Lieutenant Suwimon Saneewong Na Ayuttaya
Lieutenant Colonel Tawiwat Veeraklaew
and Khemarin Vichianlerd

Joint Post Graduated Program (SWU-CRMA)
Department of Mechanical Engineering,
Chulachomklao Royal Military Academy
and Srinakharinwirot University

A Study on the Comparison between Minimum Jerk and Minimum Energy of Nonlinear Dynamic Systems

Abstract

This paper deals with the problem of finding the external solutions of nonlinear dynamic systems by using the minimum energy and minimum jerk. Both the minimum energy consumption and smoothness, which is quantified as a function of jerk, are generally needed in many dynamic systems. This research paper proposes a simple yet very interesting relationship between the minimum jerk and minimum energy approaches in designing the time-dependent system yielding an alternative optimal solution. Morever, this objective is needed in nonlinear dynamic systems. Extremal solutions for the cost functions of jerk and energy are found using the dynamic optimization methods together with the numerical approximation. After comparison, the conclusion is that both results from the minimum jerk and energy are quite similar when the total energy comsumptions are compares.

1. Introduction

Nowadays advanced mobile machines are designed so that they are either optimized on their energy consumption or on their greatest smoothness of motion. Consequently, the trajectory planning and designs of these mobile machines and robots are done exclusively through many approaches such as the minimum energy and minimum jerk. Nevertheless, in some applications, the robot is needed to work very smoothly in order to avoid damaging the specimen that the robot is handling while consuming least amount of energy at the same time. In other words, we may want to minimize jerk of the movement of the robot as to give a smoothest motion as well as optimize that robot in the energy consumption issue.

The general form of the dynamic problems is consists of the equation of motion, the initial conditions, and the boundary conditions. The problems consider the two-point-boundary-value problem is considered. Each of the problems may contain many possible solutions depending on the objective of application. Obviously, the robot that aims to run at mininum energy will be designed to have the lowest actuator inputs during the motion. This is basically the optimization problem of the dynamic systems.

The object of this research is to searching for the relationship between the minimum jerk and minimum energy by using the optimization method so that this new alternative can be put into applications. We know that the minimum energy is the function of the acceleration so we expect the minimization of jerk to reveal relatively similar result concerning the energy consumption issue.

2. Problem Statement

Dynamic systems can be described as the first order derivative term of state as

$$\dot{x}_i = f_i(x_1,\ldots,x_n,u_1,\ldots u_m,t); \quad i = 1,\ldots,n \quad (1)$$

where, $\chi \in R^n$, $u \in R^m$ and t are state, control inputs, and time respectively.

The problem of interest is to find the states $x(t)$ and control inputs $u(t)$ that make our system operates according to the desired objective of minimum energy or minimum jerk with fixed end time and fixed end points. Therefore, states and control inputs that serve the necessary condition must also be able to bring the system from initial conditions $x(t_0)$ at initial time t_0 to the end point $x(t_f)$ at time t_f.

The optimal control problem of minimum energy will take the form of

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2 \, dt \; ; \quad i = 1,\ldots,m \quad (2)$$

where u_i is the control inputs, which can be force or torque applied to the system, J is the cost function of the energy consumption

The same kind of concept is used in the minimum jerk problem. It is well known that jerk is the change of input force with respect to time. It is, thus, the third derivative with respect to time of

$\dddot{x}$, or first order derivative of control input u. Therefore,

$$Jerk = \dddot{x} = \dot{u} \ . \qquad (3)$$

or

$$\dot{u} = \tilde{u}\,, \qquad (4)$$

so (1) becomes

$$\dot{x}_i = f_i(x_1,...,x_{n+m},\tilde{u}_1,...\tilde{u}_m,t);\ i = 1,..,n+m \quad (5)$$

We will treat $\tilde{u}$ as a variable and as the control input of our dynamic system. Consequently, (2) can be rewritten for the objective function of the minimum jerk problem as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} \tilde{u}_i^2 dt\,. \qquad (7)$$

where J is the cost function of the jerk.

