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Robust Kalman filtering with generalized Laplace observational noises[*]

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Abstract

It is well known that the Kalman filter fails to provide optimal state estimates in the sense of minimum mean of squared state error, when measurement outliers occur in a linear stochastic system. This is primarily due to the usual Gaussian assumptions made on the measurement noise term in the state space model. In this manuscript, robust filters are derived by using generalized Laplace measurement noise with single and multi scale factors to replace Gaussian assumptions. The performance of the proposed robust filters is compared to the Kalman filter and other robust filters through Monte-Carlo simulations.

Key Words: recursive estimation, measurement outliers, heavy-tailed distribution

Introduction

In real-time applications, data is frequently collected one at a time. Hence, any suitable technique proposed to analyze real-time data is required to utilize the information from the current measurement and the prior knowledge of the system. To deal with problems of linear filtering and prediction, a linear discrete-time stochastic systems possesses properties of observability and reachability, represented by

$$\underline{X}_{t+1} = A_t\underline{X}_t + \underline{W}_t \quad (1)$$

$$\underline{Y}_t = C_t\underline{X}_t + \underline{V}_t \quad (2)$$

where $\underline{V}_t$ and $\underline{W}_t$ are the measurement noise and the system noise with the covariance matrices Σ_V and Σ_W, respectively. Also, both noise terms are assumed to be independent over time. The most celebrated Kalman filter (Kalman, 1960) was proposed as a recursive

[*] This article is to develop the robust Kalman filtering with single and multi scale factors to alleviate the effect of measurement outliers by assuming the measurement noise is multivariate generalized Laplace distributed.

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estimation procedure. By means of orthogonal projections, the optimal filter was derived in the sense of minimum average loss when a linear Gaussian stochastic system is considered. The traditional Kalman recursive formula of the state estimate and its covariance are respectively given as

$$\underline{\hat{X}}_{t|t} = \underline{\hat{X}}_{t|t-1} + K_t\left(\underline{Y}_t - C_t\underline{\hat{X}}_{t|t-1}\right)$$
$$P_{t|t} = (I - K_tC_t)P_{t|t-1}$$

where
$$\underline{\hat{X}}_{t|t-1} = A_t\underline{\hat{X}}_{t-1|t-1}$$
$$P_{t|t-1} = A_{t-1}P_{t-1|t-1}A_{t-1}' + \Sigma_W$$
$$K_t = P_{t|t-1}C_t'\left(C_tP_{t|t-1}C_t' + \Sigma_V\right)^{-1}$$

Unfortunately, in the presence of outliers, the noise terms are obviously non-Gaussian. This causes the Kalman filter to be no longer optimal.

To overcome the problem of having to deal with a non-Gaussian measurement noise term, robust filtering algorithms have been constructed and, consequently, many different robust filtering theories have emerged. One class of approaches considers the noise term to be distributed as a non-Gaussian heavy-tailed random variable to account for the outliers. For example, in early work, the distribution of the noise term was assumed to be a mixture of Gaussian components (Sorenson and Alspach, 1971; Pena and Guttman, 1988; Yatawara, Abraham, and MacGregor, 1991), a mixture of Student-t distributions (Meinhold and Singpurwalla, 1989), or a generalized Gaussian distribution (Niehsen, 2002).

An alternative technique was proposed by using a weighting factor in the covariance matrix of the measurement noise term. The weight is used to balance the disparity between the measurement and the predicted state estimate. Ting, Theodorou, and Schaal (2007) constructed a filter by using a weighted least squares-like approach with a single weight for all variables. Kovacevic, Banjac, and Kovacevic (2016) also proposed the robust filter using the recursive weighted least squares. The suitable single weight was obtained by Huber's robust estimation method. A drawback of these filters is that all variables, although some may not have outliers in them, get treated by the same weighting factor. To improve the filter's performance, Yang and Cui (2008) developed a robust filter with multi weights.

In this manuscript, a robust filter is developed by assuming that the measurement noise term follows a multivariate generalized Laplace (MGL) distribution, which is a class of symmetric multivariate models depending on a shape parameter. By a scale factor depending on an estimated value of the shape parameter at each time, the noise covariance matrix of the measurement noise term can be calculated adaptively such that the proposed robust

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filter could account for measurement outliers in the system. Moreover, algorithms for a such robust filtering are provided considering single and multiple scale factors.

Multivariate Generalized Laplace Distribution

The multivariate generalized Laplace (MGL) distribution was introduced by Ernst (1998) as a class of symmetric elliptically contoured models consisting of several distributions depending on the value of a shape parameter. This shape parameter λ distinguishes between members of the family such as the Laplace $(\lambda = 1)$, the normal $(\lambda = 2)$ and the uniform $(\lambda \to \infty)$ distributions.

