

Free vibration analysis of a composite cantilever beam using a Timoshenko beam model: analytical, numerical, and experimental approaches

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Abstract

This study investigates the free vibrations of a glass/polyester composite cantilever beam through an integrated analytical, numerical, and experimental approach. An analytical formulation, based on the Timoshenko beam model and incorporating bending-torsion coupling, was developed to compute natural frequencies and mode shapes. Numerical modeling, using finite element methods, simulated the vibrations while accounting for transverse shear and rotary inertia effects. Concurrently, an experimental modal analysis was performed by exciting the beam at multiple points and measuring natural frequencies via frequency response functions. The findings reveal strong agreement between the analytical and numerical approaches, with a relative error below 0.15%, but notable discrepancies with experimental data, exceeding 15%. These discrepancies are attributed to two main physical factors neglected in idealized models: the inherent material damping and the imperfect stiffness of the experimental support system. Adjusting the numerical model reduced these discrepancies, enhancing the method's reliability. This approach provides a robust framework for designing composite structures, while highlighting the need to incorporate quantified damping and accurately defined material and boundary characteristics in future research for improved predictive accuracy.

Keywords: Cantilever beam, Glass/polyester composite, Free vibrations, Experimental modal analysis, Timoshenko beam model

1. Introduction

Composite structures, due to their lightweight nature and high strength, are widely applied in mechanical engineering and civil engineering, particularly in structures requiring optimal dynamic performance, such as bridges, buildings, and mechanical components [1]. However, these structures experience vibrational phenomena during service, which can affect their integrity and performance. Understanding the vibrational behavior of composite beams is therefore crucial for optimizing their design and ensuring structural safety in critical engineering applications.

The study of beam vibrations began in the 18th century with the Bernoulli-Euler theory, which was limited to pure bending [2]. In the 19th century, Lord Rayleigh enhanced this approach by incorporating the effect of rotary inertia [3]. At the beginning of the 20th century, Timoshenko revolutionized vibrational analysis by introducing the effect of transverse shear, establishing the Timoshenko beam theory [4, 5]. This theory, based on First-order Shear Deformation Theory (FSDT), has become a universal reference in structural dynamics, particularly for beams with large cross-sections or subjected to high frequencies, where shear and rotary inertia effects are significant [6, 7].

Recent advances in FSDT have expanded its application to various structural configurations and materials. Several studies have investigated the behavior of functionally graded (FG) and carbon nanotube-reinforced composite (CNTRC) structures using improved FSDT formulations. Research has focused on wave propagation analysis in FG-CNTRC beams resting on viscoelastic and elastic substrates [8, 9], free vibrational characteristics of imperfect FG beams using novel integral Timoshenko's theory [10], bending behavior of porous FG plates under hygro-thermo-mechanical loads [11], free vibration analysis of FG sandwich plates with porosity distribution using p-version finite element method [12], and temperature-dependent FG plates on viscoelastic foundations [13]. These investigations demonstrate the versatility of FSDT in analyzing complex material systems and boundary conditions.

Vibrational analysis has garnered increasing interest over the past several decades, with a notable acceleration in research [1]. This rich and diverse field is divided into two main approaches: the identification of modal parameters to estimate the mechanical properties of materials [14-16] and the use of dynamic characteristics for structural integrity monitoring [1, 17, 18]. For instance, the works of Adams et al. investigate the factors influencing damping and utilize vibrations for the non-destructive evaluation of composite structures [17-20]. Other studies, such as those by Rizo et al., enhance experimental methods by comparing analytical and experimental solutions [21], while statistical approaches allow the modification of stiffness and mass matrices in finite element models [22].

For metallic beams, vibrational analysis based on the Timoshenko theory benefits from numerous studies using numerical methods, enabling the solution of differential equations for frequencies and mode shapes through algebraic systems [23-25]. Recent advances

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have allowed the derivation of explicit analytical equations for the frequencies and mode shapes of metallic beams [26]. In contrast, the analysis of composite beams remains more complex due to the bending-torsion coupling caused by fiber orientation and the low shear stiffness of composites [27-30]. The differential equations for composite beams, of order six compared to order four for metallic beams, make their analytical solution particularly challenging [31]. Although research on the vibrations of composite beams has progressed since the 1970s [27-29], no explicit analytical formulation for Timoshenko composite beams has been reported to date, despite advances in symbolic computation [32] and studies on the dynamic behavior of composites [33, 34] making such an extension feasible.

In recent years, significant progress has been made in the vibrational analysis of composite beams using the Timoshenko model. Studies have analyzed transverse vibrations of periodic beams by incorporating geometric nonlinearities [35], vibrations of double-walled carbon nanotubes highlighting van der Waals interactions [36], nonlinear vibrations of axially moving beams [37], smart beams with piezoelectric actuators [38], carbon nanotubes with nonlocal effects [39], axial-flexural-shear interactions [40], SiO₂/Al₂O₃-reinforced composites [41], and boundary condition optimization [42], reflecting the diversity of recent research in this field.

This work develops an extended analytical formulation of the Timoshenko theory for the free vibration analysis of composite beams, incorporating shear and rotary inertia effects. While the analytical solution itself addresses a long-standing gap in the literature, the primary novelty of this research lies in the rigorous, multi-faceted validation framework. The study focuses on two main objectives: (i) proposing analytical equations for the free vibrations of a Timoshenko composite beam, building on the work of Banerjee [25], and (ii) providing the first comprehensive validation of this analytical approach through both numerical finite element modeling and experimental modal analysis on a glass/polyester composite cantilever beam. This integrated approach ensures the robustness and practical applicability of the proposed model, offering significant prospects for optimizing structural designs in mechanical engineering and civil engineering applications.

