Invited Article

A Formal Derivation of Permutation Symbol Identities Frequently Used in Physics

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Abstract

We rearrange a formal derivation of some permutation symbol identities which are useful in physics.

Keywords: Permutation symbol, Levi-Civita tensor, Reduced permutation symbol, Alternating multilinear form of vectors, Area element

Introduction

The relations between permutation symbols and Kronecker deltas are core identities in calculations involving permutation symbols in physics. There are various mathematical physics textbooks that prove these identities by cases [1-5]. Here, we rearrange a formal derivation on some permutation symbol identities, which are well known and useful in physics, by means of stepwise equations, avoiding proof by cases as much as possible. There are textbooks that provide good accounts of permutation symbols and vector cross products. We use some of their arguments to rearrange our arguments here; namely, [6,7] for relations between permutation symbols, determinants, alternating multilinear form of vectors, and cross products; [2] for reduced permutation symbols; [8,9] for products and contractions of Levi-Civita tensors; and [10] for identities involving metric tensor and general cross products. Our presentation here is intended as a simple derivation for educational physics not as a derivation based on differential geometry. Readers interested in studying permutation symbols in the context of differential geometry may start with [9].

Determinant and permutation symbols

We start with various definition formulas for the determinant of a $N\times N$ real matrix $A=\left[A^i_j\right]$,

$$\begin{aligned}
&\det A=\left[j_1j_2..j_N\right]A^1_{j_1}A^2_{j_2}..A^N_{j_N}\\
&=\left[i_1i_2..i_N\right]\left[j_1j_2..j_N\right]A^{i_1}_{j_1}A^{i_2}_{j_2}..A^{i_N}_{j_N}\quad(\text{no sum } i_1,i_2,..,i_N)\\
&=\frac{1}{N!}\left[i_1i_2..i_N\right]\left[j_1j_2..j_N\right]A^{i_1}_{j_1}A^{i_2}_{j_2}..A^{i_N}_{j_N}\\
&=\left[j_1j_2..j_N\right]\left[i_1i_2..i_N\right]A^{i_1}_{j_1}A^{i_2}_{j_2}..A^{i_N}_{j_N}\quad(\text{no sum } j_1,j_2,..,j_N)\\
&=\left[i_1i_2..i_N\right]A^{i_1}_1A^{i_2}_2..A^{i_N}_N=\det A^{\mathrm{T}}
\end{aligned}\tag{1}$$

where $N=1,2,3,\ldots$; $i=1,2,..,N$; $j=1,2,..,N$ and a permutation symbol is defined by

$$\begin{aligned}
&\left[123..N\right]\equiv 1,\\
&\left[i_1i_2..i_N\right]=\begin{cases}0\ ;\ \text{repeated indices}\\ 1\ ;\ \text{even permutations of }\left[1,2,..,N\right]\\ -1\ ;\ \text{odd permutations of }\left[1,2,..,N\right]\end{cases}.
\end{aligned}\tag{2}$$

We use Einstein's summation convention except that there is a special mention. Consider real matrices $A=\left[a^i_j\right]$ and $B=\left[b^i_j\right]$ where the column and row elements are components of real vectors $\vec{a}(i)=a^i_j\vec{e}^{\,j}$ and $\vec{b}(j)=b^i_j\vec{e}_i$ respectively in N-dimensional Euclidean space $\mathbf{E}^N$. The $\{\vec{e}_i\}_{i=1}^N$ and $\{\vec{e}^{\,i}\}_{j=1}^N$ are general bases which are not necessarily orthonormal but $\vec{e}^{\,i}\cdot\vec{e}_j=\delta^i_j$. The matric tensor is $\mathbf{g}=\left[g_{ij}\right]=\left[\vec{e}_i\cdot\vec{e}_j\right]$, $\mathbf{g}^{-1}=\left[g^{ij}\right]=\left[\vec{e}^{\,i}\cdot\vec{e}^{\,j}\right]$ and $g\equiv\det\mathbf{g}=\mathrm{sgn}(g)|g|$. The $\varepsilon_{i_1..i_N}=\sqrt{|g|}\left[i_1..i_N\right]$ and $\varepsilon^{i_1..i_N}=\dfrac{\mathrm{sgn}(g)}{\sqrt{|g|}}\left[i_1..i_N\right]$ are Levi-Civita tensors [8-11]. According to Eq. (1), we have