Let $\underline{Y}$ be a $k \times 1$ random vector, $\underline{\mu}$ be a $k \times 1$ vector of constants, and $\Sigma = [(\sigma_{ij})]$ be a $k \times k$ non-negative definite matrix. Suppose the random vector $\underline{Y}$ has an MGL distribution with the mean vector $\underline{\mu}$, the scale parameter matrix Σ, and the shape parameter λ, denoted by $\underline{Y} \sim MGL_k\left(\underline{\mu}, \Sigma, \lambda\right)$, with the joint density of $\underline{Y}$ defined as

$$f\left(\underline{y}\right) = \frac{\lambda\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}\Gamma\left(\frac{k}{\lambda}\right)} |\Sigma|^{-\frac{1}{2}} exp\left\{-\left[\left(\underline{y} - \underline{\mu}\right)' \Sigma^{-1} \left(\underline{y} - \underline{\mu}\right)\right]^{\frac{\lambda}{2}}\right\} \quad (3)$$

where $\Gamma(\cdot)$ denotes a gamma function. The mean vector and the covariance matrix of the MGL random vector $\underline{Y}$ are given respectively by

$$E\left(\underline{Y}\right) = \underline{\mu} \quad \text{and} \quad Cov\left(\underline{Y}\right) = \frac{\Gamma\left(\frac{k+2}{\lambda}\right)}{k\Gamma\left(\frac{k}{\lambda}\right)}\Sigma.$$

Furthermore, let a partitioned random vector

$\underline{Y}'_{k\times 1} = \left(\underline{Y}_{1(1\times k_1)}, \quad \underline{Y}_{2(1\times(k-k_1))}\right)$ having a mean vector $\underline{\mu}'_{k\times 1} = \left(\underline{\mu}_{1(1\times k_1)}, \quad \underline{\mu}_{2(1\times(k-k_1))}\right)$, and scale parameter matrix

$\Sigma = \begin{bmatrix} \Sigma_{11(k_1\times k_1)} & \Sigma_{12(k_1\times(k-k_1))} \\ \Sigma_{21((k-k_1)\times k_1)} & \Sigma_{22((k-k_1)\times(k-k_1))} \end{bmatrix}$ be given. Then, the conditional MGL density of $\underline{Y}_1 \middle| \underline{Y}_2 = \underline{y}_2^*$ is defined as

$$f\left(\underline{y}_1 \middle| \underline{y}_2 = \underline{y}_2^*\right) = \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k-k_1}{\lambda}\right)}{\pi^{\frac{k_1}{2}}\Gamma\left(\frac{k}{\lambda}\right)\Gamma\left(\frac{k-k_1}{2}\right)} |\Sigma_{11.2}|^{-\frac{1}{2}}$$

$$\times\, exp\left\{-\left[\left(\underline{y}_1-\underline{\mu}_{1.2}\right)'\Sigma_{11.2}^{-1}\left(\underline{y}_1-\underline{\mu}_{1.2}\right)+\left(\underline{y}_2^*-\underline{\mu}_2\right)'\Sigma_{22}^{-1}\left(\underline{y}_2^*-\underline{\mu}_2\right)\right]^{\frac{\lambda}{2}}+\left[\left(\underline{y}_2^*-\underline{\mu}_2\right)'\Sigma_{22}^{-1}\left(\underline{y}_2^*-\underline{\mu}_2\right)\right]^{\frac{\lambda}{2}}\right\} \quad (4)$$

where $\underline{\mu}_{1.2}=\underline{\mu}_1+\Sigma_{12}\Sigma_{22}^{-1}\left(\underline{y}_2^*-\underline{\mu}_2\right)$ and $\Sigma_{11.2}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. The mean vector and the covariance matrix of $\underline{Y}_1\left|\underline{Y}_2=\underline{y}_2^*\right.$ are given respectively by

$$E\left(\underline{Y}_1\left|\underline{Y}_2=\underline{y}_2^*\right.\right)=\underline{\mu}_{1.2} \quad \text{and} \quad Cov\left(\underline{Y}_1\left|\underline{Y}_2=\underline{y}_2^*\right.\right)=\frac{\Gamma\left(\frac{k_1+2}{\lambda}\right)}{k_1\Gamma\left(\frac{k_1}{\lambda}\right)}\Sigma_{11.2}.$$

See further details on the elliptically contoured distribution in Fang, Kotz, and Ng (1990).

MGL filter

Consider a linear discrete-time stochastic system given by equations (1) and (2). In this derivation, $\underline{V}_t$ is assumed to be distributed as a zero mean MGL random vector with a scale parameter Σ_V and a shape parameter λ_V, denoted by $\underline{V}_t\sim MGL_k\left(\underline{0},\Sigma_V,\lambda_V\right)$ and the system noise term $\underline{W}_t$ is assumed to be a Gaussian random vector, denoted by $\underline{W}_t\sim MGL_r\left(\underline{0},\Sigma_W,2\right)$. The scale parameter matrices Σ_V and Σ_W are known positive definite. Also, both noise terms are assumed to be independent over time point. By least squares technique (Kalman, 1960; Duncan and Horn, 1972), an unbiased minimum variance state estimate and its covariance can be easily derived. These provide a recursive formulae of the MGL filter as given by the following equations.

$$\underline{\hat{X}}_{t|t}=\underline{\hat{X}}_{t|t-1}+K_t\left(\underline{Y}_t-C_t\underline{\hat{X}}_{t|t-1}\right) \quad (5)$$

$$P_{t|t}=(I-K_tC_t)P_{t|t-1} \quad (6)$$

where

$$\underline{\hat{X}}_{t|t-1}=A_t\underline{\hat{X}}_{t-1|t-1}$$

$$P_{t|t-1}=A_{t-1}P_{t-1|t-1}A_{t-1}'+\tfrac{1}{2}\Sigma_W$$

$$K_t=P_{t|t-1}C_t'\left(C_tP_{t|t-1}C_t'+\frac{\Gamma\left(\frac{k+2}{\lambda_V}\right)}{k\Gamma\left(\frac{k}{\lambda_V}\right)}\Sigma_V\right)^{-1}$$

In practice, the shape parameter of the measurement noise term λ_V has to be estimated from data at each time point.

