Subalgebras of algebras determined by polymorphisms of unary central relations or by polymorphisms of non-trivial equivalence relations[*]

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Abstract

The set of all clones on an arbitrary set forms a complete lattice under the set inclusion. Its co-atoms are called maximal clones. All maximal clones on a finite set are classified by I. G. Rosenberg into 6 classes by 6 types of relations. Two of them are non-trivial equivalence relations and central relations. Consider a finite algebra that all fundamental operations are polymorphisms of non-trivial equivalence relations. We describe all subalgebras of such an algebra. Moreover, all subalgebras of a finite algebra that all fundamental operations are polymorphisms of unary central relations are investigated.

Key words: algebra, clone

Introduction and Preliminaries

For each n-ary operation f on a set A and each h-ary relation ρ on A, we say that f preserves ρ (or f is a polymorphism of ρ) if $\left(f(a_1^1,\ldots,a_n^1),\ldots,f(a_1^h,\ldots,a_n^h)\right)\in\rho$ whenever $(a_i^1,\ldots,a_i^h)\in\rho$ for all $1\leq i\leq n$. For a set Q of relations on a set A, the set of all polymorphisms of all relations in Q is denoted by $Pol_A Q$. It is clear that $Pol_A Q=\bigcap_{\rho\in Q} Pol\{\rho\}$.

The set of all finitary operations on a set A is denoted by O_A and the set of all projections on a set A is denoted by J_A. A clone on a set A is a subset of O_A which is closed under composition and contains all projections. Then an arbitrary intersection of clones is a clone. For any subset F of O_A, the clone on A generated by F, denoted by $<F>$, is an intersection of all clones that contain F. The set of all clones on a set A is denoted by

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$Clone(A)$. It forms a complete lattice under the set inclusion where the greatest element is O_A and the least element is J_A. The co-atoms of the lattice $(Clone(A);\subseteq)$ are called maximal clones. A subset C of O_A is a clone on A if and only if $C = Pol_A Q$ for some set Q of relations on A; see Pöschel (1980). I. G. Rosenberg (1970) classified all maximal clones on a finite set into 6 classes of 6 types of relations. Two of them are non-trivial equivalence relation and central relation.

A binary relation ρ on a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive. The diagonal relation $\Delta_A := \{(a,a) \mid a \in A\}$ and the universal relation $\nabla_A := A \times A$ are equivalence relations and said to be trivial. An h-ary relation ρ on a set A is said to be totally reflexive if it contains a set $\{(a_1,\ldots,a_h) \in A^h \mid |\{a_1,\ldots,a_h\}| < h\}$. An h-ary relation ρ is said to be totally symmetric if for any permutation s on $\{1,\ldots,h\}$, we have $(a_1,\ldots,a_h) \in \rho$ if and only if $(a_{s(1)},\ldots,a_{s(h)}) \in \rho$. A center $C(\rho)$ is a set of all $a \in A$ such that $(a,a_2,\ldots,a_h) \in \rho$ for all $a_2,\ldots,a_h \in A$. We say that a relation ρ on A is a central relation if it is totally reflexive, totally symmetric and $\phi \neq C(\rho) \subset A$. In the case that ρ is a unary relation on A, ρ is a central relation if $\phi \neq \rho \subset A$; see Lau (2006).

An algebra $\underline{A} = (A; F^{\underline{A}})$ is a pair consisting of a nonempty set A and a set $F^{\underline{A}}$ of operations on A. The set A is called the universe of $\underline{A}$ and each element in $F^{\underline{A}}$ is called a fundamental operation of $\underline{A}$. The clone on A generated by $F^{\underline{A}}$ is called a clone of term operations denoted by $T(\underline{A})$, and its elements are called term operations of $\underline{A}$. A nonempty subset B of A is a subuniverse of $\underline{A}$ if it is closed under all fundamental operations of $\underline{A}$. If B is a subuniverse of $\underline{A}$, then $\underline{B} = (B; F^{\underline{B}})$, where $F^{\underline{B}}$ is the set of restrictions of all operations in $F^{\underline{A}}$ to B, is called a subalgebra of $\underline{A}$. The set of all subuniverses of $\underline{A}$ together with empty set is denoted by $Sub(\underline{A})$ and it forms a complete lattice under set inclusion; see Denecke and Wismath (2002).