$$
\begin{aligned}
&\operatorname{sgn}(g)\det A\det B \\
&=\det\left[\vec{a}(1),..,\vec{a}(N)\right]\det\left[\vec{b}(1),..,\vec{b}(N)\right] \\
&=a_{i_1}^{1}..a_{i_N}^{N}\varepsilon^{i_1..i_N}\varepsilon_{j_1..j_N}b_1^{j_1}..b_N^{j_N} \\
&=a_{i_1}^{1}..a_{i_N}^{N}\operatorname{sgn}(g)[i_1..i_N][k_1..k_N]\delta_{j_1}^{k_1}..\delta_{j_N}^{k_N}b_1^{j_1}..b_N^{j_N} \\
&=a_{i_1}^{1}..a_{i_N}^{N}\operatorname{sgn}(g)\begin{vmatrix}\delta_{j_1}^{i_1} & \cdots & \delta_{j_N}^{i_1}\\ \vdots & \cdots & \vdots \\ \delta_{j_1}^{i_N} & \cdots & \delta_{j_N}^{i_N}\end{vmatrix}b_1^{j_1}..b_N^{j_N} \\
&=\operatorname{sgn}(g)\begin{vmatrix}a_i^1b_1^i & \cdots & a_i^1b_N^i\\ \vdots & \cdots & \vdots \\ a_i^Nb_1^i & \cdots & a_i^Nb_N^i\end{vmatrix} \\
&=\operatorname{sgn}(g)\begin{vmatrix}\vec{a}(1)\cdot\vec{b}(1) & \cdots & \vec{a}(1)\cdot\vec{b}(N)\\ \vdots & \cdots & \vdots \\ \vec{a}(N)\cdot\vec{b}(1) & \cdots & \vec{a}(N)\cdot\vec{b}(N)\end{vmatrix}.
\end{aligned}
\tag{3}
$$

The $\det\left[\vec{b}(1),..,\vec{b}(N)\right]$ is an alternating multilinear form of N vectors. The RHS of Eq. (3) is Gram determinant [6]. Since a_j^i and b_j^i are arbitrary in Eq. (3), we conclude that

$$
\boxed{
\begin{aligned}
&\varepsilon^{i_1..i_N}\varepsilon_{j_1..j_N} \\
&=\operatorname{sgn}(g)[i_1..i_N][j_1..j_N] \\
&=\operatorname{sgn}(g)\begin{vmatrix}\delta_{j_1}^{i_1} & \cdots & \delta_{j_N}^{i_1}\\ \vdots & \cdots & \vdots \\ \delta_{j_1}^{i_N} & \cdots & \delta_{j_N}^{i_N}\end{vmatrix} \\
&=\operatorname{sgn}(g)[i_1..i_N][k_1..k_N]\delta_{j_1}^{k_1}..\delta_{j_N}^{k_N}.
\end{aligned}
}
\tag{4}
$$

We have simply derived the Eq. (4) formally from Eq. (3) not a proof by cases as shown in various textbooks [1-5]. The Eq. (4) is the core identity to prove various permutation symbol identities. We can also write

$$
\begin{aligned}
&\left[j_1..j_N\right] \\
&=\left[1..N\right]\left[j_1..j_N\right]=\begin{vmatrix} \delta^1_{j_1} & \cdots & \delta^1_{j_N} \\ \vdots & \cdots & \vdots \\ \delta^N_{j_1} & \cdots & \delta^N_{j_N} \end{vmatrix}, \\
&\left[i_1..i_N\right] \\
&=\left[i_1..i_N\right]\left[1..N\right]=\begin{vmatrix} \delta^{i_1}_1 & \cdots & \delta^{i_1}_1 \\ \vdots & \cdots & \vdots \\ \delta^{i_N}_N & \cdots & \delta^{i_N}_N \end{vmatrix}.
\end{aligned} \tag{5}
$$

