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On the Determinants of Complex-Valued Fibonacci Circulant Matrices

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Abstract

Circulant matrices are interesting due to their rich algebraic structures and various applications. Fibonacci circulant matrices, circulant matrices whose first row is a list of consecutive Fibonacci numbers, have been studied since 1970. In this paper, the notions of complex-valued Fibonacci circulant matrices are introduced. The explicit formula for the determinant of such matrices are completely determined. Known results on the determinants of Fibonacci circulant matrices can be viewed as special cases. At the end, some open problems are discussed.

Keywords: determinants/ circulant matrices/ Fibonacci numbers/ Fibonacci functions

1 Introduction

A *circulant matrix* is an $n \times n$ matrix whose rows are composed of right cyclically shifted versions of a list $(a_0, a_1, \cdots, a_{n-1})$, i.e., it is of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{bmatrix}.$$

Such matrices have extensively been studied since their first appearance in the paper by Catalan [3]. P. J. Davis published the book "Circulant Matrices" which summarizes the algebraic structures, properties and some applications of circulant matrices in [5]. These matrices are interesting due to their rich algebraic structures and various applications (see [5],

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[6], [12], [16], [17], and references therein). Such matrices have been applied to various disciplines such as such as image processing, communications, signal processing, networked systems and coding theory. Determinants of matrices have wide applications and have been extensively studied (see [4], [11], [14], and references therein). Especially, the determinants of circulant matrices have been studied (see, for example, [5], [10], and [8]).

The *Fibonacci sequence* F_r is defined by the recurrence relation

$$F_r = F_{r-1} + F_{r-2}$$

for all $r \geq 3$ with the initial values $F_1 = 1$ and $F_2 = 1$. A *Fibonacci circulant matrix*, a circulant matrix whose first row is given by consecutive Fibonacci numbers, has been studied in [13] and the determinants of $n \times n$ Fibonacci circulant matrices have been determined.

As a generalization of the Fibonacci sequence, Parker and Halsey (see [15] and [8]) introduced a real-valued Fibonacci function. In [9], a complex-valued Fibonacci function was introduced. Some properties of real-valued and complex-valued Fibonacci functions have been further studied in [1], [9], and [7]. In this paper, we focus on a complex-valued Fibonacci function and introduce the notions of complex-valued Fibonacci circulant matrices. Some open problems on real-valued Fibonacci circulant matrices will be proposed.

Some basic properties of matrices and Fibonacci functions are recalled in Section 2. Due to the importance of determinants, the explicit formulas for the determinants of complex-valued Fibonacci circulant matrices are determined in Section 3. Some remarks on the determinants of real-valued Fibonacci circulant matrices are given in Section 4 as well as the determinants of Fibonacci left circulant matrices .

2 Preliminaries

In this section, some basic properties of matrices are recalled and the notions of complex-valued Fibonacci circulant matrices are introduced.

2.1 Circulant Matrices2

Given a positive integer n, denote by $M_n(\mathbb{C})$ the set of all $n \times n$ complex matrices over $\mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers.

A matrix $A \in M_n(\mathbb{C})$ is called a *circulant matrix,* if each row of A is rotated one element to the right relative to the preceding row. Precisely, a circulant matrix is of the form

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$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{bmatrix} =: \mathrm{cir}(a_0, a_1, \cdots, a_{n-1}),$$

for some $(a_0, a_1, \cdots, a_{n-1}) \in \mathbb{C}^n$.

The determinants of $n \times n$ circulant matrices have been determined in terms of the n^{th} roots of unity and the elements in the first row (see [5], [10], and [8]). For each integer $0 \le k < n$, let

$$\omega_k = \cos\frac{2k}{n}\pi + i\sin\frac{2k}{n}\pi.$$

Then ω_k's are the n^{th} th roots of unity.

The determinant of a circulant matrix is given as follows.