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Algorithms of MGL filters

The MGL filter with a single scale factor (MGLF-S) is constructed by using only one factor for all variables. Consider an innovation, defined by $\underline{S}_t = \underline{Y}_t - \hat{\underline{Y}}_t = \underline{Y}_t - C_t\hat{\underline{X}}_{t|t-1}$. If $\underline{S}_t$ is Gaussian, this implies that the system has no outliers in the measurement noise term. Thus, to deal with measurement outliers, $\underline{S}_t$ is assumed to follow an MGL distribution with a time-varying shape parameter $\lambda_{S(t)}$, denoted by $\underline{S}_t \sim MGL_k(\underline{0}, \Sigma_S, \lambda_{S(t)})$ where Σ_S is the scale parameter matrix in case when $\underline{S}_t$ is supposed to be Gaussian, given by $\Sigma_S = 2\left(C_t P_{t|t-1} C_t' + \frac{1}{2}\Sigma_V\right)$. This leads to MGLF-S obtained by replacing λ_V with $\hat{\lambda}_{S(t)}$, where $\hat{\lambda}_{S(t)}$ is the maximum likelihood estimator of $\lambda_{S(t)}$, obtained by maximizing the negative logarithm of the likelihood as in equation (3), i.e.

$$\frac{d}{d\lambda_{S(t)}}\left[log\left(\lambda_{S(t)}\Gamma\left(\frac{k}{\lambda_{S(t)}}\right)\right) + \left(\underline{S}_t'\Sigma_S^{-1}\underline{S}_t\right)^{\frac{\lambda_{S(t)}}{2}}\right] = 0$$

where $\Sigma_S = 2\delta^2\left(C_t P_{t|t-1} C_t' + \frac{1}{2}\Sigma_V\right)$ and δ is a known constant, named a tolerance factor and used to improve the performance of the filter.

To develop the MGL filter with multi scale factors (MGLF-M), the j^{th} element of an innovation is obtained to construct a diagonal matrix of the scale factors. Define a partitioned k-variate MGL random vector $\underline{S}_t = (S_{jt}, \underline{S}_t^*)'$ where $\underline{S}_t^* = \left(S_{1t}, S_{2t}, \dots, S_{(j-1)t}, S_{(j+1)t}, \dots, S_{kt}\right)$ is a vector of the rest of innovations excluding S_{jt} with the corresponding block scale parameter matrix $\Sigma_S = \begin{bmatrix}\sigma_{jj} & \Sigma_j^{*'} \\ \Sigma_j^* & \Sigma^*\end{bmatrix}$ for $j = 1,2,\dots,k$. Then, the conditional density function of S_{jt} given $\underline{S}_t^* = \underline{s}_t^*$ is distributed as a zero mean univariate generalized Laplace random variable with scale parameter $\sigma_j^2 = \sigma_{jj} - \Sigma_j^{*'}\Sigma^{*-1}\Sigma_j^*$ and shape parameter $\lambda_{S(t)}$, denoted by $S_{jt}\big|\underline{S}_t^* = \underline{s}_t^* \sim MGL_1\left(0, \sigma_j^2, \lambda_{S_j(t)}\right)$, where the conditional variance of $S_{jt}\big|\underline{S}_t^* = \underline{s}_t^*$ is given by $\sigma^2_{S_{jt}|\underline{S}_t^*=\underline{s}_t^*} = \frac{\Gamma\left(\frac{3}{\lambda_{S_j(t)}}\right)}{\Gamma\left(\frac{1}{\lambda_{S_j(t)}}\right)}\sigma_j^2$. In practice, the $\lambda_{S_j(t)}$ are replaced by the maximum likelihood estimates $\hat{\lambda}_{S_j(t)}$ for $j = 1,2,\dots,k$, which are obtained by maximizing the negative logarithm of the likelihood of $S_{jt}\big|\underline{S}_t^* = \underline{s}_t^*$ with $\sigma_j^2 = 2\delta^2\left(\sigma_{jj} - \Sigma_j^{*'}\Sigma^{*-1}\Sigma_j^*\right)$ that follows the conditional density in equation (4).

The followings are outlines of the algorithms which can be used to implement the MGL filter with single scale factor and with multi scale factors.

1. Enter the initial estimates $\underline{\hat{X}}_{t|t-1}$ and $P_{t|t-1}$.
2. Collect a new measurement $\underline{Y}_t$.

3. **Approximating the shape parameter**

Compute an innovation, $\underline{S}_t = \underline{Y}_t - \underline{\hat{Y}}_t = \underline{Y}_t - C_t\underline{\hat{X}}_{t|t-1}$ where $\underline{S}_t$ is distributed as an MGL random vector, denoted by $\underline{S}_t \sim MGL_k\left(\underline{0}, \Sigma_S, \lambda_{S(t)}\right)$.

a. **Single scale factor**

Obtain a maximum likelihood estimator $\hat{\lambda}_{S_j(t)}$ with the scale parameter matrix Σ_S defined by $\Sigma_S = 2\delta^2\left(C_t P_{t|t-1} C_t' + \frac{1}{2}\Sigma_V\right)$ where δ is a known constant tolerance factor.

b. **Multi scale factors**

Obtain maximum likelihood estimators $\hat{\lambda}_{S_j(t)}$ for $j = 1,2,\dots,k$ corresponding to a conditional random variable S_{jt} given $\underline{S}_t^* = \underline{s}_t^*$ with the scale parameter σ_j^2 defined by $\sigma_j^2 = 2\delta^2\left(\sigma_{jj} - \Sigma_j^{*'}\Sigma^{*-1}\Sigma_j^*\right)$.

