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Mathematical aspects of electrical network connections

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Abstract

In this paper, we survey the role of mathematics in electrical network connections. We discuss the behavior of current flows, voltages and impedances, mainly for series-parallel networks. In both one-port and multiport electrical networks, currents are governed by Maxwell's power principle. The joint impedances of the networks, given in terms of series and parallel sums, satisfy the series-parallel inequality. An abstract idea can be formulated in functional analysis in which any network connection is viewed as a binary operation for positive operators satisfying certain algebraic, order and topological properties.

Keywords: Electrical network connection, Maxwell's power principle, Parallel sum, Positive definite matrix, Positive operator

1. Introduction

In electrical engineering, an electrical network is an interconnection of physically electrical components (e.g. batteries, resistors, capacitors, inductors, switches) or a model of such an interconnection, consisting of electrical elements (e.g. voltage/current sources, resistances, inductances). This paper provides a discussion of the flows of currents through an electrical network obeying Ohm's law and Kirchhoff's voltage/current laws:

Ohm's law: the current through an electrical device is the ratio between the voltage (electrical potential difference) dropped on this device and its impedance.

Kirchhoff's current law: the sum of currents meeting at a node is zero.

Kirchhoff's voltage law: the sum of voltages in a closed electrical circuit is zero.

The main concern here is the joint impedance of series-parallel network. Various properties of series and parallel additions and their physical interpretations are investigated.

For one-port network, the impedance of the network can be described by the notion of parallel sum for scalars. Algebraic properties of this operation were investigated in [1]. The current flow in the network is governed by the so called Maxwell's power principle. Elementary algebra and calculus shows that the joint impedance of the network satisfies the series-parallel inequality [2].

The analysis will become more complicated in the case of multiport electrical networks. Here, the joint impedance of the network is represented in terms of matrix. Many authors discussed the role of linear algebra and matrix theory for network synthesis, focused on series-parallel connections (see e.g. [3-5]). The main tool for analyzing multiport electrical networks is the notion of parallel sum for positive definite matrices, introduced in a seminal paper [6]. It turns out that the flows of electrical currents satisfy Maxwell's principle and series-parallel inequality as in one-port case [6]. The theory of parallel sums was then discussed by many authors (see e.g. [7-8]). This motivated the study of mathematical operations derived from electrical networks, such as parallel subtraction (see e.g. [9-10]), hybrid connection (e.g. [11-12]), Wheatstone bridge connection ([13]), shorted operator ([14-17]).

To extend the idea of network connections, the perspective of functional analysis is an appropriate framework. The joint impedance of the network can be viewed as a positive operator acting on a Hilbert space. Currents and power dissipations are described by vectors and inner products on that Hilbert space. The notion of parallel sum for positive operators was considered in [15]. Algebraic, order and topological properties of the parallel sum were discussed in [15] and [18-20]. It turns out that the series-parallel inequality and Maxwell's principle also hold in this setting [19]. Notice that the parallel sum was characterized via a set of certain properties in [18]. Applications of parallel sum also go to the area of matrix/operator inequalities and equations (see e.g. [21-23]).

More abstractly, the idea of connection, introduced by Kubo and Ando [24], is a suitable generalization of series and parallel connections. A general connection is a binary operation for positive operators satisfying certain algebraic, order and topological properties. Series connection and parallel connection are typical examples of this concept. This beautiful theory was developed by many mathematicians; see e.g. [24-28]. Every connection can be realized as a weighted series connection of weighted parallel connections.

This paper is organized as follows. The next section is an analysis of one-port electrical networks. The third section

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deals with the role of linear algebra in multiport network connections. Analysis of electrical connections is presented in the language of functional analysis in the fourth section. A general setting for network connection is settled in the fifth section. Finally, we summarize the role of mathematics for electrical network connections.

2. One-port electrical networks

Consider a simple electrical circuit consisting of a battery of fixed voltage E and a resistor of resistance R, as shown in Figure 1.