Theorem 2.1 ([10, Theorem 3.4]) *Let* $(a_0, a_1, \cdots, a_{n-1}) \in \mathbb{C}^n$. *Then*

$$\det(\mathrm{cir}(a_0, a_1, \cdots, a_{n-1})) = \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j \omega_j^k\right).$$

2.2 Fibonacci Circulant Matrices

Some properties of Fibonacci circulant matrices studied in [13] are recalled. The Fibonacci numbers can be represented using the Binet's formula (see [13] and [7])

$$F_r = \frac{\alpha^r - (-\alpha)^{-r}}{\sqrt{5}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ for all natural numbers r. It is not difficult to see that the first 7 terms of the Fibonacci sequence are as follows

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \text{ and } F_7 = 13.$$

An $n \times n$ *Fibonacci circulant matrix* is a circulant matrix whose first row is given by n consecutive Fibonacci numbers. Precisely, an $n \times n$ matrix is called *Fibonacci circulant*, if it is of the form

$$\mathrm{cir}(F_r : n) := \mathrm{cir}(F_r, F_{r+1}, \cdots, F_{r+n-1})$$

for some positive integers r.

It can be easily seen that

$$\text{cir}(F_4 : 4) := \text{cir}(F_4, F_5, F_6, F_7) = \begin{bmatrix} 3 & 5 & 8 & 13 \\ 13 & 3 & 5 & 8 \\ 8 & 13 & 3 & 5 \\ 5 & 8 & 13 & 3 \end{bmatrix}.$$

The determinant of $\det(\text{cir}(F_4 : 4))$ can be computed using cofactor expansion and we have $\det(\text{cir}(F_4 : 4)) = -18067$. Once n or r becomes larger, the computation for $\det(\text{cir}(F_r : n))$ is more difficult and tedious. In [13], the closed form for the determinant of a Fibonacci circulant matrix is determined as follows.

Theorem 2.2 *Let* n *and* r *be natural numbers. Then*

$$\det(\text{cir}(F_r : n)) = \frac{(F_r - F_{n+r})^n - (F_{n+r-1} - F_{r-1})^n}{1 - L_n + (-1)^n}.$$

where $L_n = F_{n-1} + F_{n+1}$ is the n^{th} Lucas number.

2.3 Complex-Valued Fibonacci Circulant Matrices

As a generalization of the Fibonacci sequence, the concept of a real/complex-valued Fibonacci function of a real variable has been introduced (see [7], [8], [9], and [15]). For each $x \in \mathrm{R}$, a real/complex-valued Fibonacci function $F : \mathrm{R} \to \mathrm{C}$ is given by

$$F(x) = \frac{\alpha^x - f(x)\alpha^{-x}}{\sqrt{5}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $f : \mathrm{R} \to \mathrm{C}$ is a real/complex-value function such that $f(n) = (-1)^n$ for all $n \in \mathrm{N}$. Using the Binet's formula, it is not difficult to see that the restriction of the above Fibonacci function to the set N of natural numbers is the Fibonacci sequence.

For each positive integer n and real numbers r and $\ell > 0$, an $n \times n$ *real/complex-valued Fibonacci circulant matrix* in the closed interval $[r, r+\ell]$ is defined to be a matrix of the form

$$\text{cir}(F(r) : \ell : n) := \text{cir}\left(F(r), F(r + \frac{\ell}{n-1}), F(r + \frac{2\ell}{n-1}), \cdots, F(r + \frac{(n-2)\ell}{n-1}), F(r+\ell) \right).$$

We note that if $\ell = n-1$ and r is a positive integer, then a real/complex-valued Fibonacci circulant matrix

$$\text{cir}(F(r) : \ell : n) := \text{cir}\left(F(r), F(r+1), F(r+2), \cdots, F(r+n-1) \right) = \text{cir}(F_r : n).$$

In [8] and [15], Parker and Halsey introduced the real-valued Fibonacci function $F_{\mathrm{R}} : \mathrm{R} \to \mathrm{R}$ by setting $f(x) = \cos \pi x$, and hence,

$$F_{\mathrm{R}}(x) = \frac{\alpha^x - (\cos \pi x)\alpha^{-x}}{\sqrt{5}}. \tag{1}$$