4. Calculating the filter gain

a. **Single scale factor**

Compute the filter gain $K_t = P_{t|t-1}C_t'\left(C_t P_{t|t-1} C_t' + \hat{\omega}\Sigma_V\right)^{-1}$ where $\hat{\omega} = \dfrac{\Gamma\left(\frac{k+2}{\hat{\lambda}_{S(t)}}\right)}{\mathrm{k}\Gamma\left(\frac{k}{\hat{\lambda}_{S(t)}}\right)}$ is a scale factor. The covariance matrix of the measurement noise term $\hat{\omega}\Sigma_V$ is also adaptively estimated at this stage.

b. **Multi scale factors**

Compute the filter gain $K_t = P_{t|t-1}C_t'\left(C_t P_{t|t-1} C_t' + \hat{\Omega}^{\frac{1}{2}}\Sigma_V\hat{\Omega}^{\frac{1}{2}}\right)^{-1}$ where $\hat{\Omega} = diag\left(\dfrac{\Gamma\left(\frac{k+2}{\hat{\lambda}_{S_1(t)}}\right)}{\mathrm{k}\Gamma\left(\frac{k}{\hat{\lambda}_{S_1(t)}}\right)}, \dfrac{\Gamma\left(\frac{k+2}{\hat{\lambda}_{S_2(t)}}\right)}{\mathrm{k}\Gamma\left(\frac{k}{\hat{\lambda}_{S_2(t)}}\right)}, \dots, \dfrac{\Gamma\left(\frac{k+2}{\hat{\lambda}_{S_k(t)}}\right)}{\mathrm{k}\Gamma\left(\frac{k}{\hat{\lambda}_{S_k(t)}}\right)}\right)$ is a diagonal matrix of

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the multi scale factors. Obtain the adaptive covariance matrix of the measurement noise term.

5. **State vector update**

Update the state estimate $\underline{\hat{X}}_{t|t}$ and state error covariance matrix $P_{t|t}$ by

$$\underline{\hat{X}}_{t|t} = \underline{\hat{X}}_{t|t-1} + K_t\left(\underline{Y}_t - C_t\underline{\hat{X}}_{t|t-1}\right)$$
$$P_{t|t} = (I - K_tC_t)P_{t|t-1}$$

6. **Time update**

Compute the one step ahead state estimate $\underline{\hat{X}}_{t+1|t}$ and state forecast error covariance matrix $P_{t+1|t}$ given by

$$\underline{\hat{X}}_{t+1|t} = A_t\underline{\hat{X}}_{t|t}$$
$$P_{t+1|t} = A_tP_{t|t}A_t' + \frac{1}{2}\Sigma_W$$

7. Let $t = t + 1$ and go to step 2.

Clearly, when the MGLF-S is used, its performance is inferior as it tends to influence some variables that are devoid of outlier effects. However, the MGLF-M may be implemented to alleviate such outlier effects. In the next section, the performances of the proposed MGL filters are investigated and compared with the classical Kalman filter as well as other robust filters through Monte Carlo simulations. In addition, the effects of the tolerance factor and the scale factor are discussed.

Performance study

The performance of the MGL filtering technique in comparison to the Kalman filter could be illustrated by using Figure 1 that belongs to the bivariate linear stochastic system with measurement outliers occurred in the first variable at time 35, 40, 55, 75, and 80 in the series of length 100. As shown in Figure 1(a), the states from the Kalman filter are significantly affected by measurement outliers, whereas the MGLF-S and MGLF-M show resistance to outliers. However, a major drawback of having a single scale factor in the MGL filter is that both the first and the second variables are down weighed by the same weighting factor even though the second variable is not affected by outliers. As shown in Figure 1(b), the values of the shape parameter in the MGL distribution used to calculate the single scale factor is less than 2 for each variable. This tends to produce under estimates for the states of the second variable which is not affected by outliers. In the MGLF-M, each variable is influenced by their own shape parameter, see Figure 1(c) for the first variable with outliers and Figure 1(d) for the second variable with no outliers. However, even under this scenario, the values of the shape parameter shown in Figure 1(d) are not significantly different from 2.

An investigation of the filter performance by a Monte Carlo simulation is established by assuming the outliers occur in the 5-variate linear discrete-time Gaussian system defined as

$$\underline{X}_{t+1} = \underline{X}_t + \underline{W}_t$$
$$\underline{Y}_t = \underline{X}_t + \underline{V}_t$$

with the usual assumptions on the parameters. A simulation consisting of 2,000 iterations with the mean of squared state error (MSSE) as a preferred criterion for comparisons was conducted under following conditions.