3. Necessary Conditions

In this research paper, we use the calculus of variations in order to solve for the optimum solutions of the dynamic system. Representing the control input with u, the principle of calculus of variations helps us solve the optimization problem by finding the time of the control input that would minimize the cost function in the form

$$J = \phi(t,x_1,...,x_n)_{t_f} + \int_{t_0}^{t_f} L(t,x_1,...,x_n,u_1,...,u_m)dt\,, \qquad (8)$$

where

$$\phi(t,x_1,...,x_n)_{t_f}\,, \qquad (9)$$

is the cost based on the final time and the final states of the system, and

$$\int_{t_i}^{t_f} L(t,x_1,...,x_n,u_1,...,u_m)dt\,, \qquad (10)$$

is an integral cost dependent on the time history of the state and control variables, therefore, the first term of (8) is omitted.

The dynamic equations is then equal to Lagrange Multipliers of the cost functional as follow:

$$J'(x_1,...,x_n,u_1,...,u_m) = \int_{t_i}^{t_f} L'(t,x_1,...x_n,u_1,...u_m)dt \quad (11)$$

where

$$L'(t,x_1,...,x_n,u_1,...,u_m) = L + \sum_{i=1}^{n} \lambda_i(f_i) \qquad (12)$$

and $\lambda_i(t)$ are Lagrange multipliers. Consequently, (11) becomes:

$$J'(x_1,...,x_n,u_1,...,u_m) = \int_{t_i}^{t_f} [L(t,x_1,...,x_n,u_1,...,u_m) + \sum_{i=1}^{n} \lambda_i(t)[\dot{x}_i - f_i(t,x_1,...,x_n,u_1,...,u_m)]]dt \qquad (13)$$

The problem with fixed end time and end points are considered. Initial time t_0, end time t_f, initial state $x(t_0)$, and final state $x(t_f)$ must be set prior to solve the problem. The differentiable functions are dependent on the boundary condition of $x(t_0)=x_0$, $x(t_f)=x_f$, $u(t_0)=u_0$ and $u(t_f)=u_f$ where time used falls in the time interval $t_i \le t \le t_f$.

Let function $L(t,x_1,...,x_n,u_1,...,u_m,\dot{x}_1,...,\dot{x}_n)$ be represented as a functional

$$J[x_1,...,x_n,u_1,...,u_m]=\int_{t_0}^{t_f} L(t,x_1,...,x_n,u_1,...,u_m,\dot{x}_1,...,\dot{x}_n)dt \quad (14)$$

Let $\chi(t_0)$ be incremented by $h_{\chi j}(t_0)$, $u(t_0)$ be incremented by $h_{uk}(t_0)$, and still satisfy the boundary conditions, then $h_{\chi j}(t_0) = h_{\chi j}(t_f) = h_{uk}(t_0) = h_{uk}(t_f) = 0$. So, the change in functional ΔJ will be

$$\Delta J = J[x_1+h_{x1},...,x_n+h_{xj},u_1+h_{u1},...,u_m+h_{uk}] - J[x_1,...,x_n,u_1,...,u_m]$$

$$= \int_{t_i}^{t_f} \left[L\begin{pmatrix} t,x_1+h_{x1},...,x_n+h_{xj},\dot{x}_1+\dot{h}_{x1},...,\dot{x}_n+\dot{h}_{xj}, \\ u_1+h_{u1},...,u_m+h_{uk},\dot{u}_1+\dot{h}_{u1},...,\dot{u}_m+\dot{h}_{uk} \end{pmatrix} - L(t,x_1,...,x_n,\dot{x}_1,...,\dot{x}_n,u_1,...,u_m,\dot{u}_1,...,\dot{u}_m) \right] dt \quad (15)$$