The complete lattice $(Sub(\underline{A});\subseteq)$ is a useful tool to check a categorical equivalence of clones; see Koppitz and Supaporn (2013). A categorical equivalence of clones was introduced by Denecke and Lüders (1995). It is developed from a categorical equivalence of varieties by consider the clone of term operations of an algebra that generates a variety. Two varieties are categorical equivalence if there is an equivalence functor between them. Many algebraic properties are preserved by an equivalence functor; see Davey and Werner (1983).

Main Results

In this section, we describe all subuniverses of a finite algebra $\underline{A}$ where $T(\underline{A}) = Pol_A Q$ such that Q is a set of unary central relations on A and a set of non-trivial equivalence relations on A, respectively.

The first theorem was appeared in Koppitz and Supaporn (2013) but with a short proof. The full proof of this theorem is allowed by the journal to be republished and it is shown as follows.

Theorem 2.1 Let Q be a set of unary central relations on a finite set A. If $\underline{A} = (A; F^{\underline{A}})$ is an algebra where $T(\underline{A}) = Pol_A Q$, then $Sub(\underline{A}) = \{A\} \cup \{\bigcap \beta \mid \beta \subseteq Q\}$.

Proof Let $S \in Sub(\underline{A})$ where $\phi \neq S \subset A$. Then there is $a \in A \setminus S$. Assume that S is not a subset of B for all $B \in Q$. Let $f : A^{|S|} \to A$ be defined by

$$f(x_1, \dots x_{|S|}) = \begin{cases} a & ; \{x_1, \dots x_{|S|}\} = S \\ x_1 & ; \{x_1, \dots x_{|S|}\} \neq S. \end{cases}$$

For each $B \in Q$ and each $x_1, \dots, x_{|S|} \in B$, we have $\{x_1, \dots, x_{|S|}\} \subseteq B$. Then $\{x_1, \dots, x_{|S|}\} \neq S$, thus $f(x_1, \dots, x_{|S|}) = x_1 \in B$. These imply that f preserves B for all $B \in Q$. Hence $f \in Pol_A Q$. By the definition of f, we have that S is not closed under the term operation f of $\underline{A}$ because $a \in f(S^{|S|})$ but $a \notin S$. So S is not a subuniverse of $\underline{A}$. This is a contradiction. Then there is $B \in Q$ such that $S \subseteq B$. Thus $\{B \mid B \in Q$ and $S \subseteq B\} \neq \phi$ and $S \subseteq \bigcap \{B \mid B \in Q$ and $S \subseteq B\}$.

Assume that $S \subset \bigcap \{B \mid B \in Q$ and $S \subseteq B\}$. Then there is $b \in \bigcap \{B \mid B \in Q$ and $S \subseteq B\}$ but $b \notin S$. Let $g : A^{|S|} \to A$ be defined by

$$g(x_1, \dots, x_{|S|}) = \begin{cases} b & , \{x_1, \dots, x_{|S|}\} = S \\ x_1 & , \{x_1, \dots, x_{|S|}\} \neq S. \end{cases}$$

For each $\overline{B} \in Q$ and each $x_1, \dots, x_{|S|} \in \overline{B}$, we have $\{x_1, \dots, x_{|S|}\} \subseteq \overline{B}$. If $\{x_1, \dots, x_{|S|}\} \neq S$, then $g(x_1, \dots, x_{|S|}) = x_1 \in \overline{B}$. If $\{x_1, \dots, x_{|S|}\} = S$, then $S \subseteq \overline{B}$ and $g(x_1, \dots, x_{|S|}) = b$, thus $g(x_1, \dots, x_{|S|}) \in \bigcap \{B \mid B \in Q$ and $S \subseteq B\} \subseteq \overline{B}$. These imply that g preserves $\overline{B}$ for all $\overline{B} \in Q$. Hence $g \in Pol_A Q$. By the definition of g, we have that S is not closed under the term operation g of $\underline{A}$ because $b \in g(S^{|S|})$ but $b \notin S$. So S is not a subuniverse of $\underline{A}$. This is a contradiction. Then $S = \bigcap \{B \mid B \in Q$ and $S \subseteq B\}$. Thus $Sub(\underline{A}) \subseteq \{A\} \cup \{\bigcap \beta \mid \beta \subseteq Q\}$.