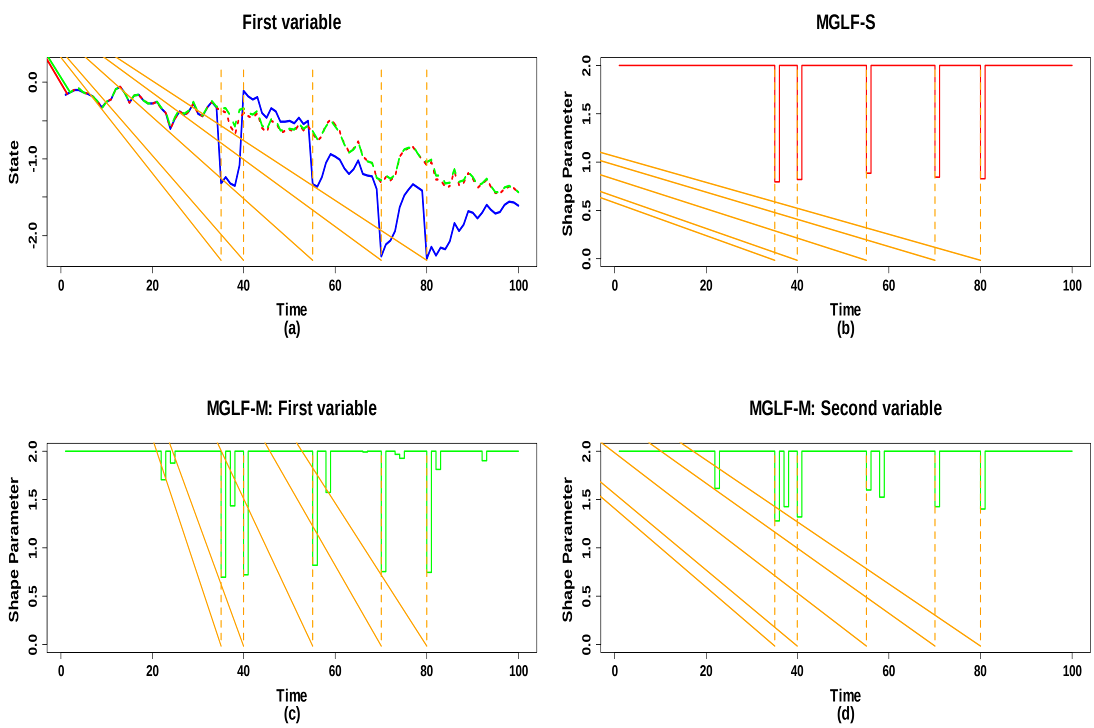

Figure 1 The states of the Kalman filter (solid line), the MGLF-S (dotted line) and the MGLF-M (dashed line) and the corresponding shape parameters.

1. System noise variances σ_W^2 = 0.01 and correlation coefficients of system noise terms ρ_W = 0.1, 0.4, and 0.8,

2. Measurement noise variances σ_V^2 = 1 and the corresponding correlation coefficients of the measurement noise terms ρ_V = 0.1, 0.4, and 0.8,

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3. Magnitude of additive measurement outliers is defined by $\delta_{outlier}\sigma_V$ where $\delta_{outlier}$ = 2, 3, 4, 5, 10, and 15,

4. Five additional measurement outliers occur only in the first variable at times 35, 40, 55, 70, and 80 in the time series of length 100,

5. The mean of squared state error in each iteration computed by

$$MSSE = \frac{1}{100}\sum_{t=1}^{100}\left(\underline{X}_t - \hat{\underline{X}}_{t|t}\right)'\left(\underline{X}_t - \hat{\underline{X}}_{t|t}\right)$$

and averaged over 2,000 iterations.

The structure of both noise covariance matrices is formed by assuming $\Sigma = \sigma^2(1-\rho)I + \sigma^2\rho J$ where I is an identity matrix and J is a matrix of ones.

Table 1 MSSE values of the filters for various correlation coefficients of the system noise terms (ρ_W), correlation coefficients of the measurement noise terms (ρ_V), and tolerance factor (δ).

$(\rho_W, \rho$	δ_{outli}	KALMAN	MGF	GGF	MGLF-S 1	MGLF-S 2	MGLF-S 3	MGLF-S 4	MGLF-S 5	MGLF-M 1	MGLF-M 2	MGLF-M 3	MGLF-M 4	MGLF-M 5
(0.1, 0.1)	2	**0.45465**	0.45467	0.47623	0.54445	0.45023	0.45421	0.45244	0.45335	0.64783	0.47062	0.45605	0.45167	0.45313
	3	0.46049	0.45967	0.46985	0.53931	0.46007	0.46141	0.46449	0.46249	0.63799	0.47600	**0.45487**	0.45851	0.45952
	4	0.47814	0.47270	0.47001	0.54531	0.46732	0.48034	0.47875	0.48451	0.63993	0.47775	**0.45845**	0.46022	0.47288
	5	0.50775	0.48423	0.47463	0.55157	0.46033	0.48944	0.50141	0.50035	0.64589	0.46903	**0.45763**	0.45914	0.47029
	10	0.69686	0.45482	0.47057	0.55091	0.45992	0.46483	0.50668	0.59004	0.64309	0.47363	**0.44657**	0.45144	0.45757
	15	0.99432	0.45309	0.46799	0.54847	0.45336	0.45531	0.46881	0.50912	0.63931	0.46544	**0.44426**	0.44090	0.44675
(0.1, 0.4)	2	0.42763	0.42751	0.47352	0.50829	0.42486	0.42527	0.42629	0.42629	0.59862	0.44308	**0.42617**	0.42495	0.42551
	3	0.43653	0.43477	0.47160	0.50923	0.42747	0.43440	0.44145	0.43786	0.59057	0.44134	**0.42430**	0.43125	0.43296
	4	0.45458	0.44324	0.47041	0.50828	0.43619	0.45387	0.46039	0.45914	0.59538	0.44710	**0.42875**	0.43189	0.43951
	5	0.48831	0.44377	0.47640	0.51685	0.43381	0.45392	0.47387	0.48153	0.59743	0.44171	**0.42754**	0.42255	0.43645