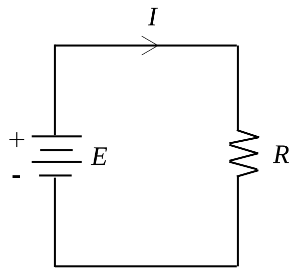

Figure 1 A simple electrical circuit

By Ohm's law, the current I flowing in the circuit is given by $I = E / R$. For the case of alternative current circuits, the voltage source generates sinusoidal waves and electrical components in the circuit may be not pure resistors (e.g. capacitors, inductors). In this case, resistances are replaced by impedances, which are complex numbers, and the Ohm's law still holds.

A one-port network is a "black box" with a single pair of input/output terminals. Consider two resistors connected in series as in Figure 2:

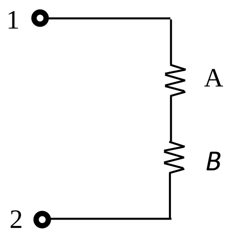

Figure 2 A series connection of two resistors

By Kirchhoff's voltage law and Ohm's law, the joint resistance R between terminals 1 and 2 is determined by $R = A + B$. Circuit equivalently (using Figure 1), two resistors together act as if they were a single resistor whose resistance is given by the *series sum* R.

Next, consider the parallel connection shown in Figure 3.

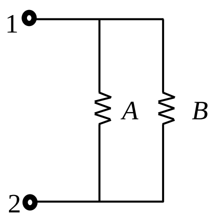

Figure 3 A parallel connection of two resistors

Using Kirchhoff's current law and Ohm's law, the joint resistance R between terminals 1 and 2 satisfies the relation

$$\frac{1}{R} = \frac{1}{A} + \frac{1}{B} \text{ or } R = \left(A^{-1} + B^{-1}\right)^{-1} = \frac{AB}{A+B}. \tag{1}$$

More precisely, the resistors together act as if they were a single resistor whose resistance is given by the *parallel sum* R, denoted by $A:B$ (see [1]). The algebraic operation $:$ is termed the *parallel addition.* The network model shows that the parallel addition is commutative and associative. Moreover, multiplication is distributive over this operation.

Consider a series-parallel connection as in Figure 4:

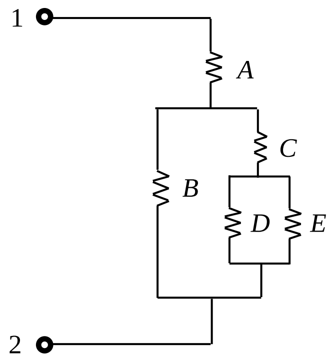

Figure 4 A series-parallel network

The joint resistance of this network is given in terms of series addition and parallel addition as follows:

$$R = A + \left[B : \left(C + \left(D : E\right)\right)\right]. \tag{2}$$

Every series-parallel connection network can be interpreted in terms of series addition and parallel addition. However, not every network is a series-parallel connection, for example, the *Wheatstone bridge connection* in Figure 5.

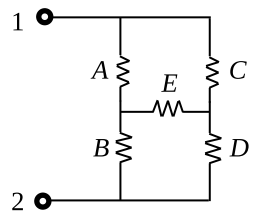

Figure 5 A Wheatstone bridge connection

In fact, a network is a series-parallel connection if and only if there is no embedded network having the Wheatstone bridge connection, see [5]. There is also a simple characterization of series-parallel connection given by [4].

Recall that the flow of currents through electrical circuits is governed by Maxwell's power principle: the current will take flow paths in such the way that the power dissipation is minimized. This principle, also known as Rayleigh's principle, is equivalent to a variational description of the parallel sum $A:B$ as follows:

$$A : B = \min_{x+y=1} Ax^2 + By^2. \tag{3}$$

The extremal characterization (3) can be derived using optimization technique in multivariable calculus. This serves an easy proof of Lehman's series-parallel inequality (see [3]) as follows.

Consider an electrical network as shown in Figure 6.

