Article

Classical Descriptions of Spin-2 Fields

Daniel Abramson
9/72 Soi Kasemsan 3, Pathumwan, Bangkok 10330, Thailand
*Corresponding author E-mail: khunchang@gmail.com

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Abstract

In quantum theory, the photon and graviton are colloquially referred to as having spin $\pm\hbar$ and $\pm 2\hbar$, respectively.[1] Spin is associated with properties of the electromagnetic and gravitational fields related to their angular momentum or the way they transform under rotations. We review some of these properties and establish connections among them that explain the terms "spin 1" and "spin 2". In particular, we show how helicity (a measure of how waves transform under rotations) is related to the eigentensors of Dirac's infinitesimal rotation operator. We also examine systems emitting gravitational radiation to analyze the angular distribution of energy and angular momentum in circularly polarized plane waves and gain insight into the spin-2 nature of the gravitational field.

Keywords: Gravitational radiation, Angular momentum, Spin, Helicity

Introduction

Gravity is often referred to as a "spin-2" field when analyzing gravitational waves in the "linearized" or weak field approximation in general relativity. Such discussions seldom convey insight into the connection between the vector or tensor nature of a field and its "spin-1" or "spin-2" characteristics. For example, Dirac [2] sketches how the infinitesimal rotation operator acts on the metric tensor, infers its eigenvalues, and concludes that

> *"the components (of the metric) that contribute to the energy thus correspond to spin 2."*

Weinberg [3] remarks that

> *"the electromagnetic wave can be decomposed into parts with helicity ±1 and 0. However, the physically significant helicities are ±1, not 0, just as for gravitational*

[1] More precisely, photons and gravitons have **helicity** $\pm\hbar$ and $\pm 2\hbar$, respectively. To quote Sidney Coleman: *"Spin is a concept that applies only to particles with mass, because only for a particle of non-zero mass can we Lorentz transform to its rest frame and there compute its angular momentum, which is its spin. For a massless particle, there is no rest frame, so we can't talk about the spin. We can however talk about its helicity, the component of angular momentum along the direction of motion."* (Chen (2019), Chap. 19, p. 400). [1]

waves they are ±2, not ±1 or 0. This is what we mean when we say, speaking classically, that electromagnetism and gravitation are carried by waves of spin 1 and spin 2, respectively."

Ohanian and Ruffini [4] state that

"circularly polarized waves carry angular momentum . . . proportional to the amount of energy carried by the wave: angular momentum = 2/ω × energy . . . (This) result cannot be obtained directly from [the plane wave solution]; this solution ignores the boundaries of the wave in the transverse direction, and it is precisely the boundary region that is crucial for the transport of angular momentum. The quantum mechanical interpretation . . . is that the quanta of the gravitational field, or gravitons, have spin 2ħ."

In this paper we explore these observations and develop insight into the description of classical fields as "spin 1" and "spin 2". In particular, we show that:

1. The metric of a gravitational wave is a linear combination of eigentensors of Dirac's infinitesimal rotation operator $\widetilde{\boldsymbol{R}}$. In this decomposition, basis eigentensors with eigenvalue $\pm\lambda$ $(\lambda = 0, 1, 2)$ have coefficients with helicity $\mp\lambda$. Notably, only the eigentensors with $\lambda = \pm 2$ represent physical waves in the spacetime geometry.

2. A monochromatic, circularly polarized gravitational wave packet carries angular momentum equal to $\pm 2/\omega$ times its energy, compared to $\pm 1/\omega$ for electromagnetic waves. This result is used to study the flux of angular momentum from a radiating system and obtain insight into the distribution of angular momentum in the radiation field.

We conclude with some open questions arising from this study.

Results and discussion

Notation

- "Geometrized units" are used throughout this paper in the analysis of both gravitational and electromagnetic phenomena. Thus, we take $c = G = 1$ as explained in the next two points.
- The speed of light c is dimensionless and equal to 1. The standard unit of time t (seconds) is replaced by x^0 (meters), defined by $x^0 = ct$, where $c = 3 \times 10^8$.
- The gravitational constant G is dimensionless and equal to 1. In SI units $G = 6.67 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$. Since time $x^0 = ct$ is measured in meters, we convert seconds to meters using $1\,\mathrm{second} = c\,\mathrm{meters}$. Hence, $G = 6.67 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}} \times (1/c^2)\,\mathrm{m^{-2} s^2}$. For G to be dimensionless, we convert mass m in kg to units of mass $\bar{m}$ measured in meters. Therefore,

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}} \times \frac{1\,\mathrm{s}^2}{c^2\,\mathrm{m}^2} \times \frac{m\,\mathrm{kg}}{\bar{m}\,\mathrm{meters}}.$$

This equals unity if $\bar{m} = m \times 6.67 \times 10^{-11} \times c^{-2}$. Therefore, the conversion formula is:

$$\bar{m}\,\mathrm{meters} = (G/c^2)_{\mathrm{SI}} \times m\,\mathrm{kg} = 7.43 \times 10^{-28} \times m\,\mathrm{kg}.$$

- Greek indices run over 0, 1, 2, 3. Latin indices run over 1, 2, 3.
- The Einstein summation convention is used, so repeated indices are summed over. Thus:

$$A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3\,, \qquad Q_{kk} = Q_{11} + Q_{22} + Q_{33}$$

- Partial derivatives may be denoted by a comma; e.g., $\phi_{,\mu} \equiv \partial\phi/\partial x^\mu$.
- The metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is used.

A. Review of gravitational waves

We review the basic theory of gravitational waves in a weak field ("linearized gravity") and properties of harmonic coordinates.

§1. The weak field approximation

We assume the curvature of spacetime is small, which means the $g_{\mu\nu}$ are approximately constant and $|g_{\mu\nu,\rho}| \ll 1$. Then Einstein's vacuum equation $R_{\rho\sigma} = 0$ becomes[2]

$$R_{\rho\sigma} = g^{\mu\nu}(g_{\rho\sigma,\mu\nu} + g_{\mu\nu,\rho\sigma} - g_{\mu\rho,\nu\sigma} - g_{\mu\sigma,\nu\rho}) = 0\,, \tag{1}$$

or using the d'Alembertian operator, $\Box\phi \equiv g^{\mu\nu}\phi_{,\mu\nu}$:

$$R_{\rho\sigma} = \Box g_{\rho\sigma} + g^{\mu\nu}(g_{\mu\nu,\rho\sigma} - g_{\mu\rho,\nu\sigma} - g_{\mu\sigma,\nu\rho}) = 0\,. \tag{2}$$

We work in harmonic coordinates which satisfy[3]

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0\,. \tag{3}$$

The Christoffel symbol is defined in terms of the metric:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2} g^{\alpha\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})\,. \tag{4}$$

Substituting this into (3) and multiplying both sides by $g_{\rho\lambda}$, we obtain a condition for harmonic coordinates in terms of the metric:

$$g^{\mu\nu}\left(g_{\mu\rho,\nu} - \tfrac{1}{2} g_{\mu\nu,\rho}\right) = 0\,. \tag{5}$$

Differentiating (5) with respect to x^σ and retaining terms up to first order in the derivatives of $g_{\mu\nu}$:

$$g^{\mu\nu}\left(g_{\mu\rho,\nu\sigma} - \tfrac{1}{2} g_{\mu\nu,\rho\sigma}\right) = 0\,. \tag{6}$$

Interchanging the indices ρ and σ in (6) and adding the equations, we obtain

$$g^{\mu\nu}(g_{\mu\nu,\rho\sigma} - g_{\mu\rho,\nu\sigma} - g_{\mu\sigma,\nu\rho}) = 0\,. \tag{7}$$

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[2]Dirac (1975), Eq. (33.1).

[3]Dirac (1975), Chap. 22.