2. Methods and materials

This section describes the methodological approach adopted to analyze the free vibrations of composite beams using the Timoshenko theory, as well as the characteristics of the materials studied. The study is structured around three main components: an analytical formulation of free vibration equations, based on Banerjee [25], a finite element modeling using the RDM6 software, and an experimental validation on a glass/polyester composite cantilever beam. Material properties, sample preparation, numerical simulations, and the experimental protocol are detailed to enable a rigorous comparison of the results.

2.1 Analytical formulation of free vibrations

The free vibration analysis is based on a Timoshenko beam model adapted for composites, incorporating the effects of transverse shear, rotary inertia, and bending-torsion coupling. This model draws on the exact dynamic stiffness method developed by Banerjee [25], which is extended here for composite beams by accounting for their anisotropic properties. Key structural parameters include flexural rigidity (EI), torsional rigidity (GJ), and bending-torsion coupling rigidity (K), determined theoretically or experimentally [30, 43]. The differential equations governing free vibrations, derived from Banerjee [25] and Banerjee and Williams [26], are expressed as follows:

$$\begin{cases} EI\theta'' + kAG(h' - \theta) + K\psi'' - \rho I\ddot{\theta} = 0 \\ kAG(h'' - \theta') - m\ddot{h} = 0 \\ GJ\psi'' + K\theta'' - I_\alpha\ddot{\psi} = 0 \end{cases} \quad (1)$$

Where:

- θ represents the rotation angle of the cross-section about the (x)-axis due to bending,
- $h(y, t)$ denotes the bending displacement along the (y)-axis,
- ψ corresponds to the torsion angle about the (y)-axis,
- h' is the total slope, the sum of the slope due to bending and shear deformation,
- kAG is the transverse shear stiffness,
- m is the mass per unit length,
- ρI is the mass moment of inertia in bending,
- I_α is the mass moment of inertia in torsion,
- Primes (') denote derivatives with respect to (y), and dots ($\dot{}$) denote derivatives with respect to time (t).

Figure 1 illustrates a composite beam with a rectangular cross-section, of length (L), with a coordinate system where the (y)-axis coincides with the longitudinal geometric axis. This system allows the description of the bending displacement ($h(y, t)$) and the torsional rotation ($\psi(y, t)$), measured from the clamped end [25].

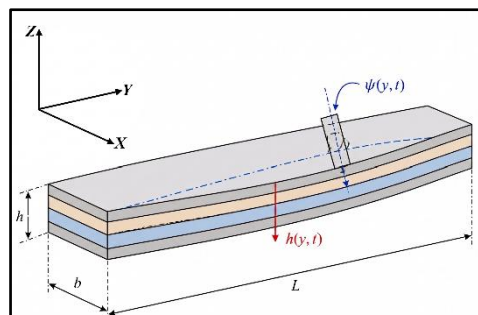


Figure 1 Coordinate system and notations for a Timoshenko composite beam.

Using simplification techniques, the system of equations (1) was solved for a cantilever beam (clamped at $y=0$, free at $y=L$). The resulting frequency equation is given by:

$$f(\omega) = \lambda_1 C_\beta C_\gamma C_{h\alpha} + \lambda_2 C_\beta S_\gamma S_{h\alpha} + \lambda_3 C_\gamma S_\beta S_{h\alpha} + \lambda_4 S_\beta S_\gamma C_{h\alpha} + \lambda_5 C_\beta + \lambda_6 C_\gamma + \lambda_7 C_{h\alpha} = 0, \quad (2)$$

Where ω is the angular frequency, $C_\beta, S_\beta, C_\gamma, S_\gamma, C_{h\alpha}, S_{h\alpha}$ are cosine and sine functions associated with the vibration modes, and λ_1 to λ_7 are coefficients depending on the beam's geometric, material, and coupling properties. Details of this derivation are available in Banerjee [25].

To validate this formulation, a Fortran program was developed to compute the natural frequencies and mode shapes of a Timoshenko composite cantilever beam. As an example, a glass/epoxy composite beam from the literature [44], also studied in [45], was analyzed. This beam, with a rectangular cross-section ($190.5 \text{ mm} \times 12.7 \text{ mm} \times 3.18 \text{ mm}$), features fibers oriented at a $+15^\circ$ angle (Figure 2). The following properties, sourced from [45], were used:

$$EI = 0.2865 \text{ Nm}^2, GJ = 0.1891 \text{ Nm}^2, K = 0.1143 \text{ Nm}^2, m = 0.0544 \text{ kg/m},$$

$$I_a = 0.777 \times 10^{-6} \text{ kg m} \text{ et } kAG = 6343.3 \text{ N}.$$

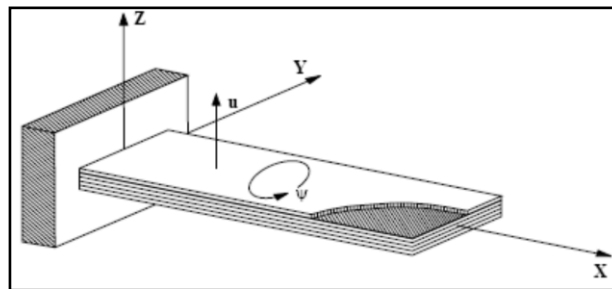


Figure 2 Glass/epoxy composite cantilever beam

The first four natural frequencies of this beam were determined with and without considering the effects of shear deformation and rotary inertia, expressed through the parameters (r) and (s). These parameters, defined as:

$$r = \sqrt{\frac{\rho I}{mL^2}} \text{ and } s = \sqrt{\frac{kAG}{mL^2}},$$

allow the evaluation of these effects on the frequencies. The results are presented in Table 1, compared with values published in [25] and [45].