	10	0.70480	0.42801	0.47140	0.51682	0.43128	0.43162	0.45686	0.50546	0.59379	0.44262	**0.41881**	0.42302	0.42603
	15	1.05263	0.42197	0.46457	0.51060	0.42572	0.42393	0.43297	0.45567	0.59085	0.43862	**0.41763**	0.41636	0.42074
(0.1, 0.8)	2	0.34119	0.33987	0.47448	0.39704	0.33834	0.34090	0.34246	0.34345	0.47580	0.35604	**0.33949**	0.33907	0.34059
	3	0.37147	0.35242	0.47660	0.41182	0.34006	0.34977	0.36922	0.36396	0.47806	0.35849	**0.34575**	0.35528	0.34897
	4	0.39990	**0.33579**	0.47332	0.40328	0.33561	0.34909	0.37563	0.39428	0.47026	0.35437	0.34785	0.36415	0.36927
	5	0.44026	**0.33392**	0.46741	0.39959	0.34306	0.34383	0.37019	0.41068	0.46227	0.36392	0.34782	0.36871	0.38553
	10	0.79567	**0.33409**	0.46866	0.39925	0.34608	0.33582	0.34428	0.35392	0.46784	0.35732	0.33845	0.35490	0.37281
	15	1.38472	0.34050	0.47149	0.41286	0.33740	0.33409	0.33641	0.34240	0.48212	0.35391	**0.33679**	0.33929	0.36174

Note that KALMAN, MGF, GGF, MGLF-S, and MGLF-M represent the Kalman filter, the mixture Gaussian filter, the generalized Gaussian filter, the robust MGL filters with a single and multi scale factors, respectively. An underlined number and a bold number show a case of minimum MSSE value when the MGLF-S with $\delta = 2$ and the MGLF-M with $\delta = 3$ are respectively considered in a comparison with KALMAN, MGF, and GGF.

Table 1 Continued.

$(\rho_W, \rho$	δ_{outli}	KALMAN	MGF	GGF	MGLF-S					MGLF-M				
					1	2	3	4	5	1	2	3	4	5
(0.4, 0.1)	2	0.42107	0.42107	0.44176	0.50892	0.42434	0.42063	0.42603	0.42230	0.59837	0.44351	0.42164	0.42566	0.42193
	3	0.43814	0.43804	0.44645	0.51905	0.43179	0.43408	0.43239	0.43387	0.60838	0.44485	**0.42847**	0.42746	0.43117
	4	0.44926	0.44331	0.43941	0.50856	0.43691	0.45121	0.45119	0.44127	0.59328	0.44597	**0.43142**	0.43378	0.43107
	5	0.46474	0.44246	0.43248	0.50343	0.43524	0.45615	0.47338	0.47250	0.58021	0.44262	**0.42599**	0.43455	0.44325
	10	0.64709	0.42359	0.43715	0.50857	0.42901	0.43680	0.47413	0.54577	0.58579	0.44008	**0.41883**	0.42344	0.42764
	15	0.93765	0.42532	0.43678	0.51522	0.42625	0.42600	0.44254	0.47214	0.59465	0.43656	**0.41556**	0.41421	0.41500

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(0.4, 0.4)	2	0.451 74	0.45 165	0.50 481	0.54 642	0.45 330	0.44 890	0.45 680	0.45 412	0.59 163	0.46 912	**0.44 978**	0.45 583	0.45 355
	3	0.467 36	0.46 614	0.50 899	0.55 446	0.45 782	0.46 473	0.46 100	0.46 412	0.59 727	0.46 790	**0.45 609**	0.45 270	0.45 897
	4	0.488 02	0.47 639	0.51 085	0.55 473	0.45 803	0.47 397	0.48 208	0.47 371	0.59 366	0.46 578	**0.45 461**	0.45 828	0.45 612
	5	0.501 82	0.46 484	0.50 380	0.54 627	0.46 429	0.47 480	0.50 481	0.50 674	0.58 574	0.47 429	**0.44 990**	0.46 048	0.46 341
	10	0.692 12	0.45 395	0.50 207	0.54 430	0.46 152	0.45 592	0.47 770	0.52 170	0.58 620	0.47 084	**0.44 537**	0.45 105	0.46 197
	15	1.000 15	0.45 240	0.49 937	0.55 080	0.45 425	0.45 710	0.46 001	0.48 207	0.59 378	0.46 498	**0.44 990**	0.44 438	0.45 422
(0.4, 0.8)	2	0.428 18	0.42 732	0.57 418	0.51 158	0.41 992	0.42 212	0.43 281	0.43 015	0.54 335	0.44 497	**0.42 690**	0.43 070	0.42 756
	3	0.446 73	**0.43 265**	0.57 447	0.51 643	0.42 369	0.43 156	0.44 015	0.44 598	0.54 237	0.44 683	0.44 109	0.44 097	0.44 207
	4	0.477 76	**0.42 735**	0.57 782	0.51 579	0.42 157	0.42 593	0.44 731	0.46 566	0.54 519	0.44 588	0.43 946	0.46 483	0.47 344
	5	0.502 79	**0.41 929**	0.56 855	0.50 412	0.43 045	0.42 709	0.44 607	0.47 439	0.52 965	0.45 393	0.44 034	0.46 817	0.48 927
	10	0.797 61	**0.41 971**	0.56 891	0.50 244	0.43 050	0.42 159	0.42 835	0.44 312	0.54 050	0.44 390	0.42 799	0.44 952	0.49 906
	15	1.277 84	**0.42 379**	0.56 812	0.52 033	0.42 229	0.42 451	0.41 773	0.43 334	0.54 720	0.44 227	0.42 697	0.42 933	0.47 086

Note that KALMAN, MGF, GGF, MGLF-S, and MGLF-M represent the Kalman filter, the mixture Gaussian filter, the generalized Gaussian filter, the robust MGL filters with a single and multi scale factors, respectively. An underlined number and a bold number show a case of minimum MSSE value when the MGLF-S with $\delta = 2$ and the MGLF-M with $\delta = 3$ are respectively considered in a comparison with KALMAN, MGF, and GGF.