Eqs. (2) and (7) give the d'Alembert equation:

$$\Box g_{\rho\sigma} = g^{\mu\nu} g_{\rho\sigma,\mu\nu} = 0\,. \tag{8}$$

§2. Linearized gravity

If the curvature of spacetime is small, we may write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}|, |h_{\mu\nu,\rho}| \ll 1$. Note that

$$(\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\rho} + h_{\nu\rho}) = \delta^{\mu}_{\rho} + O(|h_{\mu\nu}|^2)\,,$$

which implies

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(|h_{\mu\nu}|^2)\,. \tag{9}$$

Then (1) becomes, neglecting terms of $O(|h_{\mu\nu}||h_{\mu\nu,\rho\sigma}|)$:[4]

$$\eta^{\mu\nu}(h_{\rho\sigma,\mu\nu} + h_{\mu\nu,\rho\sigma} - h_{\mu\rho,\nu\sigma} - h_{\mu\sigma,\nu\rho}) = 0 \tag{10}$$

and (8) becomes

$$\eta^{\mu\nu} h_{\rho\sigma,\mu\nu} = 0\,. \tag{11}$$

The weak field approximation is referred to as **linearized gravity**, since (10) and (11) are systems of linear differential equations for $h_{\mu\nu}$ or $g_{\mu\nu}$.

§3. Harmonic coordinates in a weak field

We can write the harmonic coordinates condition (5), keeping terms up to order $|h_{\mu\nu}|$:

$$h^{\nu}_{\rho,\nu} - \tfrac{1}{2} h_{,\rho} = 0 \tag{12}$$

where

$$h \equiv h^{\mu}_{\mu} = \eta^{\mu\nu} h_{\mu\nu}$$

is the trace of $h_{\mu\nu}$. Writing $h_{,\rho} = \eta^{\nu}_{\rho} h_{,\nu}$ and raising the index ρ, (12) becomes

$$h^{\nu\rho}{}_{,\nu} - \tfrac{1}{2}\eta^{\nu\rho} h_{,\nu} = 0\,. \tag{13}$$

Define

$$\psi_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h\,, \tag{14}$$

so (13) becomes

$$\psi^{\mu\nu}{}_{,\nu} = 0\,. \tag{15}$$

[4]We can reasonably assume the second derivatives $h_{\mu\nu,\rho\sigma}$ are bounded. We ignore pathological waves such as

$$f(t,x) = \epsilon (t-x)^2 \exp[-(t-x)^2] \sin[1/(t-x)]$$

that have the property that $|f|$, $|f_{,\alpha}| = O(\epsilon)$ but $|f_{,\alpha\beta}|$ is unbounded. Such waves are unphysical.

This is analogous to the Lorenz condition ${A^\nu}_{,\nu} = 0$ for the electromagnetic 4-potential $A^\nu = (\phi, \mathbf{A})$, where ϕ and $\mathbf{A}$ are the electric and magnetic vector potentials.

The trace of $\psi_{\mu\nu}$ has the property that

$$\psi \equiv \psi^\mu_\mu = \eta^{\mu\nu}\psi_{\mu\nu} = \eta^{\mu\nu}\left(h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h\right) = h - 2h = -h \tag{16}$$

so we can easily recover $h_{\mu\nu}$ from $\psi_{\mu\nu}$ via

$$h_{\mu\nu} = \psi_{\mu\nu} + \tfrac{1}{2}\eta_{\mu\nu}h = \psi_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\psi\,. \tag{17}$$

Because of (16), $\psi_{\mu\nu}$ is called the **trace-reversed metric perturbation**.

§4. The field equations in linearized gravity

Putting $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ into (4) and keeping terms up to first order in $|h_{\mu\nu}|$ and $|h_{\mu\nu,\lambda}|$, we have:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\eta^{\alpha\lambda}(h_{\lambda\mu,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda})\,. \tag{18}$$

Inserting this into the expression for the Ricci tensor[5] $R_{\mu\nu}$ and dropping the $\Gamma\Gamma$ terms, we obtain:

$$R_{\mu\nu} = \tfrac{1}{2}(\Box h_{\mu\nu} + h_{,\mu\nu} - h^\alpha_{\mu,\alpha\nu} - h^\alpha_{\nu,\alpha\mu})\,,$$

$$R = \eta^{\mu\nu}R_{\mu\nu} = \Box h - {h^{\alpha\beta}}_{,\alpha\beta}\,.$$

Therefore, the Einstein tensor takes the rather complicated-looking form:

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \tfrac{1}{2}(\Box h_{\mu\nu} + h_{,\mu\nu} - h^\alpha_{\mu,\alpha\nu} - h^\alpha_{\nu,\alpha\mu} - \eta_{\mu\nu}\Box h + \eta_{\mu\nu}{h^{\alpha\beta}}_{,\alpha\beta})\,. \tag{19}$$

Using the harmonic coordinates condition (12) this simplifies to

$$2G_{\mu\nu} = \Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = \Box\psi_{\mu\nu}\,.$$

Therefore, the Einstein field equation $G_{\mu\nu} = -8\pi T_{\mu\nu}$ in linearized gravity takes the form

$$\Box\psi_{\mu\nu} = -16\pi T_{\mu\nu}\,. \tag{20}$$

Note that the vacuum equations (8) and (11) can easily be recovered from (20):

$$0 = \Box\psi_{\mu\nu} = \Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = \Box h_{\mu\nu} + \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\Box\psi_{\alpha\beta} = \Box h_{\mu\nu}$$

§5. The wave equation and wave vector

By (11), $g_{\rho\sigma}$ satisfies a wave equation with velocity $v = 1$:

$$\eta^{\mu\nu}g_{\rho\sigma,\mu\nu} = g_{\rho\sigma,00} - \nabla^2 g_{\rho\sigma} = 0\,. \tag{21}$$

[5] Dirac (1975), Eq. (14.4).

The wave equation

$$\frac{1}{v^2}\frac{\partial^2 f}{\partial(x^0)^2} - \nabla^2 f = 0 \tag{22}$$

has a *plane wave* solution $f(x) = f(k_\sigma x^\sigma) = f(k_0 x^0 - \mathbf{k} \bullet \mathbf{x})$.

Note that we use $\eta^{\mu\nu}$ to raise and lower indices, so $k_0 = k^0$ and $k_m = -k^m$. By convention we assume $k_0 > 0$. This is a wave traveling in the $\mathbf{k} = (k^1, k^2, k^3)$ direction. Inserting this solution into (22), we obtain the relation

$$k_0^2 - v^2(k_1^2 + k_2^2 + k_3^2) = 0\,, \tag{23}$$

or $v = k_0/k$, where $k = |\mathbf{k}|$.

A **monochromatic plane wave** is a solution of the form

$$f(x) = A\sin(k_\sigma x^\sigma) + B\cos(k_\sigma x^\sigma)\ . \tag{24}$$

k_0 is the **angular frequency**, usually designated ω.

We call $\mathbf{k}$ and $k^\mu = (k^0, \mathbf{k})$ the **wave vector** and **wave 4-vector** respectively. The quantity

$$\xi = k_\sigma x^\sigma = k_0 x^0 - \mathbf{k} \bullet \mathbf{x}$$

is the **phase**, which marks where $\exp(ik_\sigma x^\sigma)$ falls on the unit circle. This is plainly coordinate invariant, hence a scalar. Since $k_\mu = \xi_{,\mu}$, plainly k_σ is a vector.

In the case of waves traveling with speed $v = 1$, (23) gives

$$k_0^2 - (k_1^2 + k_2^2 + k_3^2) = \eta_{\rho\sigma} k^\rho k^\sigma = k_\sigma k^\sigma = 0 \tag{25}$$

so k^σ is a null vector.