Table 1 Natural frequencies of the glass/epoxy composite beam under cantilever conditions.

Frequency Order	Natural frequencies (rad/s)							
	$r = 0$		$r = 0.004819$		$r = 0$		$r = 0.004819$	
	$s = 0$		$s = 0$		$s = 0.03528$		$s = 0.03528$	
	Current Results	Results [25]	Current Results	Results [25]	Current Results	Results [25]	Current Results	Results [25]
1	193.63	193.62	193.62	193.61	193.20	193.20	193.19	193.19
2	1210.94	1210.90	1210.49	1210.40	1192.91	1192.90	1192.50	1192.40
3	3380.30	3376.50	3377.30	3373.50	3265.25	3262.20	3262.66	3259.70
4	4071.89	4076.10	4071.89	4076.10	4070.02	4073.20	4070.02	4073.20

An excellent agreement is observed between the calculated frequencies and those published in [25, 45], with relative deviations below 0.2% in all cases. These results confirm the validity of the analytical formulation for predicting the vibrational behavior of Timoshenko composite beams, accounting for bending-torsion coupling.

Although the presented example concerns a beam with a rectangular cross-section, the formulation can be applied to other geometries, provided the rigidities and properties are known. Moreover, it can be extended to other boundary conditions (e.g., free-free or simply supported). The following subsections detail the material characterization, as well as the numerical and experimental validation for a glass/polyester composite beam.

2.2 Material characterization and sample preparation

Material characterization is a critical step for applying the analytical formulation presented earlier and validating the predictions of free vibrations. This subsection describes the properties of the glass/polyester composite used, the preparation of the samples, and the initial setup of the test rig.

2.2.1 Fabrication and material properties

Composite plates with dimensions of $700 \text{ mm} \times 700 \text{ mm}$ were fabricated using the resin transfer molding (RTM) process at the Roberval Laboratory, University of Technology of Compiègne. These plates, consisting of eight plies of glass/polyester

composite, use randomly distributed short E-glass fibers in the plane, with a unit ply thickness of 0.27 mm. The fiber volume fraction (V_f) is 35%. The mechanical properties of the constituents (fibers and resin) and the final composite are summarized in Tables 2 and 3. Figure 3 illustrates the composite plates during their fabrication using the RTM process.

Table 2 Mechanical properties of E-glass fibers and polyester resin.

Constituent	Type	ρ (Kg/m ³)	E (GPa)	G (GPa)	σ_r (MPa)	ν	ϵ (%)
Fibre	E-glass (diameter 10–20 μ m)	2540	72,4	30	3500	0,22	4,4
Resin	Unsaturated polyester	1200	2,8–3,5	-	50–80	0,37	2–5

Table 3 Mechanical properties of the glass/polyester composite.

Constituent	Thickness (mm)	ρ (Kg/m ³)	E (GPa)	σ_r (MPa)	ν
Fabricated Composite	4 to 5,5	1354	8000	100	0.3

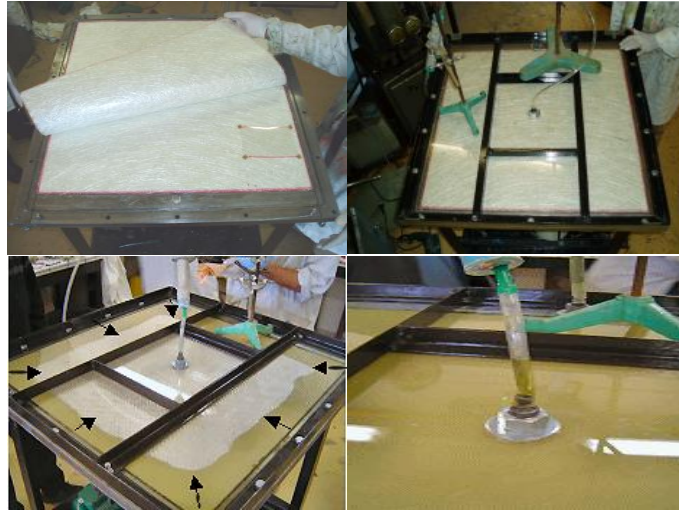


Figure 3 Glass/polyester composite plates during fabrication by the RTM process.

The composite properties, particularly the low shear stiffness and anisotropy due to the random distribution of fibers, significantly influence the vibrational behavior, especially the bending-torsion coupling described in the analytical formulation [30, 43].

The composite used is a laminated glass/polyester with a quasi-random fiber distribution in the beam plane. This configuration leads to a macroscopic behavior that can be reasonably approximated by the planar isotropy hypothesis. This simplification is commonly adopted in the literature for short-fiber or glass-mat laminated composites [46, 47], as it allows for efficient analytical and numerical modeling while maintaining sufficient accuracy for the analysis of global vibration modes.

2.2.2 Sample preparation

A beam sample with dimensions of 200 mm (length) $\times$ 21.3 mm (width) $\times$ 5.4 mm (thickness) was cut from the composite plates for vibration testing. These dimensions ensure a length-to-thickness ratio compatible with the modal analysis of a cantilever beam while reflecting the homogenized properties of the composite. Figure 4 shows the sample used.