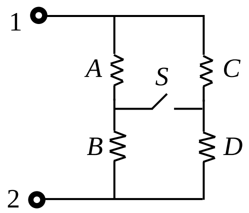

Figure 6 An electrical network for proving the series-parallel inequality

When the switch S is open, the joint resistance is given by

$$R_o = (A+C):(B+D). \tag{4}$$

On the other hand, when S is closed, the joint resistance becomes

$$R_c = (A:C)+(B:D). \tag{5}$$

Since the current takes the path of least resistance (that is, least power) and there is less constraint with the switch close, we arrive at the Lehman's series-parallel inequality:

$$(A:C)+(B:D) \le (A+B):(C+D). \tag{6}$$

Alternatively, the series-parallel inequality can be expressed as

$$\frac{AC}{A+C}+\frac{BD}{B+D} \le \frac{(A+B)(C+D)}{A+B+C+D}. \tag{7}$$

It is worth nothing that the connection in Figure 6 corresponds to replacing the resistor E in the Wheatstone bridge connection in Figure 5 with a switch. Let R_w be the joint resistance of Wheatstone bridge. By Rayleigh's principle, we obtain

$$R_c \le R_w \le R_0. \tag{8}$$

3. Multiport electrical networks

A multiport network consists of several pairs of input/output terminals. In this section, we analyze multiport electrical networks using linear algebra.

Consider an electrical network with two pairs of terminals as in Figure 7:

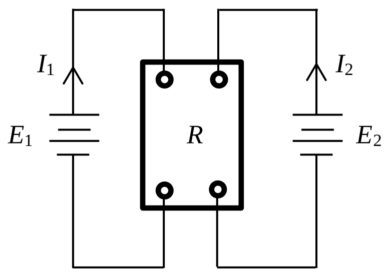

Figure 7 A two-port electrical network

The first pair of terminals is in circuit 1 and the second one is in circuit 2. Then the currents and voltages in these circuits are related by the following system of linear equations:

$$\begin{aligned} E_1 &= R_{11}I_1 + R_{12}I_2 \\ E_2 &= R_{21}I_1 + R_{22}I_2 \end{aligned}. \tag{9}$$

Rewrite these equations in vector/matrix form as

$$E = RI \tag{10}$$

where

$$E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},\ R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},\ I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.$$

If the resistor box R contains interconnected resistors, then $R_{12} = R_{21}$, that is R is a symmetric matrix. Moreover, the conservation law of energy implies that R is a positive semidefinite matrix, more precisely $x^T Rx \ge 0$ for all $x \in \mathrm{R}^2$. In what follows, a positive semidefinite symmetric matrix will be called a *resistance matrix*.

Resistor boxes can be added in series as is show in Figure 8.

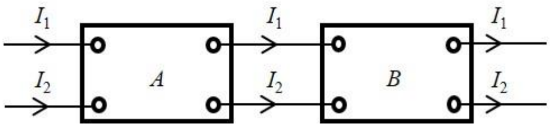

Figure 8 A series connection of two-port networks

Here, we assume that the current I_1 in the first circuit of box A is the same as the current in the first circuit of the box B. It is similar for the current I_2. This can be achieved via use of isolation transformers.

If A and B are the resistance matrices of these networks, then the joint resistance matrix R is given by $R = A + B$. In other words, series connection of resistance boxes corresponds to addition of their resistance matrices. Figure 9 gives a symbolic meaning of the series addition of resistor boxes.

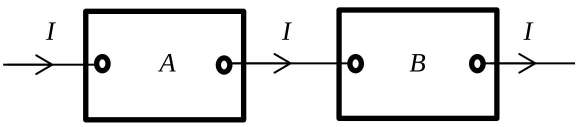

Figure 9 A symbolic meaning of the series addition

From the relation (10), any current vector I can be an input of a resistor box. However, not every voltage vector E can be an input if the resistance matrix R is not invertible. In any cases, the following fact relates the range spaces of the series connection of the networks.

$$\mathrm{Range}(A+B) = \mathrm{Range}(A) + \mathrm{Range}(B) \tag{11}$$

Now, consider the case when we connect the resistor boxes in parallel as in Figures 10 and 11.

