. \qquad (58)$$

as illustrated in fig. 4

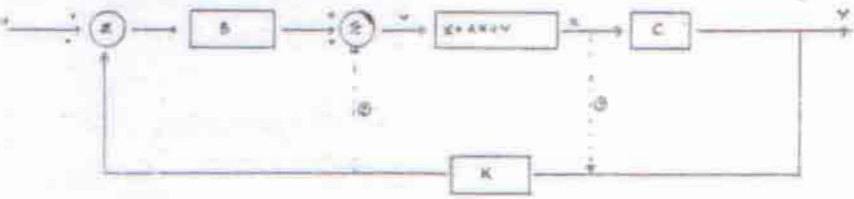

Fig 4. Proportional Feedback from Output to Input.

Dashed lines (1) Feedback from State to Input

(2) Feedback from Output to State.

If all states are available for feedback, we may set C = I in the control law and obtain the closed loop system

$$\dot{x} = \left(A + BK\right)x + Bu_{ext} \qquad (59)$$
$$y = C'x$$

This feedback from state to input is known as <u>state variable feedback.</u>

There is another class of feedback known as output-to-state feedback. This type of feedback is possible if all states can be controlled separately and the closed loop system can be obtained by setting $B = I$ as.

$$\dot{x} = \left(A + KC'\right) x + Bu_{ext} \qquad (60)$$
$$y = C'x$$

The effects of feed back on the controllability and observability of a system are as follows :—

Theorem 6.1 Controllability is invariant under feedback from state to input and observability is invariant under feedback from output to state. Both controllability and observability are invariant under output to input feedback.

The proof of this theorem is left for interested readers.

In the design of control system, a common criterion is to have a stable operating system. By means of feedback control law it is possible to prescribe the system poles to any desired positions in the complex plane. [10—11] This useful result can be stated as follows :—

Theorem 6.2 If the system

$$\dot{x} = Ax + Bu \qquad (61)$$
$$y = Cx$$

is controllable, then, for any pre—assigned configuration of n poles in the complex plane (with the restriction that complex poles must occur in conjugate pairs), there exists a state—to—input feedback matrix K, such that the closed loop system of equation (59) has its poles in the given locations. Dually, if the system of equation (61) is observable there exists an output—to—state feedback matrix, K_1 such that the closed loop system of equation (60) has its poles in the given locations

Proof We will establish the result for only a single input–single output confrollable

$$\dot{x} = Ax + Bu \qquad (62)$$
$$y = C'x$$

The reader is invited to verify that the transformation matrix of the form

$$Q^{-1} = \left[A^{n-1}b \,\middle|\, A^{n-2}b \,\middle|\, \ldots\ldots Ab \,\middle|\, b\right] \begin{bmatrix} 1 & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ p_{n-1} & 1 & 0 \ldots\ldots\ldots\ldots 0 \\ p_{n-2} & p_{n-1} & 1 \ldots\ldots\ldots\ldots 0 \\ \vdots & & \\ p_1 & p_2 & p_3 \ldots\ldots p_{n-1} \end{bmatrix} \qquad (63)$$

exists such that the system

$$\dot{\hat{x}} = QAQ^{-1}\,\hat{x} + Qbu = \hat{A}\,\hat{x} + \hat{b}u \qquad (64)$$

is in standard controllable form, where

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \ldots\ldots & 0 \\ 0 & 0 & 1 \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 \ldots\ldots & 1 \\ -p_0 & -p_1 & -p_2 \ldots\ldots & -p_{n-1} \end{bmatrix}, \hat{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad (65)$$

and $p_0\ p_1 s + p_2 s^2 + \ldots\ldots p_{n-1} s^{n-1} + s^n$ is the characterictic polynomial of A. For the feedback of the form

$$u = \hat{k}'\hat{x} + u_{ext}$$

where $\hat{k}' = \left(k_0\ k_1 \ldots\ldots\ldots k_{n-1}\right)$ (66)

We can see that the closed loop system becomes

$$\dot{\hat{x}} = \left(\hat{A} + \hat{b}\hat{k}'\right)\hat{x} + \hat{b}u_{ext} \qquad (67)$$

where

$$\hat{A} + \hat{b}\hat{k}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ | & | & | & & | \\ 0 & 0 & 0 & \cdots & 1 \\ -(p_0+k_0) & -(p_1+k_1) & -(p_2+k_2) & \cdots & -(p_n+k_n) \end{bmatrix} \tag{68}$$

The characteristic polynomial of equation (68) is $(p_0+k_0) + (p_1+k_1)s + (p_2+k_2)s^2 + \cdots + (p_{n-1}+k_{n-1})s^{n-1} + s^n$ and its coefficients may be specified to give any desired positions of poles.