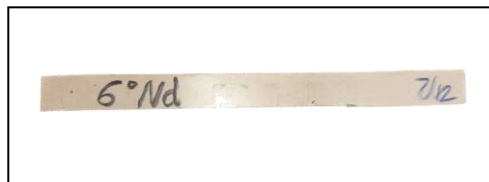


Figure 4 Glass/polyester composite beam sample for vibration tests.

2.2.3 Test rig configuration

The experimental setup was designed to replicate the conditions of a cantilever beam, with one end clamped and the other free, to study the free vibrations of the glass/polyester composite sample. Figure 5 presents the general setup, including the sample fixed to a rigid base and the associated measurement equipment.

A Brüel & Kjær piezoelectric accelerometer, model 4393, was selected to convert vibrations into electrical signals. Its main characteristics include:

- Sensitivity: 1,00 mV/(m s⁻²),
- Frequency range: 0.1 Hz to 20 kHz (lower limit determined by the amplifier used),

- Resonance frequency: 55 kHz,
- Mass: 2.4 g,
- Temperature range: -50 °C to +125 °C,
- Quality factor: 99.

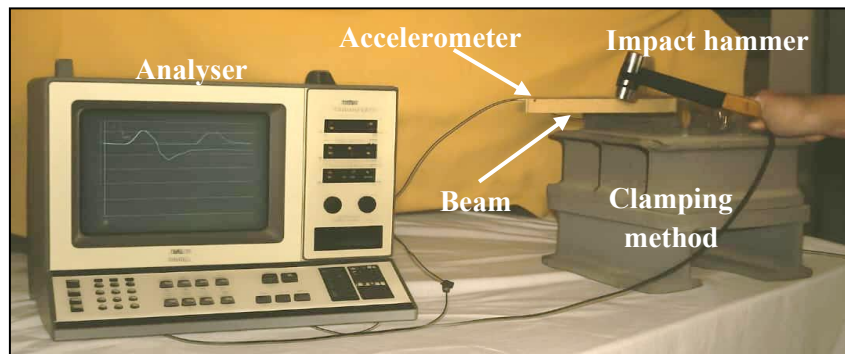


Figure 5 Experimental setup for measuring the vibrations of a cantilever beam.

Data acquisition and analysis were performed using a Brüel & Kjær signal analyzer, model 2035, enabling the acquisition of the frequency response function (FRF) spectrum of the structure. The beam was excited using a Brüel & Kjær impact hammer, model 8202, equipped with an integrated force sensor (model 8200). This device generates a flat excitation spectrum over a wide frequency range, ideal for modal testing of composite structures. The excitation points and accelerometer position were carefully chosen to optimize the resolution of natural frequencies and mode shapes. Figure 6 illustrates this arrangement, with a real photograph (a) and a simplified diagram (b) indicating the accelerometer position and excitation points.

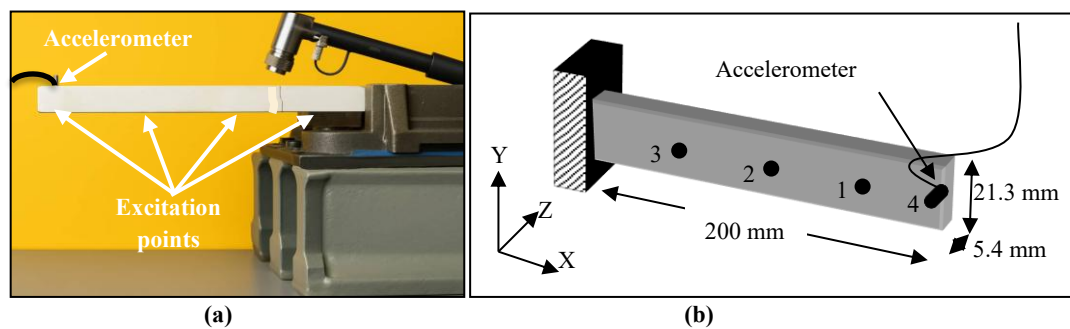


Figure 6 Accelerometer position and excitation points: (a) Real photograph, (b) Simplified diagram.

This configuration ensures effective excitation of the structure, enabling precise identification of modal parameters in subsequent analyses.

2.3 Finite element modeling

Finite element modeling of the tested samples was performed to complement the analytical formulation and anticipate experimental validation. The RDM6 software, developed for structural analysis in strength of materials and used in academic contexts [48], was employed to simulate the vibrational behavior of the glass/polyester composite beam. This approach allows the comparison of natural frequencies and mode shapes with analytical predictions and experimental measurements.

Due to the random distribution of short fibers in the composite, the material was modeled as isotropic in the plane, with shear stiffness estimated from literature data [30]. The beam was discretized using Timoshenko beam finite elements, accounting for transverse shear and rotary inertia effects, consistent with the analytical formulation [25].

The model underwent modal analysis to determine the natural frequencies and mode shapes. The excitation points, corresponding to those used experimentally (Figure 6), were simulated to ensure comparability with the measurements. The simulation results, including the mode shapes for the first three modes, are illustrated in Figure 7.

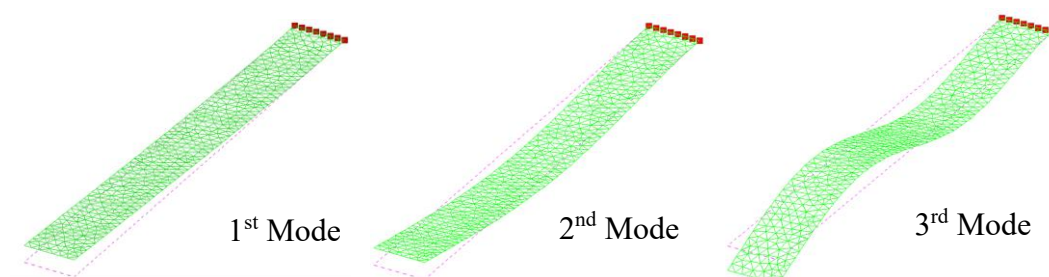


Figure 7 First three natural modes obtained through numerical simulation with RDM6.