Let $N=2,3,\ldots$; $i,j=1,2,..,N$; $\left[12..N\right]\equiv 1$; $p=1,2,..,P<N$ and reduced permutation symbol for all different indices in standard order is defined by

$$
\begin{aligned}
&\left[j_1<j_2<..<j_p<..<j_P\right] \\
&=\left[12..<j_1<..<..<j_p<..<..<j_P<..N\right]\equiv 1,
\end{aligned} \tag{6}
$$

for example, $\left[13\right]=\left[1\,2\,3\right]\equiv 1$. For arbitrary fixed indices $j_1..j_{i-1}j_{j+1}..j_N$, consider

$$
\begin{aligned}
&\left[j_1..j_i..j_N\right]\overset{(1)}{=}(-1)^{i-1}\left[j_i\ \boxed{j_1..j_{i-1}\ j_{i+1}..j_N}\right] \\
&\overset{(2)}{=}\begin{cases} 0\ ;\ \text{repeated indices} \\ (-1)^{i-1}(-1)^{n\left(\boxed{j_p\neq j_i}\right)}\left[j\ \boxed{12..j-1<j+1..N}\right] \end{cases} \\
&\overset{(3)}{=}\begin{cases} 0\ ;\ \text{repeated indices} \\ (-1)^{i-1}(-1)^{n\left(\boxed{j_p\neq j_i}\right)}(-1)^{j-1}\left[\boxed{12..j-1}\ j\ \boxed{j+1..N}\right] \end{cases} \\
&\overset{(4)}{=}\begin{cases} 0\ ;\ \text{repeated indices} \\ (-1)^{i+j_i}(-1)^{n\left(\boxed{j_p\neq j_i}\right)}(-1)^{n\left(\boxed{j_p\neq j_i}\right)}\left[\boxed{j_1..j_{i-1}\ j_{i+1}..j_N}\right] \end{cases} \\
&\overset{(5)}{=}\begin{cases} 0\ ;\ \exists l\neq i,\left(j_i=j_l\right) \\ (-1)^{i+j_i}\left[j_1..j_{i-1}\,j_{i+1}..j_N\right];\ j_i\neq j_{l\neq i} \end{cases}.
\end{aligned} \tag{7}
$$

The index $j_i=j$ is paired with i such that (i,j) may correspond to the row and column indices of a matrix element A^i_j. Step (1), we permute j_i to the forefront. Since it is ambiguous to describe a permutation between repeated indices, we separate repeated indices cases in Eq. (7). Step (2), for the non-zero case with all different indices, we permute the indices in $\boxed{..}$

pair by pair which creates each (-1), until we reach the standard order $12..j-1<j+1..N$ and obtain a factor $(-1)^{n\left(\boxed{j_p\neq j_i}\right)}$ where $n\left(\boxed{j_p\neq j_i}\right)$ is the even or odd number of permutations from $\boxed{j_1..j_{i-1}\,j_{i+1}..j_N}$ to the standard order $\boxed{12..j-1<j+1..N}$. Step (3), we permute j back into the standard order $12..j-1<j<j+1..N$ and obtain a factor $(-1)^{j-1}$. Step (4), we permute the indices back to the order $\boxed{j_1..j_{i-1}\,j_{i+1}..j_N}$ as if no j_i existing that create the same factor $(-1)^{n\left(\boxed{j_p\neq j_i}\right)}$. Step (5), we obtain the reduced permutation symbol $[j_1..j_{i-1}\,j_{i+1}..j_N]$ of which sign relates to $[12..N]\equiv 1$ by Eq. (6) and then we include the repeated indices cases and separate only the case $\exists l\neq i,(j_i=j_l)$. For $j_1..j_{i-1}\,j_{i+1}..j_N$ are fixed arbitrary indices, we then do a contraction,

$$\boxed{\begin{aligned}&\sum_{k_i=1}^{N}A^i_{k_i}[j_1..j_{i-1}\,k_i\,j_{i+1}..j_N]\\&=A^i_{j_i}(-1)^{i+j_i}[j_1..j_{i-1}\,j_{i+1}..j_N];\\&\forall l\neq i,(j_i\neq j_l)\text{ and no sum }j_i.\end{aligned}} \tag{8}$$

