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The Comparison of Using Minimum Energy and Minimum Jerk with given both end of Control Input in Dynamic Systems

Abstract

This paper deals with the problem of finding the extremal solutions of the dynamic systems in terms of minimum energy and minimum jerk with given both end of control input. In general, energy, time, velocity and displacement are used as an objective function; however, this paper is emphasizing the minimum jerk of linear equation problem. Moreover, this objective is needed in many dynamic systems such as automobile and robot that work with the fragile equipments. Not only the minimum jerk with given both end of control input in dynamic systems become an objective of this work, but also the results are used to compare with the minimum energy problem. After comparison, the conclusion is that both results from the minimum jerk and minimum energy are quite different when the total energy consumptions are compared because of boundary condition. The result from experimentation can solve the extremum linear equation problem and save the energy of work.

Key words : Minimum Jerk, Minimum Energy, Dynamic Optimization, Control Input, Boundary Condition

1. Introduction

Nowadays advanced mobile machines are designed so that they are either optimized on their energy consumption or on their greatest smoothness of motion. Consequently, the trajectory planning and designs of these mobile machines and robots are done exclusively through many approaches such as the minimum energy and minimum jerk.(1) Nevertheless, in some applications, the robot is needed to work very smoothly in order to avoid damaging the specimen that the robot is handling while consuming least amount of energy at the same time. In other words, we may want to minimize jerk with given both end of control input. The problem has two-point boundary value problem condition, initial condition and boundary condition. The movements of the robot give a smoothest motion as well as optimize that robot in the energy consumption issue.

The general form of the dynamic problems consists of the equation of motion, the initial conditions, and the boundary conditions. The problems consider the two-point-boundary-value (TPBV) problem(2) is considered.

Each of the problems may contain many possible solutions depending on the objective of application. Obviously, the robot that aims to run at minimum energy will be designed to have the lowest actuator inputs during the motion. This is basically the optimization problem of the dynamic systems.

The object of this research is to search for the relationship between the minimum jerk and minimum energy by using the optimization method with boundary condition so that this new alternative can be put into applications. We know that the minimum energy is the function of the acceleration so we expect the minimization of jerk to reveal relatively similar result concerning the energy consumption issue.

2. Problem Statement

Dynamic systems(3) of linear or non linear programming problem can be described as the first order derivative term of state as

$$\dot{x}_i = f_i(x,u,t) \; ; \; x_0(t_0) = x_0 \;\; , \; i = 1,\ldots,n \qquad (1)$$

where $x \in R^n$ are states and $u \in R^m$ are control inputs of the problem. x_0 is initial condition of state t_0 and $f(x,u,t)$ are the function of states control input and time. The problem of interest is to find the states $x(t)$ and control inputs $u(t)$ that make our system operates according to the desired objective of minimum energy or minimum jerk with given both end of control input.

2.1 Minimum Energy Problems

The optimal control problem of minimum energy will take the form of

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2 dt \; ; \quad i = 1,\dots,m \qquad (2)$$

where u_i is the control inputs, which can be force or torque applied to the system , J is the cost functional of the energy consumption

2.2 Minimum Jerk Problems

The same kind of concept is used in the minimum jerk problem. It is well known that jerk is the change of input force with respect to time. It is, thus, the third derivative with respect to time of x, or first order derivative of control input u. Therefore,

$$u = \overline{x} \qquad (3)$$

From (1) can write

$$\dot{x} = f(x, \overline{x}, t) \qquad (4)$$

By $$\overline{x} = \tilde{u} \qquad (5)$$

and $$Jerk = \dddot{x} = \dot{u} = \tilde{u} \qquad (6)$$

so (4) becomes

$$\dot{x}_i = f_i(x_1,\dots,x_{n+m}, \tilde{u}_1, \dots \tilde{u}_m, t); \; i = 1,\dots,n+m \qquad (7)$$

We will treat $\tilde{u}$ as a variable and as the control input of our dynamic system. Consequently, (2) can be rewritten for the objective function of the minimum jerk problem as

$$J = \int_{t_0}^{t_f} \sum \tilde{u}_i^2 dt, \; i = 1,\dots,m \qquad (8)$$

where J is the cost function of the jerk.