§6. The metric derivative $u_{\mu\nu}$

Consider a plane wave solution $g_{\mu\nu}$ to (21) with wave 4-vector k_σ. Then $g_{\mu\nu}$ is a function of the variable $\xi = k_\sigma x^\sigma$. We define $u_{\mu\nu} = dg_{\mu\nu}/d\xi$, so that

$$g_{\mu\nu,\rho} = \frac{dg_{\mu\nu}}{d\xi}\frac{\partial\xi}{\partial x^\rho} = u_{\mu\nu}k_\rho\ .$$

In general, $u_{\mu\nu}$ is not a tensor. If it were, then $g_{\mu\nu,\rho}$ would be a tensor by the quotient theorem (since k_ρ is a vector), which it is not.[6] Indeed, $g'_{\mu\nu}$ will not be a function of the variable $\xi' = \xi$ under an arbitrary coordinate transformation; hence, $u'_{\mu\nu}$ is not even defined.

[6] A partial derivative of the metric transforms:

$$g'_{\mu\nu,\sigma} = \frac{\partial x^\lambda}{\partial x'^\sigma}\frac{\partial}{\partial x^\lambda}\left(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}\right) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\lambda}{\partial x'^\sigma}g_{\alpha\beta,\lambda} + g_{\alpha\beta}\frac{\partial}{\partial x'^\sigma}\left(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\right).$$

Thus, $g_{\mu\nu,\sigma}$ cannot transform as a tensor for arbitrary coordinate transformations.

Note that we can always recover $g_{\mu\nu}$ from $u_{\mu\nu}$ via

$$g_{\mu\nu} = \int u_{\mu\nu}(\xi)d\xi \tag{26}$$

where the constant of integration can be determined by the value of $g_{\mu\nu}(x)$ at any x; for example, $g_{\mu\nu} \to \eta_{\mu\nu}$ at infinity.

§7. An important coordinate transformation

Consider the coordinate transformation

$$x'^{\mu} = x^{\mu} + b^{\mu}(x)\,, \qquad |b_{\mu,\rho}|, |b_{\mu,\rho\sigma}| \ll 1 \tag{27}$$

The product

$$(\delta^{\alpha}_{\mu} - b^{\alpha}{}_{,\mu})\frac{\partial x'^{\mu}}{\partial x^{\lambda}} = (\delta^{\alpha}_{\mu} - b^{\alpha}{}_{,\mu})(\delta^{\mu}_{\lambda} + b^{\mu}{}_{,\lambda}) = \delta^{\alpha}_{\lambda} + O(|b_{\alpha,\mu}|^2)$$

shows that

$$\frac{\partial x^{\alpha}}{\partial x'^{\mu}} = \delta^{\alpha}_{\mu} - b^{\alpha}{}_{,\mu} + O(|b_{\alpha,\mu}|^2)$$

The reason for assuming $|b_{\mu,\rho\sigma}| \ll 1$ will appear shortly. Keeping terms up to order $|b_{\alpha,\mu}|$:

$$g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} = g_{\alpha\beta}(\delta^{\alpha}_{\mu} - b^{\alpha}{}_{,\mu})\left(\delta^{\beta}_{\nu} - b^{\beta}{}_{,\nu}\right) = g_{\mu\nu} - g_{\mu\beta}b^{\beta}{}_{,\nu} - g_{\alpha\nu}b^{\alpha}{}_{,\mu}\,.$$

In the weak field approximation, the metric commutes with partial derivatives. More precisely, if $g_{\mu\beta} = \eta_{\mu\beta} + h_{\mu\beta}$ with $|h_{\mu\nu}| \ll 1$, then $g_{\mu\beta}b^{\beta}{}_{,\nu} = b_{\mu,\nu} + O(|h_{\mu\nu}||b_{\beta,\nu}|)$. Thus, to first order in the small quantities $|h_{\mu\nu}|$ and $|b_{\beta,\nu}|$, we have

$$g'_{\mu\nu} = g_{\mu\nu} - b_{\mu,\nu} - b_{\nu,\mu}\,. \tag{28}$$

This gives the transformation of the metric under the coordinate transformation (27).

Note that if $b_{\mu} = b_{\mu}(\xi)$, then

$$b_{\mu,\nu} = \frac{db_{\mu}}{d\xi}\frac{\partial \xi}{\partial x^{\nu}} = \dot{b}_{\mu}(\xi)k_{\nu}$$

where a dot $\dot{\square}$ denotes differentiation with respect to ξ. Hence, $g'_{\mu\nu}$ is also a function of ξ. Moreover:

$$\begin{aligned}\frac{dg'_{\mu\nu}}{d\xi} &= \frac{d}{d\xi}\left(g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\right) = \frac{dg_{\alpha\beta}}{d\xi}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} + 2g_{\alpha\beta}\frac{d}{d\xi}\left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\right)\frac{\partial x^{\beta}}{\partial x'^{\nu}} \\ &= \frac{dg_{\alpha\beta}}{d\xi}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} + 2g_{\alpha\beta}\frac{d\left(\delta^{\alpha}_{\mu} - \dot{b}^{\alpha}k_{\mu}\right)}{d\xi}\left(\delta^{\beta}_{\nu} - \dot{b}^{\beta}k_{\nu}\right) \\ &= \frac{dg_{\alpha\beta}}{d\xi}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} - 2\ddot{b}_{\beta}k_{\mu}\left(\delta^{\beta}_{\nu} - \dot{b}^{\beta}k_{\nu}\right).\end{aligned}$$

Hence,

$$u'_{\mu\nu} = u_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} + O(\ddot{b}_\mu) .$$

Note that $b_{\mu,\rho} = \dot{b}_\mu k_\rho$ and $b_{\mu,\rho\sigma} = \ddot{b}_\mu k_\rho k_\sigma$. Therefore, the conditions $|b_{\mu,\rho}| \ll 1$, $|b_{\mu,\rho\sigma}| \ll 1$ are equivalent to $|\dot{b}_\mu| \ll 1$, $|\ddot{b}_\mu| \ll 1$. We also have $|u_{\mu\nu}| \ll 1$, since $g_{\mu\nu,\sigma} = u_{\mu\nu} k_\sigma$ and $|g_{\mu\nu,\sigma}| \ll 1$. Therefore, for $u_{\mu\nu}$ to transform as a tensor with respect to the coordinate change (27), we require $\ddot{b}_\mu = b_{\mu,\rho\sigma} = 0$. In other words, we must have $b_\mu(\xi) = A_\mu \xi + B_\mu$, where A_μ and B_μ are constant vectors; that is, b_μ must be linear in ξ.

Note that (27) and (28) leave $R_{\mu\nu}$ unchanged. To see this, we have from (1) and (28):

$$R'_{\rho\sigma} = g'^{\mu\nu} \left(g_{\rho\sigma,\mu\nu} + g_{\mu\nu,\rho\sigma} - g_{\mu\rho,\nu\sigma} - g_{\mu\sigma,\nu\rho} \right) . \tag{29}$$

To find an expression for $g'^{\mu\nu}$, consider the product

$$\left(g_{\mu\lambda} - b_{\mu,\lambda} - b_{\lambda,\mu} \right) \left(g^{\lambda\nu} + g^{\nu\sigma} b^\lambda{}_{,\sigma} + g^{\lambda\sigma} b^\nu{}_{,\sigma} \right) = \delta^\nu_\mu + O(|b_{\nu,\sigma}|^2) .$$

This shows that the inverse of $g'_{\mu\nu} = g_{\mu\nu} - b_{\mu,\nu} - b_{\nu,\mu}$ is, to first order in $|b_{\nu,\sigma}|$:

$$g'^{\mu\nu} = g^{\mu\nu} + g^{\nu\sigma} b^\mu{}_{,\sigma} + g^{\mu\sigma} b^\nu{}_{,\sigma} .$$

This can be written more neatly as

$$g'^{\mu\nu} = g^{\mu\nu} + b^{\mu,\nu} + b^{\nu,\mu} \tag{30}$$

where

$$b^{\mu,\nu} \equiv \frac{\partial b^\mu}{\partial x_\nu} = \frac{\partial b^\mu}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x_\nu} = g^{\sigma\nu} b^\mu{}_{,\sigma} .$$

The last equality follows by comparing the two expressions

$$dx^\sigma = \frac{\partial x^\sigma}{\partial x_\nu} dx_\nu , \qquad dx^\sigma = g^{\sigma\nu} dx_\nu .$$

Thus, from (29), we find that $R'_{\rho\sigma} = R_{\rho\sigma}$ to first order in $|b_{\mu,\nu}|$ and $|g_{\rho\sigma,\mu\nu}|$.

