On the Relationship between Incidence Systems and Multipartite Graphs.*

Passawan Noppakaew**

Abstract

This article presents the connection between incidence systems and multipartite graphs by using category theory. There is a fully faithful functor from the category of incidence systems to the category of multipartite graphs. Therefore, multipartite graphs are good models representing incidence systems, and vice versa. Due to this connection, some properties of incidence systems and geometries can be explained by graph theory, and in the same way, some properties of multipartite graphs can be explained by building theory.

1 Introduction

In general, an incidence system involves a number of geometric objects with different types and symmetric binary relations which hold between different-type objects. Thus incidence system is a main concept in combinatorial geometry. By elaborating on Reye's work (Reye, 1882: 92-96), the abstract and general notion of an incidence system was firstly developed by E. H. Moore in 1896 (Moore, 1896: 264-290). Incidence geometry is incidence system with a given additional property. As incidence geometry is related to the most basic discrete mathematical structures of geometric configurations, it can be found almost everywhere in mathematics. Incidence geometry generalizes not only projective and affine geometry but also the geometries induced on some spaces equipped with some additional structures, such as a quadratic form. According to the applications of discrete structures in theories from many fields such as design theory, coding theory, and matroid theory, incidence geometry interests not only mathematicians but also statisticians, computer scientists, physicists and engineers.

This article presents the connection between incidence systems and multipartite graphs by using category theory and investigates the property of multipartite graphs which will correspond to incidence geometries. The connection shows that multipartite graphs are convenient models representing incidence systems so that all results of incidence systems may be entirely presented in the language of graph theory, and in the other way round. In the same way, incidence systems are models representing multipartite graphs so that all results of

* This article aims to give basic information and knowledge of incidence geometries in the aspect of graph theory, which may be useful for researchers who work in related disciplinaries.

** Department of Mathematics, Faculty of Science, Silpakorn University, Thailand. E-mail: noppakaew_p@su.ac.th

Veridian E-Journal, Science and Technology Silpakorn University
Volume 3 Number 5 September– October2016 (ISSN 2408 – 1248)

multipartite graphs may be explored by using the theory for incidence systems which comes from building theory (Brown, 1988; Ronan, 1989).

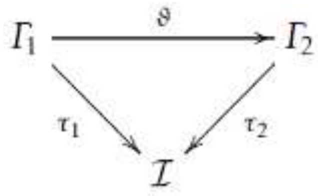
2 Incidence Systems and Geometries

In a set theoretical view point, incidence systems and geometries are defined by (Moore, 1896):

Definition 1 *An **incidence system** over a (usually finite) set* I *is a set* Γ *equipped with a reflexive and symmetric relation* $*$ *on* Γ, *called **incidence relation**, and a surjective map* $\tau:\Gamma\to\mathrm{I}$ *, called **type map**, such that for each* $v_1,v_2\in\Gamma$,

$$\text{if } v_1 * v_2 \text{ and } \tau(v_1)=\tau(v_2), \text{ then } v_1=v_2.$$

An ***incidence morphism*** $\vartheta:\Gamma_1\to\Gamma_2$ of incidence systems over I is a morphism from Γ_1 to Γ_2 preserving the incidence relation and types, i.e., for all $v_1,v_2\in\Gamma_1$, if $v_1 * v_2$ then $\vartheta(v_1)*\vartheta(v_2)$ and the following diagram commutes.



The *rank* of Γ is $|\mathrm{I}|$, which is the cardinality of I .