In [9], Horadam and Shannon introduced the complex-valued Fibonacci function of a real variable $F_{\mathrm{C}} : \mathrm{R} \to \mathrm{C}$ by setting $f(x) = e^{i\pi x}$ and have

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$$F_C(x) = \frac{\alpha^x - e^{i\pi x}\alpha^{-x}}{\sqrt{5}}. \quad (2)$$

In this paper, the main focus is to determine the determinant of a complex-valued circulant matrix defined by the complex-valued Fibonacci function in (2). Some open problems on the determinant of a real-valued circulant matrix defined by the real-valued Fibonacci function in (1) will be proposed.

3 The Determinants of Real-Valued Fibonacci Circulant Matrices

In this section, complex-valued Fibonacci circulant matrices are studied. The explicit formula for the determinant of complex-valued Fibonacci circulant matrices is determined.

The following lemma is helpful in determining the determinant of a complex-valued Fibonacci circulant matrix.

Lemma 3.1 ([13, Equation (4)]) *Let* n *be a positive integer and let* x *and* y *be complex numbers. Then*

$$\prod_{k=0}^{n-1}(x - y\omega_k) = x^n - y^n.$$

The determinant of a complex-valued Fibonacci circulant matrix can be determined in terms of α and some values of F_C, where F_C is defined in (2), as follows.

Theorem 3.2 *Let* n *be a positive integer and let* r *and* ℓ *be real numbers such that* $\ell > 0$. *Then*

$$\det(\mathrm{cir}(F_C(r):\ell:n)) = \frac{\left(F_C(r) - F_C(\frac{\ell n}{n-1}+r)\right)^n - e^{i\pi\frac{\ell n}{n-1}}\left(F_C(\ell+r) - F_C(r-\frac{\ell}{n-1})\right)^n}{1-\alpha^{\frac{\ell n}{n-1}} - e^{i\pi\frac{\ell n}{n-1}}\alpha^{-\frac{\ell n}{n-1}} + e^{i\pi\frac{\ell n}{n-1}}}.$$

Proof. For each integer $0 \le j < n$, let

$$a_j := F_C\left(\frac{j\ell}{n-1}+r\right) = \frac{\alpha^{\frac{j\ell}{n-1}+r} - e^{i\pi(\frac{j\ell}{n-1}+r)}\alpha^{-(\frac{j\ell}{n-1}+r)}}{\sqrt{5}}.$$

Then

$$\det(\mathrm{cir}(F_C(r):\ell:n)) = \det(\mathrm{cir}(a_0, a_1, \cdots, a_{n-1})).$$

By Theorem 2.1, we have Equation (3).

$$\det(\mathrm{cir}(F_C(r):\ell:n)) = \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}\left(\frac{\alpha^{\frac{j\ell}{n-1}+r} - e^{i\pi(\frac{j\ell}{n-1}+r)}\alpha^{-(\frac{j\ell}{n-1}+r)}}{\sqrt{5}}\right)\omega_k^j\right)$$

$$= \prod_{k=0}^{n-1}\frac{1}{\sqrt{5}}\left(\sum_{j=0}^{n-1}\left(\alpha^r\left(\alpha^{\frac{\ell}{n-1}}\omega_k\right)^j - e^{i\pi(\frac{j\ell}{n-1}+r)}\alpha^{-r}\left(\alpha^{-\frac{\ell}{n-1}}\omega_k\right)^j\right)\right)$$

$$= \prod_{k=0}^{n-1}\frac{1}{\sqrt{5}}\left(\frac{\alpha^r(1-(\alpha^{\frac{\ell}{n-1}}\omega_k)^n)}{1-\alpha^{\frac{\ell}{n-1}}\omega_k} - \frac{e^{i\pi r}\alpha^{-r}(1-(e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k)^n)}{1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k}\right)$$