§8. Harmonic coordinates by a coordinate change $x'^\mu = x^\mu + b^\mu(x)$

It is always possible to find coordinates in which (3) holds locally.[7] In the weak field case, we can also find harmonic coordinates using the transformation (27) with an appropriate choice of b^μ. We calculate:

[7]Weinberg (1972), §7.4.

$$g'^{\mu\nu}\left(g'_{\rho\mu,\nu}-\tfrac{1}{2}g'_{\mu\nu,\rho}\right)=g'^{\mu\nu}\left[g_{\rho\mu,\nu}-b_{\rho,\mu\nu}-b_{\mu,\rho\nu}-\tfrac{1}{2}(g_{\mu\nu,\rho}-b_{\mu,\nu\rho}-b_{\nu,\mu\rho})\right]$$
$$=(g^{\mu\nu}+b^{\mu,\nu}+b^{\nu,\mu})\left(g_{\rho\mu,\nu}-\tfrac{1}{2}g_{\mu\nu,\rho}-b_{\rho,\mu\nu}\right)$$
$$=g^{\mu\nu}\left(g_{\rho\mu,\nu}-\tfrac{1}{2}g_{\mu\nu,\rho}\right)-\Box b_\rho+O(|b_{\mu,\nu}||g_{\mu\nu,\rho}|)+O(|b_{\mu,\nu}||b_{\rho,\mu\nu}|)\,.$$

If we choose b_ρ such that

$$\Box b_\rho=g^{\mu\nu}\left(g_{\rho\mu,\nu}-\tfrac{1}{2}g_{\mu\nu,\rho}\right)\,, \tag{31}$$

then

$$g'^{\mu\nu}\left(g'_{\rho\mu,\nu}-\tfrac{1}{2}g'_{\mu\nu,\rho}\right)=0$$

to first order in $|b_{\mu,\nu}|$, $|b_{\rho,\mu\nu}|$ and $|g_{\mu\nu,\rho}|$. The coordinates $x'^\mu=x^\mu+b^\mu(x)$ are harmonic since they satisfy (5).

Note that if x^μ are harmonic coordinates, then $x'^\mu=x^\mu+b^\mu(x)$ are also harmonic providing that $\Box b_\rho=0$.

B. Polarization of gravitational waves

We analyze the polarization of gravitational plane waves and deduce that the metric $g_{\mu\nu}$ (or metric derivative $u_{\mu\nu}$) has only two independent components.

§1. Components of $\boldsymbol{u_{\mu\nu}}$

Putting $g_{\mu\nu}=g_{\mu\nu}(k_\sigma x^\sigma)$ into the harmonic coordinates condition (5), we obtain

$$u_{\nu\rho}k^\nu-\tfrac{1}{2}uk_\rho=0 \tag{32}$$

where $u\equiv g^{\mu\nu}u_{\mu\nu}=u^\mu_\mu$. The equation $\Box g_{\mu\nu}=\Box u_{\mu\nu}=0$ consists of 10 equations in 10 independent components $u_{\mu\nu}$. The 4 equations (32) for $0\le\rho\le3$ reduce this number to 6. But it turns out that there are only two distinct degrees of freedom that represent the propagation of *physical* waves, as opposed to *coordinate waves*.

§2. Coordinate waves

Wave-like behavior of $g_{\mu\nu}$ may arise from the propagation of a physical disturbance in the spacetime geometry or from a choice of coordinates, or both. Consider flat spacetime with cartesian coordinates $x'^\mu=(t',x',y',z')$.[8] Let x^μ be another coordinate system defined by $t=t'$, $x=x'$, $y=y'$, and $z=z'+(\varepsilon/\omega)\cos\xi$, where $\xi=\omega(t-z)$ and $|\varepsilon|\ll1$. Then:

[8]We will occasionally write $x^\mu=(t,x,y,z)$, but the time coordinate is always measured in units of distance.

$$\begin{aligned} ds^2 &= dt^2 - dx^2 - dy^2 - [dz - \varepsilon \sin\xi\,(dt - dz)]^2 \\ &= \eta_{\mu\nu} dx^\mu dx^\nu - 2\varepsilon \sin\xi\, dtdz + 2\varepsilon\sin\xi\, dz^2 + O(\varepsilon^2) \\ &= \left(\eta_{\mu\nu} + h_{\mu\nu}\right) dx^\mu dx^\nu + O(\varepsilon^2)\,, \end{aligned}$$

where

$$h_{\mu\nu} = \varepsilon \sin\xi \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

The oscillatory behavior of the metric could be mistaken for a wave propagating in the $\hat{\boldsymbol{z}}$ direction, but this is a *flat* spacetime metric. This **coordinate wave** is a manifestation of the coordinate system.

§3. Polarization of $\boldsymbol{g_{\mu\nu}}$ waves

Let x^μ be harmonic coordinates. Consider the coordinate change (27) with $b^\mu = b^\mu(\xi)$. From (28) we have:

$$g'_{\mu\nu} = g_{\mu\nu} - \dot{b}_\mu k_\nu - \dot{b}_\nu k_\mu\,,$$

therefore,

$$u'_{\mu\nu} = u_{\mu\nu} - \ddot{b}_\mu k_\nu - \ddot{b}_\nu k_\mu\,. \tag{34}$$

The coordinates x'^μ are harmonic, from the remark at the end of section A §8 and the fact that

$$\Box b_\rho = \eta^{\mu\nu} b_{\rho,\mu\nu} = \eta^{\mu\nu}\ddot{b}_\rho k_\mu k_\nu = \ddot{b}_\rho k_\mu k^\mu = 0\,.$$

Consider a monochromatic plane wave traveling in the $\hat{\boldsymbol{z}}$ direction with $k_\sigma = \omega(1,0,0,-1)$. The harmonic coordinates condition (32) gives the following relations:

$$\begin{aligned} u_{00} + u_{03} &= \tfrac{1}{2}u\,, \qquad & u_{02} + u_{23} &= 0\,, \\ u_{01} + u_{13} &= 0\,, \qquad & u_{03} + u_{33} &= -\tfrac{1}{2}u\,. \end{aligned} \tag{35}$$

Since $u_{\mu\nu} = u_{\nu\mu}$, we focus on the components with $\mu \le \nu$ (the upper triangle). Subtracting the last equation in (35) from the first gives $u_{00} - u_{33} = u = u_{00} - u_{11} - u_{22} - u_{33}$. Hence,

$$u_{11} + u_{22} = 0\,. \tag{36}$$

Adding the first and last equations in (35) gives

$$2u_{03} = -(u_{00} + u_{33})\,. \tag{37}$$

Eqs. (36)-(37) allow us to express u_{13}, u_{23}, u_{03} and u_{22} in terms of the other components:

$$\begin{aligned} u_{13} &= -u_{01}\,, \qquad & u_{23} &= -u_{02}\,, \\ u_{22} &= -u_{11}\,, \qquad & u_{03} &= -\tfrac{1}{2}(u_{00} + u_{33})\,. \end{aligned} \tag{38}$$