In particular, the identity map $\mathrm{id}:\Gamma\to\Gamma:v\mapsto v$ is an incidence morphism on Γ and the composition of any two incidence morphisms $\vartheta:\Gamma\to\Gamma'$ and $\vartheta':\Gamma'\to\Gamma''$, defined by

$$\begin{array}{rccc}\vartheta'\circ\vartheta: & \Gamma & \to & \Gamma'' \\ & v & \mapsto & \vartheta'(\vartheta(v))\end{array}$$

is again an incidence morphism. Moreover, this composition is associative. Therefore the incidence systems over I together with their incidence morphisms form a category called the ***category of incidence systems*** over I , denoted by $\mathbf{ISys}_{\mathrm{I}}$. However, such a category is depend on I . In order to be able to construct the category of all incidence systems, the definition of incidence morphisms is generalized by Noppakaew et al. (Noppakaew et al., 2013) in 2013. They noticed that, given an incidence system Γ over I and a map $\rho:\mathrm{I}'\to\mathrm{I}\,\mathrm{j}$ the set

$$\rho^*\Gamma := \coprod_{i\in\mathrm{I}'}(\rho^*\Gamma)_i ,$$

the disjoint union of $(\rho^*\Gamma)_i := \Gamma_{\rho(i)} = \{v\in\Gamma \mid \tau(v)=\rho(i)\}$ (for any $i,j\in\mathrm{I}$, the sets $\Gamma_{\rho(i)}$ and $\Gamma_{\rho(j)}$ are considered to be distinct and disjoint even though $\rho(i)=\rho(j)$), together with the restriction of the incidence relation of Γ to $\rho^*\Gamma$ and the type function $\rho^*\tau:\rho^*\Gamma \to \mathrm{I}'$, sending any element $v \in (\rho^*\Gamma)_i$ to i, forms an incidence system, called the ***pullback incidence system*** of Γ over I' induced by ρ. From this observation, they then proposed the definition of incidence morphisms between two incidence systems over the different sets.

Definition 2 *Let Γ_1 and Γ_2 be incidence systems over I_1 and I_2, respectively. A **generalized incidence morphism** $\Theta:\Gamma_1 \to \Gamma_2$ over an map $\rho:\mathrm{I}_1 \to \mathrm{I}_2$ is an incidence morphism $\vartheta:\rho^*\Gamma_1 \to \Gamma_2$. In particular, if ϑ is injective (resp. surjective), then so is Θ.*

When ρ is the inclusion map, Θ may be considered to be a projection of the incidence system Γ_1 to Γ_2. Noppakaew et al. already explained the projection idea in their work and gave some examples.

Indeed, an incidence morphism between two incidence systems over I is a generalized incidence morphism over the identity map $\mathrm{id}:\mathrm{I} \to \mathrm{I}$. Noppakaew et al. proved that the incidence systems over arbitrary sets together with their generalized incidence morphisms over arbitrary maps form a category called the ***category of incidence systems***, denoted by $\mathbf{ISys}$.

Definition 3 *Let Γ be an incidence system over I. A **flag** F of Γ is a set of mutually incident elements of Γ and we say that F is of **type** $\tau(F) := \{\tau(v) \mid v \in F\}$ and of **rank** $|\tau(F)|$.*

*A **residue** of a flag F, denoted by $\mathrm{Res}(F)$, is a subset of Γ consisting of all $v \in \Gamma - F$ such that $v * v'$ for all $v' \in F$.*

One can easily see that, given a flag F of an incidence system Γ over I, both F and $\mathrm{Res}(F)$ are incidence systems over $\tau(F)$ and $\mathrm{I} - \tau(F)$, respectively.

Definition 4 *An **Incidence geometry** Γ over I is an incidence system over I such that every maximal flag of Γ is of type I.*

As incidence geometries are incidence systems, therefore the incidence geometries together with generalized incidence morphisms form a category called the ***category of incidence geometries***, denoted by $\mathbf{IGeo}$.

3 Incidence Systems from a Graphical Viewpoint

The most common and familiar incidence systems are the ones of rank 2. They can be viewed as incidence structures between “points" and “lines", each of which can be uniquely represented by Levi graph (Coxeter, 1950: 413–455). In this graph, points are represented by black vertices and lines by white vertices. An edge joins a black vertex and a white vertex if and only if the corresponding point and line are incident, i.e. the point lies on the line.