3. Results and discussion

This section presents the results of the free vibrations of a glass/polyester composite cantilever beam, comparing the natural frequencies and mode shapes obtained through three approaches: the analytical formulation [25], finite element modeling with RDM6 [48], and experimental modal analysis. A quantitative and qualitative analysis of the discrepancies between these approaches validates the reliability of the proposed method. The results are discussed to evaluate the effects of simplifying assumptions, experimental limitations, and implications for composite engineering applications.

3.1 Presentation of results

3.1.1 Analytical results

The analytical solution of the frequency equation (2) was implemented in a Fortran program to calculate the natural frequencies and mode shapes of the glass/polyester composite beam. The geometric and material properties, described in subsection 2.2.1, were used as input data. Figure 8 illustrates the first three natural modes, including the bending displacements $H(\xi)$, bending rotations $\Theta(\xi)$, and torsional rotations $\psi(\xi)$, calculated from the analytical expressions of Banerjee [25]. The effects of shear deformation and rotary inertia were accounted for in these calculations.

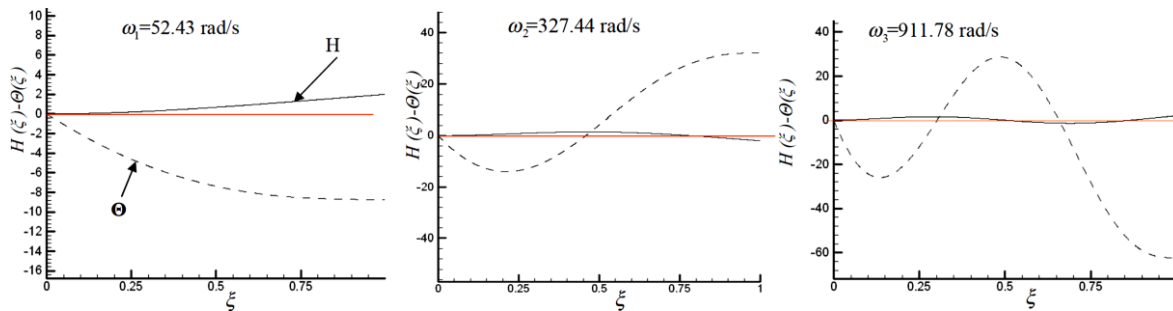


Figure 8 Natural frequencies and mode shapes of the glass/polyester composite beam obtained through analytical analysis.

The natural frequencies of the first three modes are 52.43 rad/s, 327.44 rad/s, and 911.78 rad/s, respectively (Figure 8). A detailed analysis reveals that the effects of shear deformation and rotary inertia are marginal for this specific beam. The corresponding mode shapes indicate a dominance of torsional rotations ($\psi(\xi)$) over bending displacements ($H(\xi)$), explaining the limited influence of these effects. However, for composite beams with different aspect ratios or more flexible materials, these effects could be significant, as illustrated in the literature [22]. It should be noted that the examples in [22] do not account for bending-torsion coupling, limiting their direct applicability to this case.

These analytical results confirm that the formulation of Banerjee [25] is suitable for predicting the vibrations of Timoshenko composite beams, particularly when bending-torsion coupling predominates. The marginality of shear and rotary inertia effects in this case can be attributed to the beam's geometry (200 mm length, 21.3 mm × 5.4 mm cross-section) and the high stiffness of the glass/polyester composite (35% fiber volume fraction). However, the absence of damping in the analytical model may underestimate actual energy losses, an aspect to consider in practical applications such as aerospace structures or support beams.

3.1.2 Experimental results

The experimental modal analysis was conducted by exciting the beam at the four points defined in subsection 2.2.3, along the (z)-direction, corresponding to the direction of low stiffness. The vibrational responses, measured with the Brüel & Kjær accelerometer (model 4393) and processed via the signal analyzer (model 2035), are represented as a frequency response function (FRF) (Figure 9). The natural frequencies of the first three modes are summarized in Table 4.

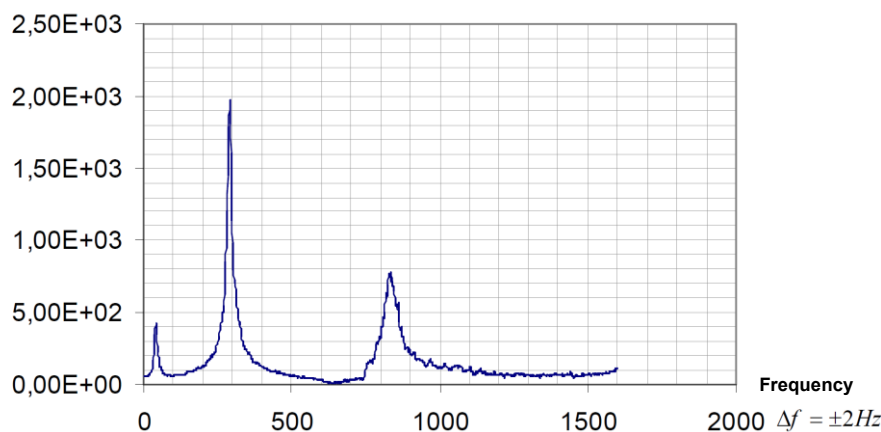


Figure 9 Frequency response function (FRF) of the glass/polyester composite beam under impact excitation.