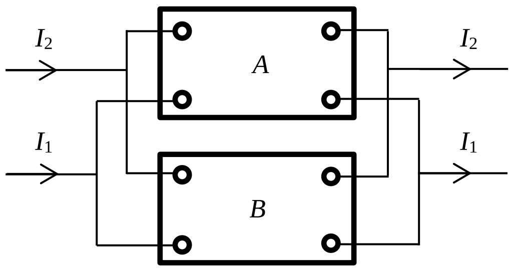

Figure 10 A parallel connection of two-port networks

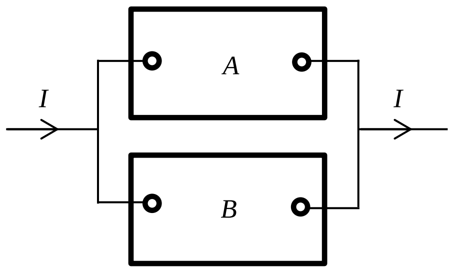

Figure 11 A symbolic form of a parallel connection

Suppose that both A and B are represented by invertible positive semidefinite matrices (that is, positive definite matrices). Then the joint resistance matrix R of the parallel connection and the resistance matrices A, B are related by

$$R^{-1} = A^{-1} + B^{-1}. \tag{12}$$

We obtain that $R = A(A+B)^{-1}B$. For simplicity, we write $A:B$ for R and call it the parallel addition of A and B, that is

$$A:B = A(A+B)^{-1}B. \tag{13}$$

Note that the relation (11) shows that $\text{Range}\,(A+B) \supseteq \text{Range}(B)$. This means that $A+B$ is invertible on its range. Hence, $(A+B)^{-1}B$ is a well-defined matrix. Thus, the parallel addition is a well-defined operation for any pair of positive semidefinite matrices. Various algebraic, order and analytic properties of this operation were investigated in [10-13]. We will discuss some of these properties.

By virtue of the network model, the parallel addition is expected to be commutative and associative. Here, we give a direct proof of commutativity:

$$\begin{aligned} A:B &= (A+B-B)(A+B)^{-1}B = B - B(A+B)^{-1}B \\ B:A &= B(A+B)^{-1}(A+B-B) = B - B(A+B)^{-1}B \end{aligned} \tag{14}$$

This implies that

$$A:B = B(A+B)^{-1}A. \tag{15}$$

From the definition (13) of parallel sum, we clearly have $\text{Range}\,(A+B) \subseteq \text{Range}\,(A)$. Similarly, the relation (15) shows that $\text{Range}\,(A+B) \subseteq \text{Range}\,(B)$. Further analysis gives

$$\text{Range}(A+B) = \text{Range}(A) \cap \text{Range}(B). \tag{16}$$

The relations (11) and (16) reveal a remarkable duality between series addition and parallel addition.

To give an application of the above duality principle, we will analyze the networks in Figures 12 and 13.

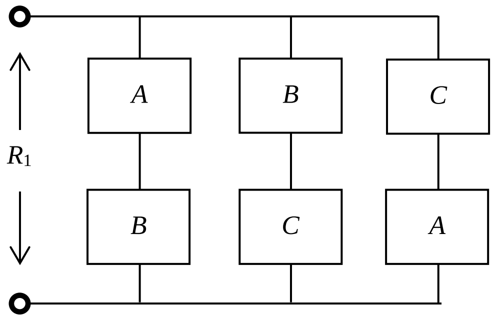

Figure 12 A series-parallel network 1

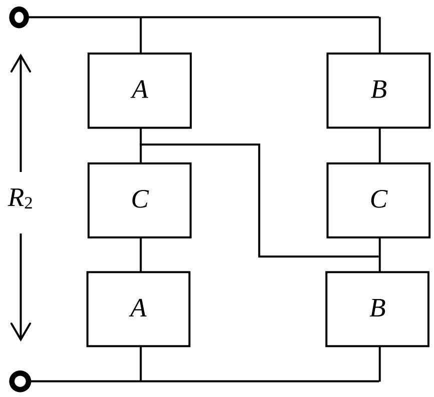

Figure 13 A series-parallel network 2

Clearly the joint resistance matrix of the first network is given by

$$R_1 = (A+B):(B+C):(C+A) \tag{17}$$

Let X, Y, Z be the range spaces of A, B, C respectively. From (11) and (17), we have