The proof of identities Eq. (7) was given in [2,6] and we have clarified in more detail and extended to be more general. We must be aware of the reduced permutation symbol in the RHS of Eq. (8), the set of indices $\{j_1,..,j_{i-1},j_{i+1},..,j_N\}\subseteq\{1,2,..,j_i-1,j_i+1,..,N\}$ and if $j_1..j_{i-1}\,j_{i+1}..j_N$ are all different indices, $[j_1..j_{i-1}\,j_{i+1}..j_N]$ is even or odd permutations with respect to the standard order $[12..j_i-1\,j_i+1..N]\equiv 1$. If $j_i=1$ or $j_i=N$, the reduced permutation symbol in standard order is $[j_i+1=2..N]\equiv 1$ or $[12..j_i-1=N-1]\equiv 1$, respectively. We then have similar identity to Eq. (5) that

$$\begin{aligned}&[j_1..j_{i-1}\,j_{i+1}..j_N]\\&=[12..j_i-1\,j_i+1..N][j_1..j_{i-1}\,j_{i+1}..j_N]\\&=\begin{vmatrix}\delta^1_{j_1}&\cdots&\delta^1_{j_{i-1}}&\delta^1_{j_{i+1}}&\cdots&\delta^1_{j_N}\\\vdots&&\vdots&\vdots&&\vdots\\\delta^{j_i-1}_{j_1}&\cdots&\delta^{j_i-1}_{j_{i-1}}&\delta^{j_i-1}_{j_{i+1}}&\cdots&\delta^{j_i-1}_{j_N}\\\delta^{j_i+1}_{j_1}&\cdots&\delta^{j_i+1}_{j_{i-1}}&\delta^{j_i+1}_{j_{i+1}}&\cdots&\delta^{j_i+1}_{j_N}\\\vdots&&\vdots&\vdots&&\vdots\\\delta^N_{j_1}&\cdots&\delta^N_{j_{i-1}}&\delta^N_{j_{i+1}}&\cdots&\delta^N_{j_N}\end{vmatrix}.\end{aligned} \tag{9}$$

Consider an example for the case $N=3$ of Eq. (8), by using Eq. (9), we obtain

$$\begin{aligned}
&[123]\equiv 1 \text{ and } [12]=[13]=[23]\equiv 1;\\
&\sum_{k_2=1}^{3} A_{k_2}^{i=2}[3k_2 1]=A_{j_2=2}^{2}[321]=A_2^2(-1)^{2+2}[31]\\
&=A_2^2[13][31]=A_2^2\begin{vmatrix}\delta_3^1 & \delta_1^1\\ \delta_3^3 & \delta_1^3\end{vmatrix}=-A_2^2=-A_2^2[13].
\end{aligned} \tag{10}$$

It may be interesting to see the trivial case $N=2$ of Eq. (8) with an example, by using Eq. (9), we obtain

$$\begin{aligned}
&[12]\equiv 1 \text{ and } [1]=[2]\equiv 1;\\
&\sum_{k_1=1}^{2} A_{k_1}^{i=1}[k_1 1]=A_{j_1=2}^{1}[21]=A_2^1(-1)^{1+2}[1]\\
&=-A_2^1[1][1]=-A_2^1\delta_1^1=-A_2^1.
\end{aligned} \tag{11}$$