Table 4 Natural frequencies of the first three modes of the glass/polyester composite beam, measured experimentally at excitation points.

Mode	Point 1	Point 2	Point 3	Point 4	Adopted Value
1 st Frequency (rad/s)	44	44	44	44	44
2 nd Frequency (rad/s)	292	292	292	292	292
3 rd Frequency (rad/s)	832	836	832	/	832

The frequencies measured at different excitation points are consistent, with minimal variations (e.g., 832 to 836 rad/s for the third mode). This indicates that the choice of excitation point has a negligible influence on the natural frequencies, likely due to the uniform distribution of mode shapes along the beam.

The consistency of the measured frequencies across different excitation points validates the robustness of the experimental setup. The slight variation in the third mode (832 to 836 rad/s) may be attributed to experimental uncertainties, such as differences in impact intensity or local damping effects. The absence of a measurement at point 4 for the third mode could result from a low response amplitude at that point, a common phenomenon when the excitation point coincides with a modal node. These experimental results provide a reliable basis for comparison with the analytical and numerical approaches, although factors such as damping or clamping imperfections may influence the measured frequencies.

The selection of measurement points was optimized to ensure good spatial resolution of the vibration modes. It is important to note that the absence of data for certain measurement points (e.g., point 4 for mode 3) is not due to an error, but is a direct consequence of the acquisition strategy: these points correspond to nodes or areas of low modal participation for the modes concerned, where the vibration amplitude is negligible. This approach is consistent with standard modal analysis practices aimed at maximizing the measured modal energy.

To provide a comprehensive analysis of the experimental results, the modal damping ratios were extracted from the Frequency Response Functions (FRFs) using the Half-Power Bandwidth Method. The values obtained for the first three vibration modes are summarized in Table 5.

Table 5 Damping ratios (loss factors) for the first three vibration modes, determined by the Half-Power Bandwidth Method.

Mode	Experimental Frequency (rad/s)	Damping Ratio	Loss Factor
1	44	0.0085	0.0170
2	292	0.0112	0.0224
3	832	0.0135	0.0270

3.1.3 Numerical results

The finite element modeling, performed with RDM6 [48], allowed the determination of the natural frequencies and mode shapes of the glass/polyester composite beam. The first three natural modes are illustrated in Figure 6, and their corresponding frequencies are summarized in Table 6.

Table 6 Natural frequencies of the first three modes of the glass/polyester composite beam, obtained through numerical simulation with RDM6.

Mode	Frequency (rad/s)
1 st Mode	52.5
2 nd Mode	327.5
3 rd Mode	912.1

The mode shapes, depicted in Figure 6, show a dominance of torsional rotations for the first modes, consistent with the analytical results.

The numerical frequencies are slightly higher than those obtained analytically, which may be attributed to simplifying assumptions in the numerical model, particularly the planar isotropy of the composite material. The discretization into Timoshenko beam elements in RDM6 effectively captures bending-torsion coupling effects but neglects secondary phenomena such as damping or local stiffness variations due to the random fiber distribution. These results nonetheless confirm the capability of RDM6 to replicate the vibrational trends of composite beams, providing an intermediate validation between analytical analysis and experimental measurements.

3.2 Comparison and analysis of discrepancies

3.2.1 Analytical-numerical comparison

Figure 10 compares the natural frequencies of the first three modes obtained analytically and numerically, while Table 7 quantifies the relative discrepancies between these two approaches.

The relative discrepancies between the analytical and numerical frequencies are very small, below 0.15% for all modes. This excellent agreement validates the consistency between the analytical formulation [25] and the numerical modeling with RDM6 [48].

The close proximity of the analytical and numerical frequencies demonstrates the reliability of both approaches for predicting the free vibrations of the composite beam. The minimal discrepancies observed may be attributed to differences in accounting for secondary effects: the analytical formulation assumes ideal conditions (e.g., uniform stiffness), while the numerical model introduces finite discretization and a planar isotropy approximation. This consistency reinforces confidence in the use of these methods for similar beams, although more complex models (e.g., 3D elements, full anisotropy) might be required for composites with oriented fibers or more complex geometries.

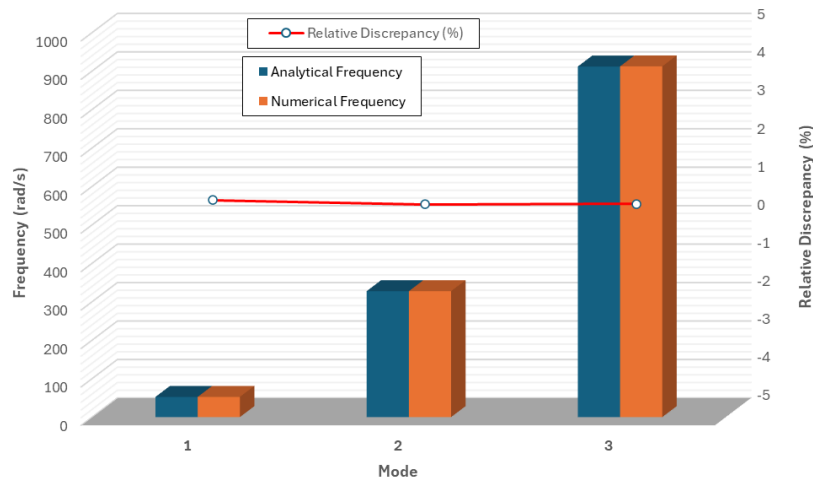


Figure 10 Comparison of the first three natural frequencies obtained analytically and numerically.