Table 1 Continued.

$(\rho_W, \rho$	δ_{outli}	KAL MAN	MGF	GGF	MGLF-S 1	MGLF-S 2	MGLF-S 3	MGLF-S 4	MGLF-S 5	MGLF-M 1	MGLF-M 2	MGLF-M 3	MGLF-M 4	MGLF-M 5
(0.8,	2	0.334	0.33	0.35	0.40	0.33	0.33	0.32	0.33	0.45	0.34	**0.33**	0.32	0.33
0.1)		45	462	044	124	456	284	948	423	739	838	**357**	883	400
	3	0.337	**0.33**	0.34	0.39	0.34	0.34	0.34	0.34	0.45	0.35	0.33	0.33	0.34
		37	**681**	414	823	631	302	235	389	070	767	796	832	188
	4	0.354	0.35	0.34	0.40	0.34	0.35	0.35	0.35	0.46	0.34	**0.33**	0.34	0.34
		24	020	851	628	162	194	609	418	049	849	**700**	162	526
	5	0.368	0.35	0.34	0.39	0.33	0.35	0.36	0.37	0.44	0.34	**0.33**	0.33	0.34
		07	021	290	628	934	993	621	352	386	477	**598**	614	992
	10	0.511	0.33	0.34	0.40	0.34	0.34	0.36	0.42	0.44	0.34	**0.33**	0.32	0.33
		21	603	501	202	163	772	911	922	953	942	**343**	974	760
	15	0.744	0.33	0.34	0.40	0.33	0.34	0.34	0.37	0.45	0.34	**0.33**	0.32	0.32
		57	735	666	530	879	152	949	204	417	470	**192**	734	640
(0.8,	2	**0.409**	**0.40**	0.46	0.49	0.42	0.42	0.41	0.41	0.49	0.43	0.42	0.40	0.41
0.4)		**42**	**942**	189	525	088	288	002	807	541	333	410	954	755
	3	0.419	**0.41**	0.46	0.49	0.43	0.42	0.42	0.41	0.49	0.44	0.41	0.42	0.41
		98	**932**	312	971	299	493	835	989	751	217	942	316	673
	4	0.439	0.43	0.46	0.51	0.42	0.42	0.44	0.43	0.50	0.43	**0.41**	0.42	0.42
		14	264	683	686	412	436	460	820	614	144	**078**	677	435
	5	0.448	0.42	0.46	0.50	0.41	0.43	0.44	0.45	0.49	0.42	**0.41**	0.41	0.42
		97	267	390	318	787	456	541	766	642	416	**805**	448	865
	10	0.584	**0.42**	0.45	0.50	0.42	0.43	0.43	0.46	0.50	0.43	0.42	0.41	0.42
		65	**051**	790	798	531	016	493	057	025	113	158	816	335
	15	0.803	**0.41**	0.46	0.50	0.42	0.42	0.42	0.43	0.50	0.42	0.41	0.41	0.41
		96	**911**	623	868	399	783	907	395	699	881	948	956	827
(0.8,	2	**0.440**	0.44	0.58	0.53	0.45	0.45	0.44	0.45	0.52	0.48	0.47	0.44	0.45
0.8)		**44**	099	962	775	256	670	186	015	476	445	119	973	149
	3	0.456	**0.45**	0.59	0.53	0.46	0.45	0.46	0.45	0.52	0.49	0.47	0.48	0.46
		45	**002**	071	903	921	529	651	575	645	642	432	527	989
	4	0.481	**0.45**	0.59	0.56	0.45	0.44	0.47	0.47	0.53	0.48	0.46	0.50	0.51
		54	**331**	685	094	091	437	577	547	809	106	744	974	174
	5	0.496	**0.44**	0.59	0.54	0.44	0.45	0.45	0.48	0.52	0.47	0.47	0.49	0.55
		25	**636**	439	101	821	394	623	615	906	700	501	853	025
	10	0.690	**0.45**	0.58	0.55	0.45	0.46	0.45	0.45	0.52	0.47	0.46	0.48	0.53
		58	**302**	565	384	728	105	311	674	980	230	897	471	456
	15	1.003	**0.44**	0.59	0.54	0.45	0.46	0.45	0.45	0.53	0.47	0.46	0.47	0.50
		66	**915**	736	242	467	001	475	229	356	001	096	265	088

Note that KALMAN, MGF, GGF, MGLF-S, and MGLF-M represent the Kalman filter, the mixture Gaussian filter, the generalized Gaussian filter, the robust MGL filters with a single and multi scale factors, respectively. An underlined number and a bold number show a case of minimum MSSE value when the MGLF-S with $\delta = 2$ and the MGLF-M with $\delta = 3$ are respectively considered in a comparison with KALMAN, MGF, and GGF.