By generalizing this idea, incidence system is conveniently described using graph theory (Scharlau, 1995; Weiss, 2004; Shult, 2010) and defined by:

Definition 5 *An incidence system* Γ *over a set* I *is an* ***undirected*** $|\mathrm{I}|$***-partite graph*** $\mathbf{G}(\Gamma)$, *i.e., a graph with* $|\mathrm{I}|$*-fold partitions of the vertex set into independent components so that edges join only two vertices from distinct components. The vertex set* $V(\mathbf{G}(\Gamma))$ *is actually* Γ *and the edge set* $E(\mathbf{G}(\Gamma))$ *is determined by the incidence relation of* Γ. *An incidence morphism is then just a type-preserving graph homomorphism.*

One can easily see that there is a one-to-one correspondence between the set of all incidence systems and the set of all undirected multipartite graphs. The category of undirected multipartite graphs can be constructed in the same manner as the category of incidence systems. As undirected graphs together with graph homomorphisms form a category called the ***category of undirected graphs***, denoted by $\mathbf{Grph}$ (Lane, 1997), and the composition of type preserving maps is still type preserving, therefore undirected k-partite graphs together with type-preserving graph homomorphisms form a category called the the ***category of undirected*** k***-partite graphs***, denoted by $\mathbf{MPGrph}_k$.

Let G be an undirected k-partite graph and ρ be a map from $\{1,2,\ldots,r\}$ to $\{1,2,\ldots,k\}$. The vertex set $V(G)$ of G is partitioned into disjoint sets $V(G)_1$, $V(G)_2$,..., $V(G)_k$. The graph ρ^*G whose the vertex set is

$$V(\rho^*G) := \coprod_{i\in\{1,2,\ldots,r\}} V(\rho^*G)_i,$$

where $V(\rho^*G)_i := V(G)_{\rho(i)}$, and the edge set is

$$E(\rho^*G) := E(G) \cap \left(V(\rho^*G) \times V(\rho^*G)\right),$$

is an r-partite graph, called the ***pullback*** r***-partite graph*** induced by ρ. The set $V(\rho^*G)_i$ and $V(\rho^*G)_j$ are considered to be distinct and disjoint for all $i, j \in \mathrm{I}$ even though $\rho(i) = \rho(j)$. In particular, if ρ is injective, then ρ^*G is a subgraph of G. Moreover, if G' is an undirected k-partite graph and $f : G \to G'$ is a type-preserving graph homomorphism, then the map $\rho^* f : \rho^*G \to \rho^*G'$ defined by

$$\rho^* f \Big|_{V(\rho^*G)_i} := f \Big|_{V(G)_{\rho(i)}} : V(G)_{\rho(i)} \to V(G')_{\rho(i)}$$

for all $i \in \{1,2,\ldots,r\}$, is well-defined and adjacency-preserving because f does. Thus $\rho^* f$ is a type-preserving graph homomorphism. Therefore ρ^* is a functor from the category $\mathbf{MPGrph}_k$ to the category $\mathbf{MPGrph}_r$.

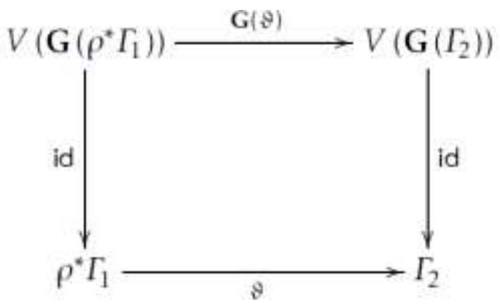
Definition 6 *Let G_1 and G_2 be undirected graphs with k and r partites, respectively. A **generalized type-preserving graph homomorphism** $F: G_1 \to G_2$ over a map $\rho: \{1,2,\ldots,r\} \to \{1,2,\ldots,k\}$ is a type-preserving graph homomorphism $f: \rho^* G_1 \to G_2$. In particular, if f is injective (resp. surjective), then so is F.*