Applying Taylor's Series to (15), disregard the higher order terms, and apply it to the problem results in

$$\delta J' = \int_{t_i}^{t_f} \sum_{j=1}^{n} \left(\frac{\partial L'}{\partial x_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_j}\right) h_{xj}\, dt + \int_{t_i}^{t_f} \sum_{k=1}^{m} \left(\frac{\partial L'}{\partial u_k} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{u}_k}\right) h_{uk}\, dt + \sum_{k=1}^{m} \left(\frac{\partial L'}{\partial \dot{u}_k} h_{uk}|_{t_f} - \frac{\partial L'}{\partial \dot{u}_k} h_{uk}|_{t_i}\right) + \sum_{j=1}^{m} \left(\frac{\partial L'}{\partial \dot{x}_j} h_{xj}|_{t_f} - \frac{\partial L'}{\partial \dot{x}_j} h_{xj}|_{t_i}\right) \quad (16)$$

Since $h_{\chi j}|_{tf} = h_{\chi j}|_{ti} = 0$ and $\frac{\partial L'}{\partial \dot{u}_k} = 0$, the last two terms of (16) become zero. In order that the cost functional of jerk in (13) can be solved for minimal solution, the condition that make $\delta J' = 0$ at arbitrary variation of $h_{\chi j}$ and h_{uk} are needed. From (16), obviously the mentioned conditions are as follow:

$$\frac{\partial L'}{\partial x_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_j} = 0, \quad (17)$$

and

$$\frac{\partial L'}{\partial u_k} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{u}_k} = 0, \quad (18)$$

for $j = 1,...,n$ and $k = 1,...,m$.

Equations (17) and (18) are the necessary conditions that will lead to solve for Lagrange multipliers $\chi_j(t)$, and control inputs $u_k(t)$. Alternatively, we can use the derived relationship below to solve for the unknowns necessary conditions:

For

$$\dot{x}_i = f_i(x_1,...,x_n,u_1,...u_m,t), \quad i = 1,...,n. \quad (19)$$

Necessary conditions are

$$\dot{\lambda}_j = -\frac{\partial L}{\partial x_j} - \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial x_j}, \quad j = 1,\ldots,n, \qquad (20)$$

$$\frac{\partial L}{\partial u_k} + \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial u_k} = 0, \quad k = 1,\ldots,m. \qquad (21)$$

4. Example Problem

The system of non-linear equation of mobile machines is given as

$$\ddot{x}_1 + \sin\theta(t) = 0. \qquad (22)$$

From equation (22) can be rewritten as the first order differential equation of the form

$$\begin{aligned} \dot{x}_1 &= x_2 = -\cos\theta(t) \\ x_1 &= \dot{x}_2 = \sin\theta(t) \end{aligned} \qquad (23)$$

where the states of the rest-to-rest system at fixed initial time $(t=0)$ and final time $(t=1)$ is specified as

$$x_1(0) = 0,\ x_2(0) = 0,\ x_1(1) = 1 \text{ and } x_2(1) = 0$$

4.1 Minimum Energy Problems

The cost function of the form

$$J = \int_{t_0}^{t_f} u_1^2 \, dt. \qquad (24)$$

In order for the cost function in (24) to be minimize, using the Calculus of Variation, a new functional $J'(\chi)$ in (13) is defined. Representing the L' with F, we will have

$$F = u_1^2 + \lambda_1(x_2 - \dot{x}_1) + \lambda_2(u_1 - \dot{x}_2). \qquad (25)$$

After (17) and (18) to the problem gives the resulting necessary conditions are

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin\theta(t) \\ u_1(t) &= \frac{\sin\theta}{2}\lambda_2 \end{aligned} \qquad (26)$$