All filters are in a comparison consisting of the Kalman filter, the robust filter with mixture Gaussian noise (MGF), the robust filter with generalized Gaussian noise (GGF) and the proposed robust filters, MGLF-S and MGLF-M. The MGF and GGF involve a heavy-tailed distributed measurement noise term. The MGF is assumed a measurement noise term has a mixture Gaussian distribution (Yatawara, Abraham, and MacGregor, 1991). In this study, the mixture measurement noises were determined by a probability of outliers occurred that sets to 0.05. The GGF is developed by using a generalized Gaussian distributed noise and assuming all measurement noises are independent (Niehsen, 2002).

The simulation results in Table 1 illustrate the effect on MSSE values of all five filters for various system noise and measurement noise correlations, ρ_W and ρ_V, and the tolerance factor δ. When $\delta_{outlier}$ is small, MSSE values of the Kalman filter do not dramatically change as those of the robust filters do. This implies that the Kalman filter is robust to small magnitudes of measurement outliers. MSSE values of the Kalman filter also tend to increase obviously when $\delta_{outlier}$ grows up. In contrast, those of all the other four robust filters are consistent to all magnitudes of measurement outliers. Further, the correlation coefficients of the noise terms influence the effectiveness of all filters, i.e. MSSE values of those filters get large when correlation coefficients of noise terms are inflated.

To compare the performance of the robust filters, a suitable tolerance factor δ might be chosen to obtain the optimal MGL filters that achieve a minimum MSSE. Based on the studied model in this investigation resulted in Table 1, in most cases when $\delta = 2$, the MGLF-S produces the lowest MSSE against all δ and the MGLF-M does as $\delta = 3$. However, the effectiveness of both MGL filters get worse when ρ_W or ρ_V become large. Meanwhile, the MGF performs well in this situation. Furthermore, when ρ_V is small or moderate, the MGLF-M with $\delta = 3$ performs superior to the MGLF-S with $\delta = 2$ in most cases.

Without the tolerance factor $(\delta = 1)$, the MGL filters have large values of MSSE since the state estimates of the filters are down weighed heavily by the scale factors although those measurements are not outliers. In addition, it could be noticed that the efficiency of the MGL filters can be improved by choosing a suitable value of the tolerance factor. For example, in a such simulation under the studied model, a value of the tolerance factor should be set to 2 or 3 as resulted in Table 1. However, a suitable value of δ could be varied by a model. Thus, an appropriate value of a such factor should be found out to attain a minimum mean square of the state error.

Conclusion

To address the linear filtering problem when measurement noise outliers occur, it was assumed that the measurement noise follows a multivariate generalized Laplace distribution, and the MGL filters depending on a single and multi scale factors were developed. The performance of the MGL filters is shown to improve significantly when the measurement noise covariance matrices are also adaptively estimated. The evidence from a Monte Carlo investigation reveals that the proposed MGL filters are in fact robust against measurement outliers. However, their performances are deteriorated by effects of both noise term correlations. MSSE values of the MGL filters tend to be extensive when a large magnitude of the tolerance factor is applied. Thus, an effectiveness of the MGL filters are relied on a chosen suitable value of the tolerance factor that achieves a minimum MSSE filter.

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References

Duncan, David B. and Susan D. Horn. (1972). “Linear dynamic recursive estimation from the viewpoint of regression analysis.” **Journal of the American Statistical Association 67:** 815-821.

Ernst, Michael D. (1998). “A multivariate generalized Laplace distribution.” **Computational Statistics 13:** 227-232.

Fang, Kai.-Tai., Samuel Kotz, and Kai W. Ng. (1990). **Symmetric Multivariate and Related Distributions.** New York: Chapman & Hall.

Kalman, Rudolf E. (1960). “A new approach to linear filtering and prediction problems.” **Journal of Basic Engineering 82(D):** 35-45.

Veridian E-Journal, Science and Technology Silpakorn University
Volume 3 Number 6 November – December 2016 ISSN 2408 - 1248

Kovacevic, Brako, Zoran Banjac, and Kostic Kovacevic. (2016). "Robust adaptive filtering using recursive weighted least squares with combined scale and variable forgetting factors." EURASIP **Journal on** Advances in Signal Processing 37: 1-22.

Meinhold, Richard J. and Nozer D. Singpurwalla. (1989). "Robustification of Kalman filter models." **Journal of the American Statistical Association 84:** 479-486.

Niehsen, Wolfgang. (2002). "Robust Kalman filtering with generalized Gaussian measurement noise." **IEEE Transactions on Aerospace and Electronic System 38:** 1409-1412.

Pena, Daniel and Irwin Guttman. (1988). "Bayesian approach to robustifying the Kalman filter." In **Bayesian Analysis of Time Series and Dynamic Models**, **227-253.** Edited by James C. Spall. New York: Marcel Dekker.

Sorenson, Harold W. and Daniel L. Alspach. (1971). "Recursive Bayesian estimation using Gaussian sums." **Automatica 7:** 465-479.

Ting, Jo-Anne, Evangelos Theodorou, and Stefan Schaal. (2007). "Learning an outlier-robust Kalman filter." CLMC Technical Report, University of Southern California, USA.

Yang, Yuanxi and Xianqiang Cui. (2008). "Adaptively robust filter with multi adaptive factor." **Survey Review 40:** 260-270.

Yatawara, Nihal, Bovas Abraham, and John F. MacGregor. (1991). "A Kalman filter in the presence of outliers." **Communications in Statistics-Theory and Methods 20:** 1803-1820.