$$\text{Range}(R_1) = (X+Y)\cap(Y+Z)\cap(Z+X) \tag{18}$$

On the other hand, the joint resistance matrix of the second network is

$$R_2 = [A:(B+C)] + [B:(A+C)]. \tag{19}$$

Hence, the range of R_2 is

$$\text{Range}(R_2) = (X\cap(Y+Z)) + [Y\cap(Z+X)]. \tag{20}$$

Now, recall that the collection of subspaces of a vector space form a modular lattice. From modular identity, we have

$$(X+Y)\cap(Y+Z)\cap(Z+X) = (X\cap(Y+Z))+[Y\cap(Z+X)] \quad (21)$$

This means that

$$\text{Range}(R_1)=\text{Range}(R_2). \quad (22)$$

Thus, various analogous procedures for constructing networks with the same range can be obtained by the duality principle.

The network connection used by Lehman [3] to obtain the series-parallel inequality for positive reals can be extended to resistor boxes. More precisely, for positive semidefinite matrices A, B, C, D of the same size, we have

$$(A:C)+(B:D)\le(A+B):(C+D). \quad (23)$$

Here, the partial order $X \le Y$ means that $B-A$ is positive semidefinite whenever X and Y are Hermitian or real symmetric matrices.

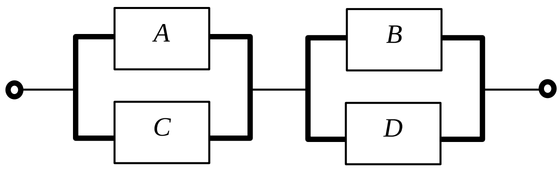

Figure 14 The joint impedance of the network with parallel first and series last

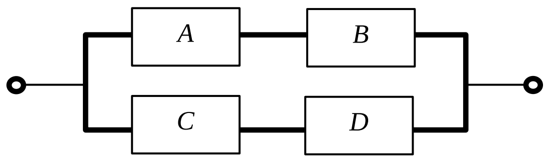

Figure 15 The joint impedance of the network with series first and parallel last

The series-parallel inequality means that the joint impedance of the network in Figure 14 is not greater than that in Figure 15.

Consider the network connections in which some circuits are put in series and some circuits are put in parallel. Such a connection is called the *hybrid connection* as shown in Figure 16.

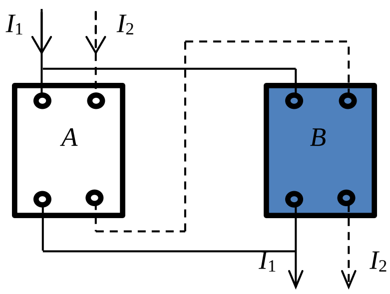

Figure 16 A hybrid connection

An elegant network synthesis of the hybrid connection can be found in [6]. The joint resistance of the hybrid connection is called the *hybrid sum*. In this type of connections, the series-hybrid inequality is valid.

4. Perspective of functional analysis in electrical network connections

Let us formulate the theory of electrical networks in the viewpoint of functional analysis. An abstract idea is to consider arbitrary networks, including infinite networks [29]. See [26, 28, 30-31] for surveys.

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, which may be of infinite-dimensional. Each current in the network considered here is given by a vector in. The impedance of this network can be represent by a positive operator which transforms any input current vector in H to an output current vector in the same space H. The voltage vector when the current x flows through the impedance operator A is given by Ax according to Ohm's law. The power dissipation of the network with impedance operator A and current vector x is given by the inner product $\langle Ax, x\rangle$. Note that the term $\langle Ax, x\rangle$ is always nonnegative due to the positivity of the operator A.