Thus, only six components of $u_{\mu\nu}$ are independent: u_{11}, u_{12}, u_{01}, u_{02}, u_{00} and u_{33}. If we make the change of coordinates (27), we can calculate $u'_{\mu\nu}$ for these six components using (34):

$$\begin{aligned} u'_{11} &= u_{11}\,, & u'_{02} &= u_{02} - \ddot{b}_2\omega\,,\\ u'_{12} &= u_{12}\,, & u'_{33} &= u_{33} + 2\ddot{b}_3\omega\,,\\ u'_{01} &= u_{01} - \ddot{b}_1\omega\,, & u'_{00} &= u_{00} - 2\ddot{b}_0\omega\,. \end{aligned}$$

Notice that only u_{11} and u_{12} are unaltered by the coordinate change. By a suitable choice of b_μ we can make the other four components vanish. Choose b_μ such that

$$\ddot{b}_1 = \frac{u_{01}}{\omega}\,, \qquad \ddot{b}_2 = \frac{u_{02}}{\omega}\,, \qquad \ddot{b}_3 = -\frac{u_{33}}{2\omega}\,, \qquad \ddot{b}_0 = \frac{u_{00}}{2\omega}\,. \tag{39}$$

Then in the x' coordinate system:

$$u'_{11} = -u'_{22} = u_{11}\,, \qquad u'_{12} = u'_{21} = u_{12}\,, \qquad \text{all other } u'_{\mu\nu} = 0\,.$$

Thus, all $u'_{\mu\nu}$ vanish except $u'_{11} = -u'_{22}$ and $u'_{12} = u'_{21}$. Hence, in the x' coordinate system, $u'_{\mu\nu}$ has only two independent components.[9]

In the x' coordinate system (henceforth dropping the primes), $u_{\mu\nu}$ is a linear combination

$$u_{\mu\nu}(x) = u_{11}(\xi)\Omega^{+}_{\mu\nu} + u_{12}(\xi)\Omega^{\times}_{\mu\nu}\,, \tag{40}$$

where $\xi = x^0 - x^3$, and

$$\Omega^{+}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Omega^{\times}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{41}$$

Equivalently, recovering $h_{\mu\nu}$ from $u_{\mu\nu}$ using (26), we have

$$h_{\mu\nu}(\xi) = h_{11}(\xi)\Omega^{+}_{\mu\nu} + h_{12}(\xi)\Omega^{\times}_{\mu\nu}\,. \tag{42}$$

$\Omega^{+}_{\mu\nu}$ and $\Omega^{\times}_{\mu\nu}$ are the two independent polarization states of $u_{\mu\nu}$ or $h_{\mu\nu}$.

The x' coordinate system is called the ***transverse traceless*** **(TT) gauge**, since in these coordinates the perturbation $h_{\mu\nu}$ is traceless and perpendicular to the direction of propagation of the wave (the fourth row and column are zeros). Note that $h_{\mu\nu}$ remains transverse and traceless regardless of the direction of the wave vector $\mathbf{k}$. (A rotation of the axes is just a change of basis, under which the trace of a matrix is invariant.)

[9] Note that the requirements $|b_\mu|, |b_{\mu,\alpha}| \ll 1$ of (27) are satisfied. Using $g_{\mu\nu,\rho} = u_{\mu\nu}k_\rho$ and the assumption $|g_{\mu\nu,\rho}| \ll 1$, we have $|u_{\mu\nu}| \ll 1$; therefore $|\ddot{b}_\mu| \ll 1$ by (39). Thus:

$$\dot{b}_\mu = \int \ddot{b}_\mu(\lambda)\,d\lambda \ll 1\,, \qquad b_\mu = \int \dot{b}_\mu(\lambda)\,d\lambda \ll 1\,, \qquad b_{\mu,\alpha} = \dot{b}_\mu k_\alpha \ll 1\,.$$

Note also that $h_{\mu\nu}$ satisfies $h_{0\nu} = h_{\mu 0} = 0$ in the TT gauge, which remains true for any $\mathbf{k}$, since matrix multiplication by a spatial rotation does not affect the first row or column.

C. Infinitesimal rotations and spin-2 fields

We introduce Dirac's infinitesimal rotation operator and obtain a relation between its eigenvalues on specific vector subspaces and the helicity of elements of the subspace. In particular, we look at gravitational plane waves (with spacetime metric $g_{\mu\nu}$) and electromagnetic plane waves (with electromagnetic 4-potential A_ν).

§1. Finite rotations

Consider a change of coordinates $x^\mu \to x'^\mu$ resulting from a rotation of the x^μ axes by an angle θ around the x^3 axis. The change of coordinates $x'^\mu = R^\mu{}_\nu(\theta) x^\nu$ is a linear transformation:

$$R^\mu{}_\nu(\theta) = \frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its inverse is obtained by replacing θ by $-\theta$:

$$R_\nu{}^\mu(\theta) = \frac{\partial x^\mu}{\partial x'^\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$R^\mu{}_\nu(\theta)$ and $R_\nu{}^\mu(\theta)$ are Lorentz transformations, also denoted by $\Lambda^\mu{}_\nu$ and $\Lambda_\nu{}^\mu$. This notation is consistent with the rules for raising and lowering indices, since

$$\Lambda_\mu{}^\rho \Lambda^\nu{}_\rho = \Lambda_\mu{}^\rho \left(\eta^{\alpha\nu}\eta_{\beta\rho}\Lambda_\alpha{}^\beta\right) = \eta^{\alpha\nu}\left(\eta_{\beta\rho}\Lambda_\alpha{}^\beta \Lambda_\mu{}^\rho\right) = \eta^{\alpha\nu}\eta_{\alpha\mu} = \delta_\mu^\nu \,.$$

(The third equality is the defining property of a Lorentz transformation.) Thus, $\Lambda_\sigma{}^\rho$ and $\Lambda^\mu{}_\nu$ (or $R_\sigma{}^\rho$ and $R^\mu{}_\nu$) are inverses.

To check how $u_{\mu\nu}$ transforms under this rotation, note that (see footnote 6):

$$g'_{\mu\nu,\sigma} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\lambda}{\partial x'^\sigma} u_{\alpha\beta} k_\lambda + g_{\alpha\beta}\frac{\partial}{\partial x'^\sigma}\left[R_\mu{}^\alpha(\theta) R_\nu{}^\beta(\theta)\right]$$

or

$$g'_{\mu\nu,\sigma} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} u_{\alpha\beta} k'_\sigma \,.$$

Comparing this with $g'_{\mu\nu,\sigma} = u'_{\mu\nu} k'_\sigma$, we deduce that

$$u'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} u_{\alpha\beta} = R_\mu{}^\alpha R_\nu{}^\beta u_{\alpha\beta} \,. \tag{43}$$

Thus, $u_{\mu\nu}$ transforms as a tensor.

Note that the rotation $R_\mu{}^\nu(\theta)$ obviously leaves $k_\sigma = \omega(1,0,0,-1)$ unchanged:

$$k'_\sigma = R_\sigma{}^\lambda k_\lambda = k_\sigma$$

Therefore, a rotation does not alter the harmonic coordinates condition (32), since

$$u'_{\nu\rho}k'^\nu = R_\nu{}^\alpha R_\rho{}^\beta u_{\alpha\beta}k^\nu = R_\rho{}^\beta u_{\alpha\beta}k^\alpha = R_\rho{}^\beta\left(\tfrac{1}{2}uk_\beta\right) = \tfrac{1}{2}uk_\rho = \tfrac{1}{2}u'k'_\rho\,.$$

Hence $u'_{\nu\rho}$ satisfies (32).