If $F_1: G_1 \to G_2$ *and* $F_2: G_2 \to G_3$ are generalized type preserving homomorphisms of multipartite graphs over the maps $\rho_1: \{1,2,\ldots,r\} \to \{1,2,\ldots,k\}$ and $\rho_2: \{1,2,\ldots,m\} \to \{1,2,\ldots,r\}$, respectively, then $F_2 \circ F_1: G_1 \to G_3$, given by

$$f_2 \circ \rho_2^* f_1 : \rho_2^*\left(\rho^* G_1\right) \to G_3,$$

where $\rho_2^* f_1 : \rho_2^*\left(\rho^* G_1\right) \to \rho_2^* G_2$ and $f_2 : \rho_2^* G_2 \to G_3$, is a generalized type-preserving graph homomorphism over the map $\rho_1 \circ \rho_2$. Moreover, this composition is associative. Therefore, undirected multipartite graphs together with their generalized type-preserving graph homomorphisms form a category called the category of undirected multipartite graphs, denoted by $\mathbf{MPGrph}$.

From the construction above, there is the functor setting up one-to-one correspondence between $\mathbf{ISys}$ and $\mathbf{MPGrph}$. Let $\Theta: \Gamma_1 \to \Gamma_2$ be a generalized incidence morphism over a map $\rho: \mathrm{I}_1 \to \mathrm{I}_2$, where Γ_1 and Γ_2 are incidence systems over I_1 and I_2, respectively. Then Θ is actually an incidence morphism $\vartheta: \rho^* \Gamma_1 \to \Gamma_2$. The map $\mathbf{G}(\vartheta): \mathbf{G}\left(\rho^* \Gamma_1\right) \to \mathbf{G}\left(\Gamma_2\right)$, where $\mathbf{G}\left(\Gamma_i\right)$ is the $|\mathrm{I}_i|$-partite graphs associated to the incidence systems Γ_i over I_i as defined in Definition 5, given by the commuting diagram



is a generalized type-preserving graph homomorphism over the map $\rho: \mathrm{I}_1 \to \mathrm{I}_2$. Therefore $\mathbf{G}$ is a functor from $\mathbf{ISys}$ and $\mathbf{MPGrph}$.

Veridian E-Journal, Science and Technology Silpakorn University
Volume 3 Number 5 September– October2016 (ISSN 2408 – 1248)

4 Incidence Systems from a Graphical Viewpoint

The functor $\mathbf{G}$ introduced in the previous section is a fully faithful functor. We have thus obtained the following.

Theorem 7 $\mathbf{ISys} \cong \mathbf{MPGrph}$.

Therefore, the properties of incidence systems can be studied through investigating the properties of their corresponding multipartite graphs, and vice versa.

Definition 8 *A flag F of an incidence system Γ over a set I is a* ***clique*** $\mathbf{G}(F)$, *i.e., a complete subgraph, of the corresponding undirected $|\mathrm{I}|$-partite graph $\mathbf{G}(\Gamma)$.*

Recall that a maximal clique is a clique that cannot be extended by including one more adjacent vertex, i.e., it is not a subset of a larger clique. A maximum clique is a clique of largest size in a given graph G and its size is denoted by $\omega(G)$. Therefore a maximum clique is always maximal. This gives us a concept of graphs corresponding to incidence geometries.

Theorem 9 *Let G be a k-partite graph such that $\omega(G)=k$. If $\mathrm{Aut}(G)$, the set of all type-preserving graph automorphisms of G, acts transitively on the cliques of the same size, then $G=\mathbf{G}(\Gamma)$ for some incidence geometry Γ over I such that $|\mathrm{I}|=k$.*

Theorem 10 *Let G be a cycle (or circular) graph, i.e., the graph consists of a single cycle, with $2n$ vertices. Then the symmetric group S_n acts transitively on the cliques of the same size of G. Consequencely, G is the bipartite graph corresponding to some incidence geometry Γ over I such that $|\mathrm{I}|=2$.*

The incidence geometry Γ whose the corresponding multipartite graph is a cycle graph with $2n$ vertices is known as a geometric configuration, called n-gon.