3. Necessary Conditions

In this paper, we can use the calculus of variations in order to solve for the optimum solutions of the indirect method. Representing the control input with u

The principle of calculus of variations helps us solve the optimization problem by finding the time of the control input that would minimize the cost function in the form

$$J = \phi(t, x_L, \dots, x_u)_{t_f} + \int_{L_o}^{L_1} L(t, x_l, \dots, x_u, u_l, \dots u_m) dt \qquad (9)$$

where

$$\phi(t, x_1, \dots, x_n)_{t_f}, \qquad (10)$$

is the cost based on the final time and the final states of the system, and

$$J' = \int_{t_0}^{t_f} \left(L' + \sum_{i=1}^{n} \lambda_i (\dot{x}_i - f_i) \right) dt \qquad (11)$$

is an integral cost dependent on the time history of the state and control variables, therefore, the first term of (9) is omitted.

The dynamic equation[4] is then equal to Lagrange Multipliers of the cost functional as follow:

$$J'(x_1, \dots, x_n, u_1, \dots, u_m) = \int_{t_i}^{t_f} L'(t, x_1, \dots, x_n, u_1, \dots, u_m) dt \qquad (12)$$

The problem with fixed end time and end points are considered. Initial time t_0, end time t_f, initial state $x(t_0)$, and final state $x(t_f)$ must be set prior to solving the problem. The differentiable functions are dependent on the boundary condition of $x(t_0)= x_0$, $x(t_f)= x_f$, $u(t_0)= u_0$ and $u(t_f)= u_f$ where time used falls in the time interval $t_i \le t \le t_f$.

Let function $F(t, x_1,\ldots,x_n,\dot{x}_1,\ldots,\dot{x}_n)$ be represented as a functional

$$J[x_1,x_2,\ldots,x_n]=\int_{t_0}^{t_f} F(t,x_1,x_2,\ldots,x_n,\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_n)dt \quad (13)$$

Let $x(t_0)$ be incremented by $h_{xj}(t_0)$, $u(t_0)$ be incremented by $h_{uk}(t_0)$ and still satisfy the boundary conditions, then $h_{xj}(t_0)=h_{xj}(t_f)=h_{uk}(t_0)=h_{uk}(t_f)=0$. So, the change in functional ΔJ will be

$$\begin{aligned}\Delta J &= J[x_1+h_1,x_2+h_2,\ldots,x_n+h_n]-J[x_1,x_2,\ldots,x_n]\\ &=\int_{t_0}^{t_f}\left[\begin{matrix}F(t,x_1+h_1,x_2+h_2,\ldots,x_n+h_n,\dot{x}_1+\dot{h}_1,\ldots,\dot{x}_n+\dot{h}_n)\\ -F(t,x_1,\ldots,x_n,\dot{x}_1,\ldots,\dot{x}_n)\end{matrix}\right]dt\end{aligned} \quad (14)$$

Using Taylor's Series to (14), disregarding the higher order terms, and applying it to the problem result in

$$\delta J=\int_{t_0}^{t_f}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}h_i+\frac{\partial F}{\partial \dot{x}_i}\dot{h}_i\right)dt \quad (15)$$

and integral by parts

$$\delta J=\int_{t_0}^{t_f}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}-\frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i}\right)h_i dt+\sum_{i=1}^{n}\left[\frac{\partial F}{\partial \dot{x}_i}h_i|_{t_f}-\frac{\partial F}{\partial \dot{x}_i}h_i|_{t_0}\right] \quad (16)$$

Since the minimum and maximum solution, the condition that make $\delta J=0$ at arbitrary variation of $h(t_0)=h(t_f)=0$. So the continuous derivative

$$\frac{\partial F}{\partial x_i}-\frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i}=0, \quad i=1,\ldots,n \quad (17)$$

4. Example Problem

The system of linear equation of mobile machines[5] is given as

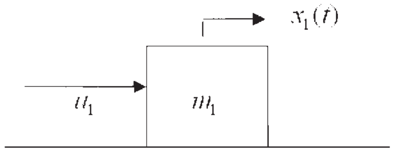

Figure 1 Dynamic system of mass m_1

Given m_1 = $1kg$. From the Newton's law, the equation of motion is

$$\ddot{x}_1=u_1 \quad (18)$$

or

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=u_1\end{aligned} \quad (19)$$

where u_1 is control input[6] of the problem. After that we can give

$$\begin{aligned}x_1(0)&=0 & x_1(1)&=1\\ \dot{x}_1(0)&=0 & \dot{x}_1(1)&=0\\ \ddot{x}_1(0)&=0 & \ddot{x}_1(1)&=0\end{aligned} \quad (20)$$