4.2 Minimum Jerk Problems

The minimum jerk problem has the exact same format as the minimum energy problem in (23). However, since the time derivative of control inputs are considered, the (23) must be rewritten as to include the consideration of jerk into the system as shown below:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin\theta(t) \\ \dot{x}_3 &= \sin\theta(t)\tilde{u}_1 \end{aligned} \qquad (27)$$

where the boundary condition from the initial $(t=0)$ and final time $(t=1)$ is the same as the minimum energy problem.

$$x_1(0) = 0,\ x_2(0) = 0,\ x_1(1) = 1 \text{ and } x_2(1) = 0$$

The jerk cost function in this problem is defined as

$$J = \int_{t_0}^{t_f} \tilde{u}_1^2 \, dt. \qquad (28)$$

Following the same procedure as used in minimum energy problem, F is given as

$$F = \tilde{u}_1^2 + \lambda_1(x_2 - \dot{x}_1) + \lambda_2(x_3 - \dot{x}_2) + \lambda_3(\tilde{u}_1 - \dot{x}_3). \quad (29)$$

Following is the necessary condition derived from (17) and (18) for the minimum jerk problem:

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \\ \dot{\lambda}_3 &= -\lambda_2 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \tilde{u}_1 \\ \tilde{u}_1(t) &= \frac{\sin\theta}{2}\lambda_3 \end{aligned} \quad (30)$$

4.3 Numerical Results

Fig.1 is a plot of the time history of the control input derived from the two problems: minimum energy and minimum jerk. For the minimum energy problem, the time-dependent control input can be solved and found to have a non-linear equation problem.

At initial time $(t=0)$, the control input rises very sharply thus its jerk is also tremendous. As time passes, the control input linearly decreases. In term of application, the considerable force and its jerk at initial time may result in the damage of specimen handled by robot or uncomfortable feeling of the passenger in the mobile machine.

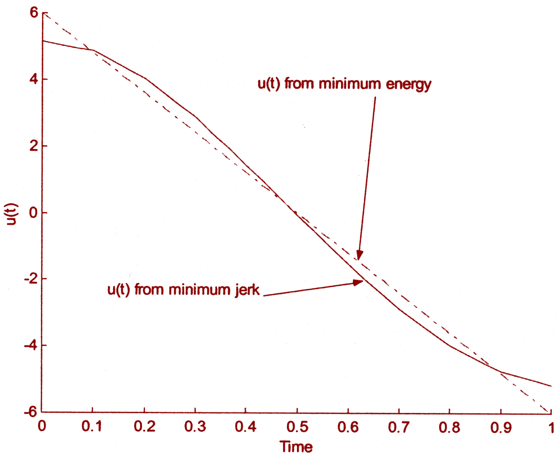

Figure 1 Comparison of the result from minimum energy and minimum jerk problem

5. Conclusion

The use of non-linear programming provides us with a numerical method that helps solving the control profile. From the visual observation, it is obviously shown that the plots of control input $u(t)$ and $\tilde{u}(t)$ are closely related. However, the minimum jerk problem has an advantage that the boundary conditions of the control inputs can be assigned to the problem.

References

(1) Agrawal SK., Fabien BC. Optimization of dynamic systems. Boston: Kluwer Academic Publishers, 1999.

(2) Bock HG. Numerical solution of nonlinear multipoint boundary value problems with application to optimal control. ZAMM, 1978. p. 58.

(3) Craig JJ. Introduction to Robotic: Mechanics and control. Addision-Wesley Publishing Company, 1986.

(4) Mark WS. Robot dynamics and control. University of Illinois at Urbana-Champaign, 1989.

(5) Kane TR., Levinson DA. Dynamics: Theory and applications. McGraw-Hill, 1985.

(6) Saneewong Na Ayuttaya S. The development of program for designing the structure of automatic car cloth cover. (Dissertation). Srinakharinwirot University, 2005.

(7) Veeraklaew T. Extensions of optimization theory and new computational approaches for higher-order dynamic systems (Dissertation). Delaware: The University of Delaware, 2000.