A rotation of coordinates can be viewed as an operator $\boldsymbol{R}(\theta)$ on scalars, vectors and tensors, with one rotation $R_\mu{}^\alpha$ for each rank. From the transformation rule for a covariant tensor $T_{\mu\nu\lambda\cdots}$, we have

$$\boldsymbol{R}(\theta)(T_{\mu\nu\lambda\cdots}) \equiv T'_{\mu\nu\lambda\cdots} = T_{\alpha\beta\gamma\cdots}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\gamma}{\partial x'^\lambda}\cdots = T_{\alpha\beta\gamma\cdots}R_\mu{}^\alpha(\theta)R_\nu{}^\beta(\theta)R_\lambda{}^\gamma(\theta)\cdots \tag{44}$$

Scalars, vectors and rank-2 tensors transform under $\boldsymbol{R}(\theta)$ as follows (in operator and component notation):

$$S' = \boldsymbol{R}(\theta)(S) \qquad S' = S$$
$$V' = \boldsymbol{R}(\theta)(V) \qquad V'_\mu = R_\mu{}^\alpha V_\alpha$$
$$U' = \boldsymbol{R}(\theta)(U) \qquad U'_{\mu\nu} = R_\mu{}^\alpha R_\nu{}^\beta U_{\alpha\beta}$$

§2. Dirac's infinitesimal rotation operator

In his treatise on quantum mechanics,[10] Dirac defined the *infinitesimal rotation operator* $\widetilde{\boldsymbol{R}}$:

$$\widetilde{\boldsymbol{R}} = \lim_{\theta\to 0}\frac{\boldsymbol{R}(\theta) - \boldsymbol{I}}{\theta}\,. \tag{45}$$

This can also be expressed

$$\widetilde{\boldsymbol{R}}\delta\theta = \boldsymbol{R}(\delta\theta) - \boldsymbol{I}\,. \tag{46}$$

We write $\widetilde{\boldsymbol{R}}$ to avoid confusion with the finite rotation operator $\boldsymbol{R}(\theta)$. Boldface symbols distinguish operators from matrices. The form which $\boldsymbol{R}(\theta)$ takes varies according to whether it operates on a vector V_μ or a tensor $U_{\mu\nu}$, as shown in (44). In particular, the operator $\boldsymbol{R}(\theta)$ on a rank-2 tensor cannot be represented by a matrix; it is an object with four indices.

For a vector $V = V_\mu$, (46) gives

[10] Dirac (1958), §25. [5]

$$\widetilde{\boldsymbol{R}}(V)_\mu = \lim_{\theta\to 0}\frac{R_\mu{}^\alpha(\theta) - \delta_\mu^\alpha}{\theta}V_\alpha = \lim_{\theta\to 0}\frac{1}{\theta}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \cos\theta - 1 & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta - 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}V$$

$$= \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}V = (\tilde{R}V)_\mu\,,$$

where $\tilde{R}$ is the matrix

$$\tilde{R} = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{47}$$

Thus, for a vector V, we have $\widetilde{\boldsymbol{R}}(V) = \tilde{R}V$. We write $\widetilde{\boldsymbol{R}}$ for the operator (45) and $\tilde{R}$ for the matrix (47) to avoid potential confusion, which will become evident in a moment.

For a tensor $U = U_{\mu\nu}$, (45) gives

$$\widetilde{\boldsymbol{R}}(U)_{\mu\nu} = \lim_{\theta\to 0}\frac{R_\mu{}^\alpha(\theta)R_\nu{}^\beta(\theta) - \delta_\mu^\alpha\delta_\nu^\beta}{\theta}U_{\alpha\beta}\,. \tag{48}$$

We are careful not to interpret $\widetilde{\boldsymbol{R}}(U)$ as the matrix product $\tilde{R}U$. The matrix product $\tilde{R}U$ has entries $(\tilde{R}U)_{\mu\nu} = \sum_\lambda \tilde{R}_{\mu\lambda}U_{\lambda\nu}$, whereas $\widetilde{\boldsymbol{R}}(U)_{\mu\nu}$ is given by (48) and requires two applications of the rotation matrix $R_\mu{}^\alpha(\theta)$.

We can find a formula for $\widetilde{\boldsymbol{R}}(U)_{\mu\nu}$ as follows. For a finite rotation, (44) gives

$$U'_{\mu\nu} = R_\mu{}^\alpha(\theta)R_\nu{}^\beta(\theta)U_{\alpha\beta}\,, \tag{49}$$

which we can write as a matrix equation:

$$U' = R(\theta)UR(\theta)^T\,. \tag{50}$$

Applying (50) and (46) with an infinitesimal rotation $\delta\theta$, we have the *matrix equation*:

$$U' - U = \left(\tilde{R}\delta\theta + I\right)U\left(\tilde{R}^T\delta\theta + I\right) - U\,.$$

So, to first order in $\delta\theta$:

$$U' - U = \left(\tilde{R}U + U\tilde{R}^T\right)\delta\theta\,. \tag{51}$$

Using (47), we calculate:

$$\tilde{R}U + U\tilde{R}^T = \begin{pmatrix}0 & U_{02} & -U_{01} & 0\\ U_{20} & U_{12} + U_{21} & U_{22} - U_{11} & U_{23}\\ -U_{10} & U_{22} - U_{11} & -U_{12} - U_{21} & -U_{13}\\ 0 & U_{32} & -U_{31} & 0\end{pmatrix}. \tag{52}$$

From (45), we have

$$\boldsymbol{R}(\delta\theta)(U_{\mu\nu}) = R_\mu{}^\alpha(\delta\theta) R_\nu{}^\beta(\delta\theta) U_{\alpha\beta} \,,$$

so with (49) and (46), we also have the *operator equation*:

$$U' - U = \boldsymbol{R}(\delta\theta)(U) - \boldsymbol{I}(U) = \widetilde{\boldsymbol{R}}(U)\delta\theta \,. \tag{53}$$

Comparing (51) and (53) we see that

$$\widetilde{\boldsymbol{R}}(U)_{\mu\nu} = \big(\tilde{R}U + U\tilde{R}^T\big)_{\mu\nu} \,. \tag{54}$$

Now it is clear why we are careful to distinguish between the operator $\widetilde{\boldsymbol{R}}$ and the matrix $\tilde{R}$. The $\widetilde{\boldsymbol{R}}$ appearing on the left-hand side of (54) is the operator of (48), not the matrix (47). Otherwise, (54) would read $(\tilde{R}U)_{\mu\nu} = (\tilde{R}U + U\tilde{R}^T)_{\mu\nu}$ — a nonsensical result.[11]

§3. Eigenvalues of $\widetilde{\boldsymbol{R}}$

If $V = V_\mu$ is a vector, then

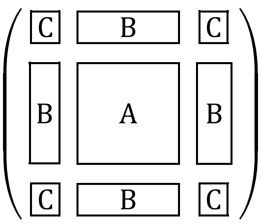
$$\widetilde{\boldsymbol{R}}(V) = \tilde{R}V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ V_2 \\ -V_1 \\ 0 \end{pmatrix}.$$

Therefore, the operator $\widetilde{\boldsymbol{R}}^2$ (and the matrix $\tilde{R}^2$) has eigenvalues -1 and 0, with corresponding eigenvectors $(0, V_1, V_2, 0)$ and $(V_0, 0, 0, V_3)$, each spanning a 2-dimensional subspace. Hence, $(i\widetilde{\boldsymbol{R}})^2 = -\widetilde{\boldsymbol{R}}^2$ has eigenvalues 1 and 0, so that $i\widetilde{\boldsymbol{R}}$ has eigenvalues ±1 and 0.