Using Eq. (7) and Eq. (8), we can write the Laplace expansion of determinant. Here we write the Laplace expansion for determinant of matrix of metric tensor for later use,

$$\begin{aligned}
&g=g_{ij}C_{ij}(\mathbf{g});\ \text{sum all } j_k \text{ and no sum for all } i_k\\
&=g_{i_k j_k}[i_1..i_k..i_N][j_1..j_k..j_N]g_{i_1 j_1}..g_{i_{k-1}j_{k-1}}g_{i_{k+1}j_{k+1}}..g_{i_N j_N}\\
&=g_{i_k j_k}(-1)^{k+i_k+k+j_k}[i_1..i_{k-1}\,i_{k+1}..i_N][j_1..j_{k-1}\,j_{k+1}..j_N]g_{i_1 j_1}..g_{i_{k-1}j_{k-1}}g_{i_{k+1}j_{k+1}}..g_{i_N j_N}\\
&=g_{i_k j_k}(-1)^{i_k+j_k}[i_1..i_{k-1}\,i_{k+1}..i_N][j_1..j_{k-1}\,j_{k+1}..j_N]g_{i_1 j_1}..g_{i_{k-1}j_{k-1}}g_{i_{k+1}j_{k+1}}..g_{i_N j_N}\\
&=g_{ij}(-1)^{i+j}G_{ij};\ \text{sum } j \text{ and no sum } i.
\end{aligned} \tag{12}$$

where $C_{ij}(\mathbf{g})$ and G_{ij} are cofactor and minor of the element g_{ij} of matrix $\mathbf{g}$, respectively. The index $i_k=i$ is paired with $j_k=j$ in $g_{i_k j_k}=g_{ij}$. Ones may use Laplace expansion like Eq. (12) to prove Eq. (7) and Eq. (8) but we consider that it has a circulation, so we prove Eq. (7) and Eq. (8) first and use them to write Eq. (12) in general.

Products and contractions of permutation symbols

For $N=2,3,\ldots$; $P=0,1,2,..,N$; and fixed arbitrary indices $(i_{N-P+1},..,i_N)$ and $(j_{N-P+1},..,j_N)$ where $\{i_{N-P+1},..,i_N\}\subseteq\{1,2,..,N\}$ and $\{j_{N-P+1},..,j_N\}\subseteq\{1,2,..,N\}$, we have

$$\boxed{\begin{aligned}&\sum_{k_i=1}^{N}..\sum_{k_{N-P}=1}^{N}[k_1..k_{N-P}\,i_{N-P+1}..i_N][k_1..k_{N-P}\,j_{N-P+1}..j_N]\\&=(N-P)!\,[i_{N-P+1}..i_N][j_{N-P+1}..j_N].\end{aligned}} \tag{13}$$

This relation is given in [8,9] where the RHS is written in terms of series of products of Kronecker deltas. Now we will prove Eq. (13) where the RHS is written in terms of reduced permutation symbols defined similarly to Eq. (6) and Eq. (9). Consider the following arguments;

(1) The term in LHS is zero if there are repeated indices in the ordered set $(k_1,..,k_{N-P})$.

(2) There is a summation of $(N-P)!$ terms for an ordered set of different indices $(k_1,..,k_{N-P})$.

(3) Every non-zero term in LHS with a fixed $(i_{N-P+1},..,i_N)$ and $(j_{N-P+1},..,j_N)$ has the same value because it is the product between 2 permutation symbols which have the same set $(k_1,..,k_{N-P})$ with all different k_m ; $m=1..N-P$. It follows that

$$\begin{aligned}&[k_1..k_{N-P}\,i_{N-P+1}..i_N][k_1..k_{N-P}\,j_{N-P+1}..j_N]\\&=[l_1<..<l_{N-P}\,i_{N-P+1}..i_N][l_1<..<l_{N-P}\,j_{N-P+1}..j_N]\\&=[i_{N-P+1}..i_N][j_{N-P+1}..j_N]\\&=\begin{vmatrix}\delta^{i_{N-P+1}}_{j_{N-P+1}} & \cdots & \delta^{i_{N-P+1}}_{j_N}\\ \vdots & \cdots & \vdots\\ \delta^{i_N}_{j_{N-P+1}} & \cdots & \delta^{i_N}_{j_N}\end{vmatrix};\text{no sum } k_m \text{ and } l_m;m=1..N-P.\end{aligned} \tag{14}$$