$$= \prod_{k=0}^{n-1}\frac{\left(\alpha^r - \alpha^{\frac{\ell n}{n-1}+r}\omega_k^n\right)\left(1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k\right) - \left(e^{i\pi r}\alpha^{-r} - e^{i\pi(\frac{\ell n}{n-1}+r)}\alpha^{-(\frac{\ell n}{n-1}+r)}\omega_k^n\right)\left(1-\alpha^{\frac{\ell}{n-1}}\omega_k\right)}{\sqrt{5}\left(1-\alpha^{\frac{\ell}{n-1}}\omega_k\right)\left(1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k\right)}$$

$$= \prod_{k=0}^{n-1}\frac{\alpha^r - e^{i\pi r}\alpha^{-r} - \alpha^{\frac{\ell n}{n-1}+r} + e^{i\pi(\frac{\ell n}{n-1}+r)}\alpha^{-(\frac{\ell n}{n-1}+r)} - \left(e^{i\pi\frac{\ell}{n-1}}\alpha^{r-\frac{\ell}{n-1}} - e^{i\pi r}\alpha^{-(r-\frac{\ell}{n-1})} - e^{i\pi\frac{\ell}{n-1}}\alpha^{\ell+r} + e^{i\pi(\frac{\ell n}{n-1}+r)}\alpha^{-(\ell+r)}\right)\omega_k}{\sqrt{5}\left(1-\alpha^{\frac{\ell}{n-1}}\omega_k\right)\left(1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k\right)}$$

$$= \prod_{k=0}^{n-1}\frac{\left(\frac{\alpha^r - e^{i\pi r}\alpha^{-r}}{\sqrt{5}}\right) - \left(\frac{\alpha^{\frac{\ell n}{n-1}+r} - e^{i\pi(\frac{\ell n}{n-1}+r)}\alpha^{-(\frac{\ell n}{n-1}+r)}}{\sqrt{5}}\right) - \left(\left(\frac{\alpha^{r-\frac{\ell}{n-1}} - e^{i\pi(r-\frac{\ell}{n-1})}\alpha^{-(r-\frac{\ell}{n-1})}}{\sqrt{5}}\right) - \left(\frac{\alpha^{\ell+r} - e^{i\pi(\ell+r)}\alpha^{-(\ell+r)}}{\sqrt{5}}\right)\right)e^{i\pi\frac{\ell}{n-1}}\omega_k}{\left(1-\alpha^{\frac{\ell}{n-1}}\omega_k\right)\left(1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k\right)}$$

$$= \prod_{k=0}^{n-1}\frac{F_C(r) - F_C(\frac{\ell n}{n-1}+r) - \left(F_C(r-\frac{\ell}{n-1}) - F_C(\ell+r)\right)e^{i\pi\frac{\ell}{n-1}}\omega_k}{\left(1-\alpha^{\frac{\ell}{n-1}}\omega_k\right)\left(1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k\right)}. \qquad (3)$$

Hence,

$$\det(\mathrm{cir}(F_C(r):\ell:n)) = \prod_{k=0}^{n-1}\frac{F_C(r) - F_C(\frac{\ell n}{n-1}+r) - (F_C(r-\frac{\ell}{n-1}) - F_C(\ell+r))e^{i\pi\frac{\ell}{n-1}}\omega_k}{\left(1-\alpha^{\frac{\ell}{n-1}}\omega_k\right)\left(1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k\right)}.$$

By Lemma 3.1, we have

$$\prod_{k=0}^{n-1}\left(F_C(r) - F_C(\frac{\ell n}{n-1}+r) - \left(F_C(r-\frac{\ell}{n-1}) - F_C(\ell+r)\right)e^{i\pi\frac{\ell}{n-1}}\omega_k\right)$$

$$= \left(F_C(r) - F_C(\frac{\ell n}{n-1}+r)\right)^n - e^{i\pi\frac{\ell n}{n-1}}\left(F_C(r-\frac{\ell}{n-1}) - F_C(\ell+r)\right)^n.$$