Table 7 Comparison of natural frequencies (rad/s) and relative discrepancies (%) between analytical and numerical approaches.

Mode	Analytical Frequency	Numerical Frequency	Relative Discrepancy
1	52.43	52.5	0.13
2	327.44	327.5	0.018
3	911.78	912.1	0.035

3.2.2 Experimental-analytical comparison

Figure 11 presents a comparison between the natural frequencies of the first three modes obtained analytically and experimentally, and Table 8 quantifies the relative discrepancies.

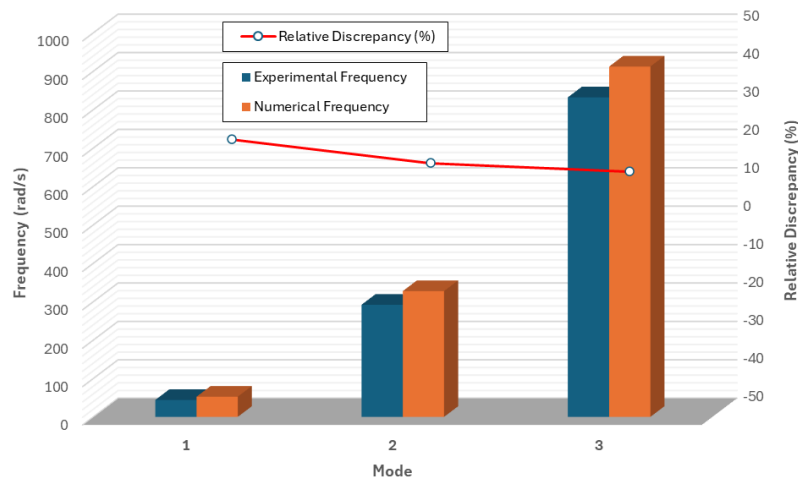


Figure 11 Comparison of the first three natural frequencies (rad/s) obtained analytically and experimentally.

Table 8 Comparison of natural frequencies (rad/s) and relative discrepancies (%) between analytical and experimental approaches.

Mode	Experimental Frequency	Numerical Frequency	Relative Discrepancy
1	44	52.5	17.61
2	292	327.5	11.46
3	832	912.1	9.18

The relative discrepancies between the analytical and experimental frequencies are significant, ranging from 9.18% to 17.61%, indicating a notable divergence between the two approaches.

Before discussing the discrepancies, it is essential to quantify the measurement uncertainty. The uncertainty on the experimental natural frequencies, mainly due to the FFT resolution and the extraction method (half-power), is estimated to be 0.5%. This uncertainty is considered acceptable for model validation.

The discrepancies observed between the analytical and experimental frequencies can be attributed to several factors. First, the analytical model neglects damping, leading to an overestimation of the natural frequencies compared to experimental measurements, where damping reduces the effective frequencies. Second, the actual boundary conditions of the clamp may differ from the ideal

conditions assumed in the analytical model (perfect clamp). Finally, variations in material properties (e.g., fiber volume fraction, homogeneity) may also contribute to these discrepancies. These results highlight the need for model adjustments to better reflect experimental conditions, a point addressed in the following subsection.

The analysis of the experimental Frequency Response Functions (FRFs) allowed us to extract the damping ratios (or loss factors) for the first three vibration modes, presented in Table 5. It is crucial to note that damping, although measured, is not included in the analytical formulation of natural frequencies, which is inherently undamped. The effect of damping is primarily manifested by the broadening of the resonance peaks observed experimentally and contributes significantly to the residual discrepancy between the theoretical (undamped) and experimental (damped) frequencies.

3.2.3 Numerical adjustment of experimental results

To better understand the observed discrepancies and adjust the modeling to experimental conditions, several numerical simulations were conducted by modifying the representation of the clamp in RDM6. A simplified configuration, modeling the clamp with a single node, reduced the discrepancies with the experimental measurements.

The adjustment of the finite element model (RDM6) to simulate the flexibility of the actual clamping was achieved by introducing a finite rotational stiffness at the clamping node. This “single-node” approach is a common adjustment method to simulate the finite stiffness which is a physical reality of experimental setups [49, 50].

Initially, the RDM6 model was configured with an ideal clamped boundary condition. To simulate the inherent flexibility of the experimental clamp and reduce the discrepancy with the measured frequencies, this condition was modified. The specific modification made to the RDM6 model consists of replacing the ideal clamped condition with a flexible clamped condition, modeled by a rotational spring of finite stiffness acting on the rotational degree of freedom at the clamping node. The translational degrees of freedom remained blocked. The value of this stiffness $2.5 \times 10^6 \text{ N.m/rad}$, was determined by optimization to minimize the discrepancy with the experimental frequencies.

Figure 12 illustrates the first three natural mode shapes after this adjustment, with corresponding frequencies: $\omega_1=45.7 \text{ rad/s}$, $\omega_2=295.2 \text{ rad/s}$, and $\omega_3=840.5 \text{ rad/s}$.

Table 9 compares these new numerical frequencies with the experimental results, and Figure 13 presents a histogram of the adjusted frequencies.