The familiar identities for the $N=3$ case follow from Eq. (13) immediately,

$$\begin{aligned}&[k_1i_2i_3][k_1j_2j_3]\\&=(3-2)![i_2i_3][j_2j_3]=\begin{vmatrix}\delta^{i_2}_{j_2} & \delta^{i_2}_{j_3}\\ \delta^{i_3}_{j_2} & \delta^{i_3}_{j_3}\end{vmatrix}.\end{aligned} \tag{15}$$

$$\begin{aligned}&[k_1k_2i_3][k_1k_2j_3]\\&=(3-1)![i_3][j_3]=2\,\delta^{i_3}_{j_3}.\end{aligned} \tag{16}$$

$$[k_1k_2k_3][k_1k_2k_3]=(3-0)!. \tag{17}$$

It may be interesting to see the trivial case $N=2$ with examples,

$$\begin{aligned}
&[k_1 i_2][k_1 j_2] \\
&=(2-1)![i_2][j_2]=\delta^{i_2}_{j_2}\,; \\
&[12]\equiv 1 \text{ and } [1]=[2]\equiv 1, \\
&[k_1 1][k_1 1]=0+[21][21]=[1][1]=\delta^1_1=1, \\
&[k_1 1][k_1 2]=[1][2]=\delta^1_2=0.
\end{aligned} \tag{18}$$

Products and contractions of Levi-Civita tensors

Consider the vectors $\vec{a}=a^i\vec{e}_i$, $\vec{b}=b^i\vec{e}_i$, $\vec{c}=c^i\vec{e}_i$, and $\vec{d}=d^i\vec{e}_i$ in $\mathbf{E}^3$. The general cross product is defined by [1,10,11]

$$\begin{aligned}
&\vec{a}\times\vec{b} \\
&=a^{i_2}b^{i_3}\vec{e}_{i_2}\times\vec{e}_{i_3}=a^{i_2}b^{i_3}\vec{e}^{\,i_1}\varepsilon_{i_1 i_2 i_3}o \\
&=a^{i_2}b^{i_3}\vec{e}^{\,i_1}[i_1 i_2 i_3]\sqrt{|g|}\,o
\end{aligned} \tag{19}$$

where o is orientation of the basis $\{\vec{e}_i\}_{i=1}^3$. Some familiar vector identities such as BAC-CAB rule [1-3] follow immediately from Eq. (15) – Eq. (17). You may wonder how to write the product Eq. (13) in terms of Levi-Civita tensors and reduced Levi-Civita tensors with induced-metric determinant as a minor of metric (12). We consider only simple cases $P=N-1$. For $N=3$ and $P=3-1=2$, consider a 2D infinitesimal area $\mathrm{d}S=\sqrt{(\mathrm{d}\vec{y}\times\mathrm{d}\vec{z})\cdot(\mathrm{d}\vec{y}\times\mathrm{d}\vec{z})}$ [1,10] constructed by noncollinear displacements $\mathrm{d}\vec{y}=\mathrm{d}y^i\vec{e}_i$ and $\mathrm{d}\vec{z}=\mathrm{d}z^i\vec{e}_i$ in $\mathbf{E}^3$, we can use Eq. (13) and Eq. (12) to obtain