By Lemma 3.1, we have

$$\prod_{k=0}^{n-1}\left(1-\alpha^{\frac{\ell}{n-1}}\omega_k\right) = 1-\alpha^{\frac{\ell n}{n-1}}$$

and

$$\prod_{k=0}^{n-1}\left(1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k\right) = 1-e^{i\pi\frac{\ell n}{n-1}}\alpha^{-\frac{\ell n}{n-1}}.$$

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Hence,

$$\prod_{k=0}^{n-1}\left(1-\alpha^{\frac{\ell}{n-1}}\omega_k\right)\left(1-e^{i\pi\frac{\ell}{n-1}}\alpha^{-\frac{\ell}{n-1}}\omega_k\right)=\left(1-\alpha^{\frac{\ell n}{n-1}}\right)\left(1-e^{i\pi\frac{\ell n}{n-1}}\alpha^{-\frac{\ell n}{n-1}}\right)$$

$$=1-\alpha^{\frac{\ell n}{n-1}}-e^{i\pi\frac{\ell n}{n-1}}\alpha^{-\frac{\ell n}{n-1}}+e^{i\pi\frac{\ell n}{n-1}}.$$

Therefore,

$$\det(\mathrm{cir}(F_C(r):\ell:n))=\frac{\left(F_C(r)-F_C(\frac{\ell n}{n-1}+r)\right)^n-e^{i\pi\frac{\ell n}{n-1}}\left(F_C(r-\frac{\ell}{n-1})-F_C(\ell+r)\right)^n}{1-\alpha^{\frac{\ell n}{n-1}}-e^{i\pi\frac{\ell n}{n-1}}\alpha^{-\frac{\ell n}{n-1}}+e^{i\pi\frac{\ell n}{n-1}}}$$

as desired. □

From Theorem 3.2, if $\ell=n-1$, then the determinant of a complex-valued Fibonacci circulant matrix in the closed interval $[r,r+n-1]$ can be given as follows.

Corollary 3.3 *Let* n *be a positive integer and let* $r\in\mathbb{R}$. *Then*

$$\det(\mathrm{cir}(F_C(r):n-1:n))=\frac{\left(F_C(r)-F_C(n+r)\right)^n-\left(F_C(n-1+r)-F_C(r-1)\right)^n}{1-(-\alpha^{-1})^n-\alpha^n+(-1)^n}.$$

Remark 3.4 *If* $\ell=n-1$ *and* r *is a positive integer, we have*

$$\det(\mathrm{cir}(F_C(r):n-1:n))=\frac{(F_r-F_{n+r})^n-(F_{n+r-1}-F_{r-1})^n}{1-L_n+(-1)^n}=\det(\mathrm{cir}(F_r:n)),$$

where $L_n=\dfrac{\alpha^n+(-\alpha)^{-n}}{\sqrt{5}}=F_{n-1}+F_{n+1}$ *is the* n^{th} *Lucas number.*

4 Conclusion and Remarks

The determinants of complex-valued Fibonacci circulant matrices have been completely determined in Section 3. Here, some remarks on other forms of Fibonacci circulant matrices are given.

4.1 Real-Valued Fibonacci Circulant Matrices

Consider a real-valued Fibonacci circulant matrix $cir(F_R(r):\ell:n)$, where $\ell>0$ and r are real numbers, n is a positive integer, and F_R is defined in (1). From the proof of Theorem 3.2, we observe that it does not work if we replace the function F_C in (2) by the function F_R since $f(x)=\cos\pi x$ is not an exponential function. Hence, the results in this paper cannot applied for this case and it is left as an open problem.

Open Problem 1. What is the formula for the determinant of $\mathrm{cir}(F_{\mathrm{R}}(r):\ell:n)$?