For each $B \in Q$, we have f preserves B for all $f \in F^{\underline{A}}$. Then B is closed under all fundamental operations of $\underline{A}$, thus B is a subuniverse of $\underline{A}$. These imply that $Q \subseteq Sub(\underline{A})$. Then $\{A\} \cup \{\bigcap \beta \mid \beta \subseteq Q\} \subseteq Sub(\underline{A})$.

Therefore, $Sub(\underline{A}) = \{A\} \cup \{\bigcap \beta \mid \beta \subseteq Q\}$.

Example 2.2 Let $A=\{a,b,c,d,e,f\}$ and $Q=\{\{a,b,c\},\{c,d,e\},\{a,e\}\}$. By the definition of $Pol_A Q$, an n-ary operation f on A is an element of $Pol_A Q$ if and only if f preserves all elements in Q ; i.e. $f(\{a,b,c\}^n)\subseteq\{a,b,c\}$, $f(\{c,d,e\}^n)\subseteq\{c,d,e\}$ and $f(\{a,e\}^n)\subseteq\{a,e\}$. By Theorem 2.1, we get $Sub(\underline{A})=\{\phi, A,\{a,b,c\},\{c,d,e\},\{a,e\},\{c\},\{a\},\{e\}\}$.

The last theorem, we describe all subuniverses of a finite algebra $\underline{A}$ where $T(\underline{A})=Pol_A Q$ such that Q is a set of non-trivial equivalence relations on A. The result surprised us that the set $Sub(\underline{A})$ does not depend on a number of non-trivial equivalence relations.

Theorem 2.3 Let Q be a set of non-trivial equivalence relations on a finite set A. If $\underline{A}=(A;F^{\underline{A}})$ is an algebra where $T(\underline{A})=Pol_A Q$, then $Sub(\underline{A})=\{\phi, A\}$.

Proof Clearly, $\{\phi, A\}\subseteq Sub(\underline{A})$. Let $S\in Sub(\underline{A})$. Assume that $S\neq\phi$ and $S\neq A$. Then $\phi\neq S\subset A$. Thus there is $a\in A\setminus S$. Let $f:A\rightarrow A$ be defined by $f(x)=a$ for all $x\in A$. For each $\theta\in Q$ and each $(x,y)\in\theta$, we have $(f(x),f(y))=(a,a)\in\theta$. These imply that f preserves θ for all $\theta\in Q$. Then $f\in Pol_A Q$. By the definition of f, we have that S is not closed under the term operation f of $\underline{A}$ because $a\in f(S)$ but $a\notin S$. So S is not a subuniverse of $\underline{A}$. This is a contradiction. Then $S=\phi$ or $S=A$. Therefore, $Sub(\underline{A})=\{\phi, A\}$.

References

[1] Davey, B. A. and H. Werner. (1983). “Dualities and Equivalences for Varieties of Algebras, Contributions to lattice theory.” **Colloq. Math. Soc.,** 8: 1-15.

[2] Denecke, K. and O. Lüders. (1995). “Category equivalences of clones.” **Algebra Universalis,** 34: 608-618.

[3] Denecke, K. and S. L. Wismath. (2002). **Universal Algebra and Applications in Theoretical Computer Science**. United States of America: Chapman & Hall / CRC.

[4] Koppitz, J. and W. Supaporn. (2013). “Category Equivalences of clone of operations preserving unary operations.” **Comptes rendus de l’Académie bulgare des sciences,** 66(2): 177-184.

[5] Lau, D. (2006). **Function algebras on finite sets: Basic course on many-valued logic and clone theory.** Berlin: Springer.

[6] Pöschel, R. (1980). **A general Galois theory for operations and relations and concrete characterization of related algebraic structures.** Berlin: Report 1980 1 Akademie der Wissenschaften der DDR Institut für Mathematik.

[7] Rosenberg, I. G. (1970). “Über die funktionale Vollständigkeit in den mehrwertigen Logiken.” **Rozpravy Československé Akad. vĕd, Ser. Math. Nat. Sci.,** 80: 3-93.