$$
\begin{aligned}
(\mathrm{d}S)^2 &= \mathrm{d}y^{i_2}\,\mathrm{d}z^{i_3}\,\mathrm{d}y^{j_2}\,\mathrm{d}z^{j_3}\left(\vec{e}_{i_2}\times\vec{e}_{i_3}\right)\cdot\left(\vec{e}_{j_2}\times\vec{e}_{j_3}\right)\\
&= \mathrm{d}y^{i_2}\,\mathrm{d}z^{i_3}\,\mathrm{d}y^{j_2}\,\mathrm{d}z^{j_3}\,\varepsilon_{i_1i_2i_3}\vec{e}^{\,i_1}\cdot\vec{e}^{\,j_1}\varepsilon_{j_1j_2j_3}\\
&= \mathrm{d}y^{i_2}\,\mathrm{d}z^{i_3}\,\mathrm{d}y^{j_2}\,\mathrm{d}z^{j_3}\,g_{i_2k_2}g_{i_3k_3}\varepsilon^{j_1k_2k_3}\varepsilon_{j_1j_2j_3}\\
&= \mathrm{d}y^{i_2}\,\mathrm{d}z^{i_3}\,\mathrm{d}y^{j_2}\,\mathrm{d}z^{j_3}\,g_{i_2k_2}g_{i_3k_3}\left[k_2k_3\right]\mathrm{sgn}(g)\left[j_2j_3\right];k_1=j_1\\
&= \mathrm{d}y^{i_2}\,\mathrm{d}z^{i_3}\,\mathrm{d}y^{j_2}\,\mathrm{d}z^{j_3}G_{i_1j_1}\left[i_2i_3\right]\left[j_2j_3\right]\mathrm{sgn}(g)\\
&= \mathrm{d}y^{i_2}\,\mathrm{d}z^{i_3}\,\mathrm{d}y^{j_2}\,\mathrm{d}z^{j_3}\sqrt{\left|G_{i_1j_1}\right|}\left[i_2i_3\right]\sqrt{\left|G_{i_1j_1}\right|}\left[j_2j_3\right]\mathrm{sgn}\left(G_{i_1j_1}\right)\mathrm{sgn}(g)\\
&= \mathrm{d}y^{i_2}\,\mathrm{d}z^{i_3}\,\mathrm{d}y^{j_2}\,\mathrm{d}z^{j_3}\,\bar{\varepsilon}_{i_2i_3}\bar{\varepsilon}_{j_2j_3}\,\mathrm{sgn}\left(G_{i_1j_1}\right)\mathrm{sgn}(g)
\end{aligned}
\tag{20}
$$

where $\mathrm{sgn}\left(G_{i_1j_1}\right)$ is the sign of minor of the element $g_{i_1j_1}$. For $P=N-1$, according to Eq. (3) and Eq. (12), we use similar arguments in Eq. (20) for a product of alternating multilinear forms with contraction, $\det\left[\vec{e}^{\,j_1},\vec{a}(2),..,\vec{a}(N)\right]\det\left[\vec{e}_{j_1},\vec{b}(2),..,\vec{b}(N)\right]$ where $\vec{e}^{\,j_1}=g^{j_1i_1}\vec{e}_{i_1}$, we conclude that

$$
\begin{aligned}
&\varepsilon_{i_1i_2..i_N}g^{i_1j_1}\varepsilon_{j_1j_2..j_N}=\bar{\varepsilon}_{i_2..i_N}\bar{\varepsilon}_{j_2..j_N}\,\mathrm{sgn}\left(G_{i_1j_1}\right)\mathrm{sgn}(g)\\
&\equiv\sqrt{\left|G_{i_1j_1}\right|}\left[i_2..i_N\right]\sqrt{\left|G_{i_1j_1}\right|}\left[j_2..j_N\right]\mathrm{sgn}\left(G_{i_1j_1}\right)\mathrm{sgn}(g),\\
&\varepsilon^{k_1i_2..i_N}\varepsilon_{k_1j_2..j_N}=\mathrm{sgn}(g)\bar{\varepsilon}^{i_2..i_N}\bar{\varepsilon}_{j_2..j_N}\\
&\equiv\mathrm{sgn}(g)\frac{\mathrm{sgn}\left(G_{i_1j_1}\right)}{\sqrt{\left|G_{i_1j_1}\right|}}\left[i_2..i_N\right]\sqrt{\left|G_{i_1j_1}\right|}\left[i_2..j_N\right].
\end{aligned}
\tag{21}
$$

Conclusion

We rearrange a formal derivation of some permutation symbol identities, which are well known and useful in physics, by stepwise equations while avoiding proof by cases as much as possible. We derive the relations between permutation symbols and Kronecker deltas formally and derive some results of products and contractions of permutation symbols in terms of reduced permutation symbols instead of writing the results in terms of series of Kronecker deltas as shown in various textbooks. We use expression for an area element to illustrate that how products and contractions of the Levi-Civita tensors lead to reduced Levi-Civita tensor with the determinant of induced metric. Our work is intended to present a formal derivation that may be useful for physics courses, rather than to present a new idea of proofs.

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