4.2 Fibonacci Left Circulant Matrices

In [2], the other form of circulant matrices has been focused on, namely, a left circulant matrices. A matrix $A \in M_n(\mathbb{C})$ is called a *left circulant matrix*, if each row of A is rotated one element to the left relative to the preceding row. Precisely, a left circulant matrix is of the form

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & a_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-1} & \cdots & a_{n-4} & a_{n-3} \\ a_{n-1} & a_0 & \cdots & a_{n-3} & a_{n-2} \end{bmatrix} =: \mathrm{lcir}(a_0, a_1, \cdots, a_{n-1}),$$

for some $(a_0, a_1, \cdots, a_{n-1}) \in \mathbb{C}^n$.

The following relation between left circulant matrices and circulant matrices is given in [2].

Lemma 4.1 ([2]) *Let* $\mathbf{c} = (c_0, c_1, c_2, \cdots, c_{n-1}) \in \mathbb{C}^n$. *Then*

$$\mathrm{lcir}(\mathbf{c}) = H\mathrm{cir}(\mathbf{c}),$$

where $H = \begin{bmatrix} 1 & O_1 \\ O_1^T & \tilde{I}_{(n-1)} \end{bmatrix}$, $\tilde{I}_{(n-1)} = \begin{bmatrix} \mathbf{0} & & 1 \\ & \iddots & \\ 1 & & \mathbf{0} \end{bmatrix}$ *is an* $(n-1)\times(n-1)$ *matrix and* $O_1 = (\underbrace{0, 0, \cdots, 0}_{n-1 \text{ copies}})$.

Since the determinant is multiplicative and $\det(H) = 1$, we have

$$\det(\mathrm{lcir}(\mathbf{c})) = \det(H\mathrm{cir}(\mathbf{c})) = \det(\mathrm{cir}(\mathbf{c})).$$

for all $\mathbf{c} \in \mathbb{C}^n$. Hence, the following corollaries follow immediately from Theorem 3.2.

Corollary 4.2 *Let* n *be a positive integer and let* r *and* ℓ *be real numbers such that* $\ell > 0$. *Then*

$$\det(\mathrm{lcir}(F_{\mathrm{C}}(r):\ell:n)) = \det(cir(F_{\mathrm{C}}(r):\ell:n))$$

$$= \frac{\left(F_{\mathrm{C}}(r) - F_{\mathrm{C}}(\frac{\ell n}{n-1} + r)\right)^n - e^{i\pi\frac{\ell n}{n-1}}\left(F_{\mathrm{C}}(r - \frac{\ell}{n-1}) - F_{\mathrm{C}}(\ell + r)\right)^n}{1 - \alpha^{\frac{\ell n}{n-1}} - e^{i\pi\frac{\ell n}{n-1}}\alpha^{-\frac{\ell n}{n-1}} + e^{i\pi\frac{\ell n}{n-1}}}.$$

Corollary 4.3 *Let* n *be a positive integer and let* $r \in \mathbb{R}$. *Then*

$$\det(\mathrm{l}cir(F_{\mathrm{C}}(r):n-1:n)) = \det(cir(F_{\mathrm{C}}(r):n-1:n))$$
$$= \frac{\left(F_{\mathrm{C}}(r)-F_{\mathrm{C}}(n+r)\right)^n - \left(F_{\mathrm{C}}(n-1+r)-F_{\mathrm{C}}(r-1)\right)^n}{1-(-\alpha^{-1})^n - \alpha^n + (-1)^n}.$$

Corollary 4.4 *Let* n *and* r *be positive integers. Then*

$$\det(\mathrm{lcir}(F_r:n)) = \det(\mathrm{cir}(F_r:n)) = \frac{(F_r - F_{n+r})^n - (F_{n+r-1} - F_{r-1})^n}{1 - L_n + (-1)^n},$$

where $L_n = \dfrac{\alpha^n + (-\alpha)^{-n}}{\sqrt{5}} = F_{n-1} + F_{n+1}$ *is the* n^{th} *Lucas number.*

We have the following open problem for the determinant of a real-valued Fibonacci left circulant matrix.

Open Problem 2. What is the formula for the determinant of $\mathrm{lcir}(F_{\mathrm{R}}(r):\ell:n)$?

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