Since the system of equation (68) and the system

$$\begin{aligned} \dot{x} &= Q^{-1}(\hat{A}+\hat{b}\hat{k}')Qx + Q^{-1}\hat{b}\, u_{ext} \\ &= (A+b\hat{k}'Q)\, x + bu_{ext} \end{aligned} \tag{69}$$

are dynamical equivalent. Then the system of equation (69) has the same set of pre-assigned poles with a feedback control law of the form

$$u = \hat{k}'Qx + u_{ext} \tag{70}$$

The dual result may be proved in an analogous manner

6.2 Dynamic Feedback

In practice, the state variables in the system are not readily accessible, thus feedback control laws directly involving the state may not be possible. Therefore we need to reconstruct the state of the system from output information using a device known as an observer, or state estimator. Intuitively it should be clear that to be able to approximate or reconstruct the state out of any output information, the system should be completely observable.

Suppose the system to be controlled is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{71}$$

Then an observer is a second order linear system

$$\dot{z} = Fz + Gu + Hy$$

$$w = Dz + Ey \qquad (72)$$

driven by the input and output of the system of equation (71) whose output, w (t), approximates x (t). With a control law of the form

$$u = Kw + u_{ext} \qquad (73)$$

the augmented closed loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A+BKEC & BKD \\ HC+GKEC & F+GKD \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ G \end{bmatrix} u_{ext}$$

In general an observer should have dimension large enough to supplement the p-dimensional information supplied by the output vector. The following theorem confirms this and establishes the existence of the state estimator.

Theorem 6.3 Let the dimension of x, y, z, w be n, p, r and n, respectively If $r > (n-p)$ and the system of equation (71) is completely observable, then there exists matrices F,G,H,D and E such that the output, w (t), of the system of equation (72) approaches the state x (t) of the system of equation (71) asymptotically as $t \to \infty$ regardless of the initial states x_0 and z_0 and extermal input u_{ext}. The dynamic behaviour of the observer (72) is determined by the poles of F which may be chosen arbitrarily.

If the feedback law (73) is applied, the poles of the (n+r) dimensional closed loop system of equation (74) consists of the poles of (A+BK) and those of F.

NB This result was first established by Luenberger (13-14) and may also be found in references (10, 12-14)

6.3 Linear Regulator

In recent years optimal control theory has provided some powerful techniques for the design of both closed-loop and open-loop control laws. The simplest example of this theory is the linear regulator, in which a feedback control law is to be designed for the system

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{75}$$

by choosing the control which minimizes a quadratic cost functional of the form

$$J = \int_0^\infty \left(u'Ru + x'Qx\right) dt \tag{76}$$

where matrices R and Q are generally positive definite and positive semidefinite, respectively. The effect of the cost functional (76) is to penalize large state and controls more heavily than small ones. Conceptually this is equivalent to saying that energy required for controlling the system as well as the energy in the system should be kept minimum.

The optimal control has the form of feedback from state to input as follows:

<u>Theorem 6.4</u> Assume that the system of equation (75) is controllable and that R and Q are positive definite and positive semidefinite, respectively. Then there exists the control

$$u = -R^{-1} B'Px(t) \tag{77}$$

minimizing the cost functional of equation (76). P is the positive definite solution of

$$A'P + PA - PBR^{-1} B'P + Q = 0 \tag{78}$$

and the closed loop system

$$\dot{x} = \left(A - BR^{-1}B'P\right)x \tag{79}$$

is asymptotically stable.

7. Summary and Conclusion

The state variable approach to finite – dimensional time invariant, linear control systems theory has been introduced. In section 2. the concepts of a dynamical system and its states were discussed. The idea of similarity transformation was also explained.

Sections 3 and 4 dealt with the solutions of homogenous and inhomogenous, linear time invariant differential equations which were used for modelling of linear time invariant dynamical systems. Solutions were obtained employing the matrix exponential e^{At}. Various representations of e^{At} were given along with Feddeeva's algorithm for generating e^{At}. The solution of the inhomogenous equation was found as a convolution of e^{At} with the forcing function.

The concepts of controllability and observability were defined and explained in detail in section 5. Duality between controllability and observability was noted. It was also seen that controllability and observabiltty were invariant under similarity transformation.

The fundamental concept of various feed back controls were introduced in section 6. Of significant in this section is the fact that if a system is controllable, pole assignment can be acheived by state variable feedback. A brief introduction to the dynamic observer and the optimization theory in the form of a linear regulator was also given in this section.

This paper was meant to be only a brief tutorial introduction to linear systems control theory. The material covered barely scratches the surface of modern control theory. The interested reader is refered to the textbooks (1-9) and the excellent articles (15-16)

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