In particular, if G is a k-partite graph such that $\omega(G)=k$ and $\mathrm{Aut}(G)$ acts transitively on the cliques of the same size, then the graph structure of G can be realized by $\mathrm{Aut}(G)$ and some of its subgroups. Let C be a maximum clique of G. Then the size of C is k. For each $v_i \in C$, denote

$$H_i := \mathrm{Stab}_{\mathrm{Aut}(G)}(v_i) = \{ g \in \mathrm{Aut}(G) \mid g \cdot v_i = v_i \}.$$

Then the set of all left (or right) cosets $\{ gH_i \mid g \in \mathrm{Aut}(G), i \in \{1,2,\dots,k\} \}$ and the non-empty intersection relation on this set correspond to the vertex set $V(G)$ and the edge set $E(G)$, respectively. Therefore all properties of G may be represented in the language of abstract group theory. The following theorem can be easily proved by using the orbit-stabilizer theorem in group theory.

Theorem 11 *Let* G *be a* k*-partite graph such that* $\mathrm{Aut}(G)$ *acts transitively on the cliques of the same size. Then* $deg(v_1)=deg(v_2)$ *if* v_1 and v_2 are vertices in the same partite, where $deg(v)$ is the degree of the vertex v in G.

On the other hand, the geometry property of incidence systems can be examined by considering the size of all maximal cliques of the corresponding multipartite graphs. There are many algorithms proposed in finding maximal cliques in a graph; one of which is proposed by Apriani et al. (Apriani et al., 2013). In order to be a geometry, an incidence system over I must have the size of each maximal clique of its corresponding multipartite graph equal to $|\mathrm{I}|$.

5 Conclusion

As discussed above, by realizing incidence systems as multipartite graphs, one can easily determine whether they are geometry by using graph theory approach. And in the other way round, by realizing multipartite graphs as incidence systems, one can understand their properties via the theory for incidence systems. This will benefit understanding the correlation between different areas of Mathematics.

References

Apriani, Wiwin. et al. (2013). "Finding Maximum Clique in a Graph." **International Journal of Science and Research** 4: 2055-2057.

Brown, Keneth S. (1988). **Buildings.** Springer, New York.

Buekenhout, Francis. (1979). "Diagrams for Geometries and Groups." **Journal of Combinatorial Theory, Series A** 27: 121-151.

Coxeter, Harold S. M. (1950). "Self-dual Configurations and Regular Graphs." **Bulletin of the American Mathematical Society** 56: 413-455.

Humphreys, James E. (1990). **Reflection Groups and Coxeter Groups.** Cambridge University Press, Cambridge.

Lane, Saunders M. (1997). **Categories for the Working Mathematician.** Springer, New York.

Moore, Eliakim H. (1896). "Tactical Memoranda I-III." **American Journal of Mathematics** 18: 264-290

Noppakaew, Passawan. (2013). **Parabolic Projection and Generalized Cox Configurations.** PhD Thesis. (under the joint supervision of David M. J. Calderbank and Alastair D. King.), University of Bath.

Veridian E-Journal, Science and Technology Silpakorn University
Volume 3 Number 5 September– October2016 (ISSN 2408 – 1248)

Pasini, Antonio. (1994). **Diagram Geometries.** Oxford University Press, Oxford.

Reye, Theodor. (1882). "Das Problem der Configuration." **Acta Mathematica** 1: 92-96

Ronan, Mark A. (1989). **Lectures on Buildings.** Academic Press, Boston.

Scharlau, Rudolf. (1995). **Buildings, Handbook of Incidence Geometry.** F. Buekenhout ed., Elsevier Science, Amsterdam. Chapter 11.

Shult, Ernest. E, (2010). **Points and Lines: Characterizing the Classical Geometries.** Universitext, Springer, Berlin.

Weiss, Robert M. (2004). **The Structure of Sherical Buildings.** Princeton University Press, Princeton.

Tits, Jacques. (1974). **Building of Spherical Type and Finite BN-Pairs.** Springer, Berlin.

Wallis, Walter D. (2000). **A Beginner's Guide to Graph Theory.** Birkhäuser, Boston.