Let A and B be invertible positive operators acting on H. Each of them describes the impedance of each subnetwork. The ordinary sum $A+B$ simply expresses of the total impedance of the series connection of these networks. The parallel sum

$$A:B=\left(A^{-1}+B^{-1}\right)^{-1} \quad (24)$$

indicates the total impedance of the parallel connection of these two networks.

The normal situation is that the impedance operators A and B are strictly positive. However, the case $A=0$ or $B=0$, that is a short circuit, can be handled by letting $A:B=0$. This motivates us to define the parallel sum for arbitrary positive operators A and B by perturbing A and B with ε and taking limit as ε tends to 0 in the strong-operator topology:

$$A:B=\lim_{\varepsilon\downarrow 0}(A+\varepsilon I):(B+\varepsilon I). \quad (25)$$

Here, I denotes the identity operator. It is clear that the parallel sum is commutative, associative and the multiplication is distributive over this operation.

In this setting, the flow of currents through the electrical network is governed by Maxwell's power principle as follows:

$$\langle (A:B)x,x\rangle=\inf\{\langle Ay,y\rangle+\langle Bz,z\rangle : y,z\in H \text{ and } y+z=x\}. \quad (26)$$

That is the current flows to each subnetwork in such a way that the total power dissipation is minimized. From this principle, one can easily derive the following properties of parallel sum:

(1) monotonicity: $A_1\le A_2, B_1\le B_2 \Rightarrow A_1:B_1\le A_2:B_2$

(2) transformer inequality: $S^*(A:B)S \le (S^*AS):(S^*BS)$ for all operators S on H

(3) continuity from above: if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n : B_n \downarrow A:B.$

Here, the notation $X_n \downarrow X$ means that $(X_n)_{n=1}^{\infty}$ is a decreasing sequence of positive operators (with respect to the positive semidefinite ordering for Hermitian operators) converging to X in the strong-operator topology.

The transformer inequality has a physical interpretation as follows. The positive operator $S^* AS$ represents the impedance of a network connected to a transformer. The transformer inequality states that the impedance of parallel connection with transformer first is greater than that with transformer last.

The Maxwell's principle (26) also leads to the series-parallel inequality as follows:

$$(A:B)+(C:D) \le (A+C):(B+D) \tag{27}$$

for all positive operators A,B,C,D.

5. General network connections

The notion of parallel sum is generalized to the concept of connection by Kubo and Ando in [24]. See [28, 30-31] for surveys of the theory and [27, 32-33] for variants of the theory.

A *connection* is a binary operation σ assigned to each pair of positive operators such that the following conditions are fulfilled for all positive operators A,B,C,D

(C1) monotonicity: $A \le B, C \le D \Rightarrow A\sigma C \le B\sigma D$

(C2) transformer inequality: $C(A\sigma B)C \le (CAC)\sigma(CBC)$

(C3) continuity from above: if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \downarrow A\sigma B.$

The property (C2) shows how network connection behaves when connected to a transformer. Typical examples of a connection are the sum (the series connection) and the parallel sum (the parallel connection).

These axiomatic properties (C1)-(C3) imply the following properties:

- positive homogeneity, i.e. $kA\ \sigma\ kB = k(A\sigma B)$ for any $k \ge 0$.
- transformer inequality (general form): $X^*(A\sigma B)X \le (X^*AX)\sigma(X^*BX)$ for any operator X.
- congruent invariance, i.e. for any invertible operator X,
$$(X^*AX)\ \sigma(X^*BX) = X^*(A\,\sigma B)X. \tag{28}$$
- concavity, i.e. for any $t \in [0,1]$,
$$[tA+(1-t)B]\ \sigma\,[tA'+(1-t)B'] \ge t(A\sigma A')+(1-t)\,(B\sigma B') \tag{29}$$

The series-parallel inequality for parallel sum is now a special case of the following inequality:

$$(A\sigma B)+(C\sigma D) \le (A+C)\sigma(B+D). \tag{30}$$

Every connection σ is associated to an *operator monotone function* $f:[0,\infty) \to [0,\infty)$ via the formula

$$A\sigma B = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2} \tag{31}$$

where A and B are strictly positive, and the formula can be extended to positive operators by a continuity argument using (C1) and (C3). See [21, 30, 34-36] for more information about the role of operator monotone functions in the theory of connections.