Now consider $\widetilde{\boldsymbol{R}}$ operating on $u_{\mu\nu}$. With $\widetilde{\boldsymbol{R}}$ as the map given by (54) and (52), $-\widetilde{\boldsymbol{R}}^2$ acts as follows:

$$\begin{pmatrix} u_{00} & u_{01} & u_{02} & u_{03} \\ u_{10} & u_{11} & u_{12} & u_{13} \\ u_{20} & u_{21} & u_{22} & u_{23} \\ u_{30} & u_{31} & u_{32} & u_{33} \end{pmatrix} \xrightarrow{-\widetilde{\boldsymbol{R}}^2} \begin{pmatrix} 0 & u_{01} & u_{02} & 0 \\ u_{10} & 2(u_{11} - u_{22}) & 2(u_{12} + u_{21}) & u_{13} \\ u_{20} & 2(u_{12} + u_{21}) & -2(u_{11} - u_{22}) & u_{23} \\ 0 & u_{31} & u_{32} & 0 \end{pmatrix}. \tag{55}$$

Since $u_{22} = -u_{11}$ and $u_{21} = u_{12}$, we see that $-\widetilde{\boldsymbol{R}}^2$ has the following effect on various blocks of $u_{\mu\nu}$:



$$\begin{aligned} &\text{Block A:} \quad u_{\mu\nu} \xrightarrow{-\widetilde{\boldsymbol{R}}^2} 4u_{\mu\nu} \\ &\text{Block B:} \quad u_{\mu\nu} \xrightarrow{-\widetilde{\boldsymbol{R}}^2} u_{\mu\nu} \\ &\text{Block C:} \quad u_{\mu\nu} \xrightarrow{-\widetilde{\boldsymbol{R}}^2} 0 \end{aligned} \tag{56}$$

[11]Notice that $\widetilde{\boldsymbol{R}}(U) = \tilde{R}U - U\tilde{R}$, the commutator of the matrices $\tilde{R}$ and U.

Therefore, $i\widetilde{\boldsymbol{R}}$ has eigenvalues ±2, ±1 and 0 when applied to Blocks A, B and C, respectively. Dirac observed that *"the components of $u_{\alpha\beta}$ that contribute to the energy* [Block A] *thus correspond to spin 2."*[12] The remainder of this section is devoted to clarifying and understanding this comment.

§4. Wave helicity[13]

The components of $u_{\mu\nu}$ transform under a finite rotation $\boldsymbol{R}(\theta)$ according to (44):

$$u'_{11} = \cos 2\theta\, u_{11} + \sin 2\theta\, u_{12}\,, \qquad u'_{01} = \cos\theta\, u_{01} + \sin\theta\, u_{02}\,, \qquad u'_{00} = u_{00}\,,$$
$$u'_{12} = -\sin 2\theta\, u_{11} + \cos 2\theta\, u_{12}\,, \qquad u'_{02} = -\sin\theta\, u_{01} + \cos\theta\, u_{02}\,, \qquad u'_{33} = u_{33}\,.$$

The other components u'_{13}, u'_{23}, u'_{03} and u'_{22} can be found using (38). Notice that

$$u'_{11} \mp i u'_{12} = e^{\pm 2i\theta}(u_{11} \mp i u_{12})$$
$$u'_{01} \mp i u'_{02} = e^{\pm i\theta}(u_{01} \mp i u_{02})\,.$$

If we define

$$F_{\pm} \equiv u_{11} \mp i u_{12}$$
$$G_{\pm} \equiv u_{01} \mp i u_{02} \tag{57}$$

then under the action of $\boldsymbol{R}(\theta)$:

$$F'_{\pm} = e^{\pm 2i\theta} F_{\pm}$$
$$G'_{\pm} = e^{\pm i\theta} G_{\pm}\,. \tag{58}$$

A wave function ϕ that transforms under a rotation $\boldsymbol{R}(\theta)$ about the direction of wave propagation according to $\phi' = e^{ih\theta}\phi$ is said to have *helicity* h.

Each component of $u_{\mu\nu}$ can be written in terms of u_{11}, u_{12}, u_{01}, u_{02}, u_{00} and u_{33} using (38). These in turn can be written in terms of $F_{\pm}$, $G_{\pm}$, u_{00} and u_{33} by (57). It follows that any $u_{\mu\nu}$ can be decomposed into a sum of waves with helicities ±2, ±1 and 0.

We saw earlier that by making a coordinate change $x'^{\mu} = x^{\mu} + b^{\mu}(\xi)$ we can arrange for all $u_{\mu\nu}$ to vanish except u_{11} and u_{12}. The waves of helicity ±1 and 0 are therefore coordinate waves; they are not physical. Only the waves of helicity ±2 are actual disturbances of the spacetime geometry.

Using (57)-(58), the components of a wave

$$u_{\mu\nu}(x) = u_{11}(\xi)\Omega^{+}_{\mu\nu} + u_{12}(\xi)\Omega^{\times}_{\mu\nu}:$$

[12]Dirac (1975), Chap. 34.

[13]Adapted from Weinberg (1972), §10.2.

transform under $\boldsymbol{R}(\theta)$ as follows

$$u'_{\mu\nu} = \tfrac{1}{2}\left[(F'_{+} + F'_{-})\Omega^{+}_{\mu\nu} + i(F'_{+} - F'_{-})\Omega^{\times}_{\mu\nu}\right]$$

$$= \tfrac{1}{2}\left[(u_{11} - iu_{12})(\Omega^{+}_{\mu\nu} + i\Omega^{\times}_{\mu\nu})e^{2i\theta} + (u_{11} + iu_{12})(\Omega^{+}_{\mu\nu} - i\Omega^{\times}_{\mu\nu})e^{-2i\theta}\right]$$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ u_{12} & -u_{11} \end{pmatrix} = R_{\mu}{}^{\lambda}(2\theta)u_{\lambda\nu}\,.$$

Thus, $\boldsymbol{R}(\theta)$ acting on $u_{\mu\nu}$ causes Block A of $u_{\mu\nu}$, $(1 \le \mu, \nu \le 2)$ to undergo a rotation by 2θ.[14] This reflects the spin-2 character of the gravitational field, which is physically determined only by the Block A components of $u_{\mu\nu}$.

§5. Polarization of electromagnetic waves

We pause to highlight the analogy with EM waves. Let $A_\nu = A_\nu(\xi)$ be a plane wave solution for the potential, with $\xi = k_\sigma x^\sigma$. Inserting this into the wave equation $\Box A_\mu = 0$ and the Lorenz condition $A^\nu{}_{,\nu} = 0$, we obtain the relations

$$k_\sigma k^\sigma = 0\,, \qquad u_\nu k^\nu = 0 \tag{59}$$

where $u_\nu \equiv dA_\nu/d\xi$. The relation $u_\nu k^\nu = 0$ reduces the number of independent components of u_ν and A_ν from 4 to 3.

The Lorenz condition is satisfied by taking any solution $\bar{A}_\nu$ and setting $A_\nu = \bar{A}_\nu - \phi_{,\nu}$ where $\Box\phi = \bar{A}^\mu{}_{,\mu}$. This leaves $F_{\mu\nu} = \bar{A}_{\mu,\nu} - \bar{A}_{\nu,\mu}$ unchanged, with $A^\nu{}_{,\nu} = 0$. We can make a further alteration of the potential:

$$A'_\nu(\xi) = A_\nu(\xi) + b_{,\nu}(\xi)$$

for some $b(\xi)$. Since b is a function of $\xi = k_\sigma x^\sigma$, we have $\Box b = 0$. This leaves $F_{\mu\nu}$ unchanged and also preserves the Lorenz condition:

$$A'^\nu{}_{,\nu} = (A^\nu + \eta^{\mu\nu} b_{,\mu})_{,\nu} = A^\nu{}_{,\nu} + \eta^{\mu\nu} b_{,\mu\nu} = 0\,.$$

Then

$$u'_\nu = u_\nu + \frac{\partial}{\partial\xi}\left(\frac{\partial b}{\partial x^\nu}\right) = u_\nu + \ddot{b}k_\nu\,.$$

The equation

$$u'_\nu = u_\nu + \ddot{b}k_\nu \tag{60}$$

reduces the number of independent components of u_ν (and A_ν) from 3 to 2.