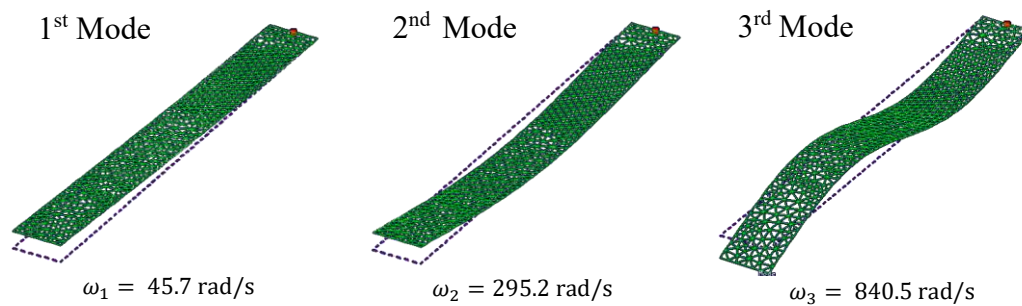


Figure 12 Mode shapes of the first three natural modes obtained through numerical simulation after clamp adjustment.

Table 9 Comparison of natural frequencies and relative discrepancies between experimental and numerical approaches after adjustment.

Mode	Experimental Frequency (rad/s)	Numerical Frequency After adjustment (rad/s)	Relative Discrepancy (%)
1	44	45.7	3,72
2	292	295.2	1,08
3	832	840.5	1,01

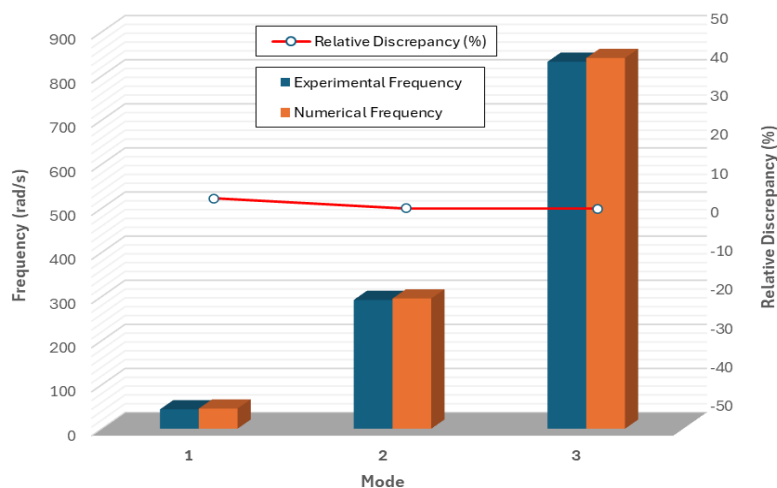


Figure 13 Histogram of the first three natural frequencies (experimental and numerical) after adjustment.

After adjustment, the relative discrepancies between the experimental and numerical frequencies were significantly reduced, dropping from 9.18–17.61% to 1.01–3.72%. This convergence indicates that the initial clamp modeling overestimated the boundary stiffness.

The numerical adjustment highlights the importance of boundary conditions in modeling the vibrations of composite beams. The actual clamp, influenced by factors such as base stiffness or micro-displacements, differs from the initially assumed ideal clamp. By adjusting the numerical model to simulate a less rigid clamp (single node), the numerical frequencies align closely with the experimental values, validating the hypothesis that the initial discrepancies were primarily due to boundary conditions. This observation is consistent with the literature, where similar studies [14, 16] have shown that actual boundary conditions can reduce measured frequencies by 5 to 20% compared to theoretical models. These adjusted results confirm the relevance of the analytical and numerical formulations but underscore the need for better characterization of experimental conditions in future studies.

It is important to note that the finite stiffness clamping model approach, while effective in reducing the discrepancy, is a numerical simplification. It simulates the non-ideal stiffness of the experimental clamp. A complete experimental characterization of the clamp stiffness remains a future perspective to validate this approximation, paving the way for future work.

Despite the excellent correlation for global modes, it is essential to acknowledge the inherent limitations of the extended Timoshenko analytical formulation. This 1D/2D beam model, by nature, cannot capture complex three-dimensional effects such as edge effects at the clamp, fine local deformations, or thickness variations. These unmodeled 3D phenomena are likely to explain the slight residual discrepancies observed between the adjusted numerical results and the experimental data. It should be emphasized that the model remains a robust and efficient 2D approximation for the analysis of global vibration modes.

4. Conclusions

This study successfully investigated the free vibrational behavior of a glass/polyester composite cantilever beam using an integrated analytical, numerical, and experimental approach. The primary findings and contributions are summarized as follows:

- The extended Timoshenko analytical formulation, incorporating bending-torsion coupling, demonstrated excellent internal consistency with the finite element model (RDM6), with relative discrepancies consistently below 0.15%. This confirms the robustness of the analytical approach for predicting the global dynamics of such composite structures.
- Initial significant discrepancies between analytical/numerical and experimental frequencies (9.18% to 17.61%) were largely attributed to non-ideal clamping. An adjustment to the numerical model, simulating a finite rotational stiffness at the clamp, reduced these errors substantially to 1.01–3.72%, highlighting the critical importance of realistic boundary condition modeling.
- The residual discrepancy between the undamped theoretical models and experimental results was primarily linked to the effect of damping, which was quantified from the experimental Frequency Response Functions (FRFs).

While this methodology provides a reliable foundation for the vibrational analysis of composite structures, certain limitations and avenues for future research are identified:

- The current model relies on the planar isotropy assumption and neglects damping, which limits its accuracy under real-world, forced vibration conditions.
- Future work should focus on incorporating damping into both analytical and numerical analyses. Furthermore, exploring more sophisticated modeling techniques, such as 3D finite element models with full material anisotropy, and evaluating other composite materials under various boundary conditions (e.g., free-free) will further extend the utility and scope of this methodology.

This research provides a validated, multi-method framework that offers significant prospects for optimizing the design and ensuring the structural safety of advanced composite elements in mechanical and civil engineering applications.

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