A connection with the property that $A\sigma A = A$ for any positive operator A is known as a *mean*. It turns out that every mean arises as a scalar multiple of a nonzero connection. For example, the harmonic mean is the twice parallel sum. For example, the harmonic mean is the twice parallel sum.

There are four ways of operations to produce new connections from the existing ones. First, an action, called the *scalar multiplication*, of a connection α with a scalar $\alpha \ge 0$ is defined by

$$\alpha\sigma:\ (A,B) \mapsto \alpha(A\sigma B).$$

The second way is to use a binary operation, called the *sum.* The sum of two connections σ and τ is

$$\sigma+\eta:\ (A,B) \mapsto (A\sigma B)+(A\eta B).$$

The third way is to use a ternary operation, called the *composition*. Given three connections σ, τ and τ, the composition of them is defined by

$$\sigma(\tau)\eta:\ (A,B) \mapsto (A\sigma B)\tau(A\eta B).$$

The fourth way to produce new connections is to take a unary operation. We respectively define the *transpose*, the *adjoint* and the *dual* of a nonzero connection σ to be the following operator connections

$$\sigma^T : (A,B) \mapsto B\,\sigma\,A, \qquad A,B \ge 0$$
$$\sigma^* : (A,B) \mapsto (A^{-1}\,\sigma B^{-1})^{-1}, \quad A,B > 0$$
$$\sigma^\perp : (A,B) \mapsto (B^{-1}\,\sigma A^{-1})^{-1}, \quad A,B > 0.$$

Here, recall that if σ is a nonzero connection, then $A\sigma B > 0$ for any $A,B > 0$ (see [37]). The connections σ^* and $\sigma^\perp$ are defined for arbitrary positive operators via a continuity argument using the monotonicity and the upper-continuity of connections. Physically, the transpose of a connection can be described as the interchange of two resistances. The dual of a connection is to "dualize" the way resistors connected, e.g. transform the series connection to the parallel connection.

Every nontrivial connection satisfies the following algebraic/order properties: positivity, strictness and cancellability (see [37-38]). The structure of the set of connections was investigated in [39]; it is a normed ordered cone.

A fundamental theorem of Kubo-Ando [24] asserts that there is a one-to-one correspondence between connections and finite positive Borel measures μ on the extended half-line $[0,\infty]$ such that

$$A\sigma B = \int_{[0,\infty]} \frac{1+\lambda}{\lambda}\{(\lambda A):B\}\, d\mu(\lambda) \tag{32}$$

for any positive operators A and B. Here, the Banach space integral is taken in the sense of Bochner; the Banach space considered here is the algebra of bounded linear operators on H. This integral can be transformed to a symmetric form as follows (see [40]):

$$A\sigma B = \int_{[0,1]} \left(\frac{1}{1-t}A : \frac{1}{t}B\right) d\tilde{\mu}(t). \tag{33}$$

For the case of series connection, the associated measure $\tilde{\mu}$ is given by $\delta_0+\delta_1$, here δ_t denotes the Dirac measure on $[0,1]$ concentrated at t. Recall that δ_t is defined for each Borel set E in $[0,1]$ by

$$\delta_t(E) = \begin{cases} 1, & t\in E \\ 0, & t\notin E \end{cases} \tag{34}$$

The associated measure of the parallel connection is the measure $(1/2)\delta_{1/2}$.

The integral representation (33) means that a general connection represents a formation of making a new impedance from two given impedances. Such a formation can be realized as a weighted series connection of weighted parallel connections. Thus, the theory of connections can be regarded as a mathematical theory of electrical network.

6. Conclusion

The treatment of calculus, linear algebra and functional analysis, applied to electrical network connections, results in the interpretations of Maxwell’s power principle, the series-parallel inequality and many physical phenomena. An abstract network connection can be viewed as a binary operation for positive operators satisfying certain order, algebraic and topological properties

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