4.1 Minimum Energy Problems

The cost function[7] of the form

$$J=\int_0^1 u_1^2\,dt \quad (21)$$

Consider the cost function in (21) to be minimizing, using the Calculus of Variation, a new functional *J'[x]* in (11) is defined. Representing the *L'* with *F*, we will have

$$F = u_1^2 + \lambda_1(x_2 - \dot{x}_1) + \lambda_2(u_1 - \dot{x}_2) \quad (22)$$

Resulting necessary conditions are

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_1 \\ u_1 &= -\lambda_2 / 2 \end{aligned} \quad (23)$$

4.2 Minimum Jerk Problems with given both end of control input.

The minimum jerk problem has exactly the same format as the minimum energy problem in (17) as shown below:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 = u \\ \dot{x}_3 &= \tilde{u}_1 = \ddot{x}_1 \end{aligned} \quad (24)$$

When we give the both end control input. The problem is stability. Consider the initial time

$$\begin{aligned} t_1 &= 0 \\ u_1 &= 0 \\ x_1 &= 0 \\ \dot{x}_1 &= 0 \\ \ddot{x}_1 &= 0 \end{aligned} \quad (25)$$

The jerk cost function in this problem is defined as

$$J = \int_0^1 \tilde{u}_1^2 \, dt = \int_0^1 \dddot{x}_1^2 dt \quad (26)$$

$\dddot{x}_1$ depends on $F(x, \dot{x}, \ddot{x}, \dddot{x}, t)$

Following the same procedure as used in minimum energy problem, F is given as

$$F = \tilde{u}_1^2 + \lambda_1(x_2 - \dot{x}_1) + \lambda_2(x_3 - \dot{x}_2) + \lambda_3(\tilde{u}_1 - \dot{x}_3) \quad (27)$$

Consider (27) with Euler- Lagrange equation.

$$\frac{\partial u}{\partial x_1} - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial \dot{x}_1}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial \ddot{x}_1}\right) - \frac{\partial^3}{\partial t^3}\left(\frac{\partial u}{\partial \dddot{x}_1}\right) = 0 \quad (28)$$

The necessary condition for the minimum jerk problem is shown below:

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \\ \dot{\lambda}_3 &= -\lambda_2 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \tilde{u}_1 \\ \tilde{u}_1 &= -\lambda_3 / 2 \end{aligned} \quad (29)$$

4.3 Numerical Results

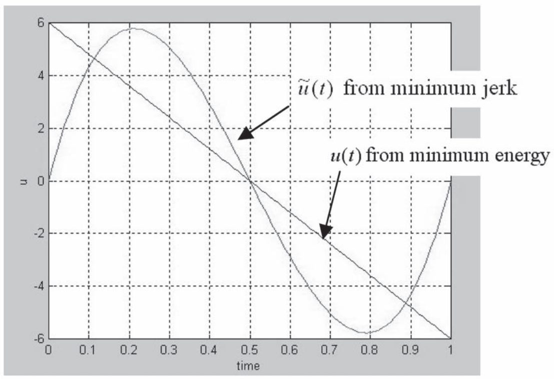

Figure 2 Comparison of the result from minimum energy and minimum jerk problem

From Figure 2 is a plot of the time history of the control input derived from the two problems: minimum energy and minimum jerk with given both end of control input.

For the minimum energy problem, the time-dependent control input can be solved at initial time ($t=0$), the control input rises very sharply thus its jerk is also tremendous. As time passes, the control input linearly decreases. In term of application, the considerable force and its jerk at initial time may result in the damage of specimen handled by robot or uncomfortable feeling of the passenger in the mobile machine.

From Figure 2, the graph from jerk moves from the origin to the end at (0,0) and (1,0) because of the boundary condition.

However, when we comparison of the cost from each problems [8], minimum jerk problem is more energy than the minimum energy, in order to control graph to boundary condition.

5. Conclusion

From the linear programming problem with a numerical method that helps solving the control profile and from the visual observation, it is obviously shown that the plots of control input $u(t)$ and $\tilde{u}(t)$ are very different. Because minimum jerk problems we give the control input both initial and end of path. We can solve the real mathematics problem for dynamic systems. Consequently, the minimum jerk problem has an advantage that the boundary conditions of the control inputs can be assigned to the problem.

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