[14] Blocks B and C of $u_{\mu\nu}$ would similarly undergo rotations by θ and 0, respectively. These are of less interest since only Block A represents a physical wave.

To illustrate this, consider a monochromatic plane wave traveling in the $\hat{\boldsymbol{z}}$ direction with $k_\sigma = \omega(1,0,0,-1)$. The condition $u_\nu k^\nu = 0$ means that $u_0 = -u_3$. This is analogous to what we did in (38) to express u_{13}, u_{23}, u_{03} and u_{22} in terms of the other six components. Hence,

$$u_\nu = u_1(0,1,0,0) + u_2(0,0,1,0) + u_3(-1,0,0,1)\,. \tag{61}$$

The vectors

$$\Omega_\nu^{(1)} = (0,1,0,0) \qquad \Omega_\nu^{(2)} = (0,0,1,0) \qquad \Omega_\nu^{(3)} = (-1,0,0,1) \tag{62}$$

are the independent polarization vectors. From (60) we have:

$$u'_1 = u_1\,, \qquad u'_2 = u_2\,, \qquad u'_3 = u_3 - \ddot{b}\omega\,.$$

By choosing $\ddot{b} = u_3/\omega$ we can make $u'_3 = 0$. Thus, only u_1 and u_2 have physical significance; $u_3(\xi)\Omega_\nu^{(3)}$ is a coordinate wave, since it can be eliminated by a change of coordinates. Therefore, only $\Omega_\nu^{(1)}$ and $\Omega_\nu^{(2)}$ appear in the physical fields.

The components of u_ν transform under a rotation $\boldsymbol{R}(\theta)$ according to $u'_\mu = [\boldsymbol{R}(\theta)(u)]_\mu = R_\mu{}^\nu(\theta)u_\nu$. Therefore:

$$\begin{aligned} u'_1 &= u_1\cos\theta + u_2\sin\theta\ , \\ u'_2 &= -u_1\sin\theta + u_2\cos\theta\ , \\ u'_0 &= u_0\,, \qquad u'_3 = u_3\,. \end{aligned}$$

If we define $G_\pm = u_1 \mp iu_2$, then $G'_\pm = e^{\pm i\theta}G_\pm$. Thus, $G_\pm$ has helicity ±1, analogous to the $G_\pm$ in (57). We can solve for u_1 and u_2 in terms of $G_\pm$ and express u_ν as a sum of waves with helicities ±1 and 0. Then only the components of u_ν with helicity ±1 are present in the physical wave, just as in the case of gravitational waves only the components with helicity ±2 have physical significance.

The components of the wave

$$u_\nu(x) = u_1(\xi)\Omega_\nu^{(1)} + u_2(\xi)\Omega_\nu^{(2)}$$

transform under $\boldsymbol{R}(\theta)$ as follows:

$$u'_\nu = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = R_\nu{}^\lambda(\theta)u_\lambda\,.$$

This reflects the spin-1 (vector) character of the electromagnetic field.

§6. Helicity from the infinitesimal rotation operator

Using (52), (53) and (54), if $u'_{\mu\nu}$ is the result of an infinitesimal rotation $\boldsymbol{R}(\delta\theta)$ operating on $u_{\mu\nu}$:

$$\delta u_{\mu\nu} = u'_{\mu\nu} - u_{\mu\nu} = \begin{pmatrix} 0 & u_{02} & -u_{01} & 0 \\ u_{02} & 2u_{12} & -2u_{11} & u_{23} \\ -u_{01} & -2u_{11} & -2u_{12} & -u_{13} \\ 0 & u_{23} & -u_{13} & 0 \end{pmatrix} \delta\theta \, .$$

where we have used the symmetry of $u_{\mu\nu}$ and the relation $u_{22} = -u_{11}$. Using the functions $F_\pm$ and $G_\pm$ defined in (57), we have:

$$\delta F_\pm = F'_\pm - F_\pm = \delta u_{11} \mp i\delta u_{12} = 2(u_{12} \pm iu_{11})\delta\theta = \pm 2iF_\pm\delta\theta$$

$$\delta G_\pm = G'_\pm - G_\pm = \delta u_{01} \mp i\delta u_{02} = (u_{02} \pm iu_{01})\delta\theta = \pm iG_\pm\delta\theta \, .$$

This implies, to first order in $\delta\theta$:

$$F'_\pm = (1 \pm 2i\delta\theta)F_\pm = e^{\pm 2i\delta\theta}F_\pm$$

$$G'_\pm = (1 \pm i\delta\theta)G_\pm = e^{\pm i\delta\theta}G_\pm$$

which is the infinitesimal version of (58). Thus, $F_\pm$ and $G_\pm$ have helicity ±2 and ±1 under infinitesimal rotations. We can recover (58) from these relations, since for a finite rotation θ, we have (writing $\delta\theta = \pm\theta/n$):

$$F'_\pm = \lim_{n\to\infty} \widetilde{\boldsymbol{R}}^n F_\pm = \lim_{n\to\infty} (1 \pm 2i\theta/n)^n F_\pm = e^{\pm 2i\theta}F_\pm \, .$$

§7. Eigenvalues of $i\widetilde{\boldsymbol{R}}$ and helicity

Consider $\widetilde{\boldsymbol{R}}$ acting on a rank-2 tensor. If Z is an eigentensor of $i\widetilde{\boldsymbol{R}}$ with eigenvalue λ, then

$$i\widetilde{\boldsymbol{R}}(Z) = i\left(\tilde{R}Z + Z\tilde{R}^T\right) = \lambda Z \, .$$

From (52) we have a system of 16 equations:

$$i\widetilde{\boldsymbol{R}}(Z) = i\begin{pmatrix} 0 & Z_{02} & -Z_{01} & 0 \\ Z_{20} & Z_{12}+Z_{21} & Z_{22}-Z_{11} & Z_{23} \\ -Z_{10} & Z_{22}-Z_{11} & -Z_{12}-Z_{21} & -Z_{13} \\ 0 & Z_{32} & -Z_{31} & 0 \end{pmatrix} = \lambda\begin{pmatrix} Z_{00} & Z_{01} & Z_{02} & Z_{03} \\ Z_{10} & Z_{11} & Z_{12} & Z_{13} \\ Z_{20} & Z_{21} & Z_{22} & Z_{23} \\ Z_{30} & Z_{31} & Z_{32} & Z_{33} \end{pmatrix} . \qquad (63)$$

The 16-dimensional vector space of real 4×4 matrices $U_{\mu\nu}$ is a direct sum of the subspaces $\mathcal{S}_A \oplus \mathcal{S}_B \oplus \mathcal{S}_C$ of matrices with entries in Blocks A, B and C; see (56). Solving (63) yields a basis of eigenmatrices $Z_1, \cdots, Z_{16}$ that spans these subspaces. The eigenmatrices Z_i have complex entries, so for real $U_{\mu\nu}$ the coefficients ζ^i in the expansion $U = \zeta^1 Z_1 + \cdots + \zeta^{16} Z_{16}$ will be complex valued. We solve for the Z_i in Blocks A, B and C separately.

Block A: We have

$$Z_{12} + Z_{21} = -i\lambda Z_{11} = i\lambda Z_{22}$$

$$Z_{22} - Z_{11} = -i\lambda Z_{12} = -i\lambda Z_{21} \, .$$

If $\lambda \neq 0$, then $Z_{11} = -Z_{22}$ and $Z_{12} = Z_{21}$. Therefore, the central 2×2 submatrix of $Z_{\mu\nu}$ must have the form

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}.$$

Hence, $2\beta = -i\lambda\alpha$ and $2\alpha = i\lambda\beta$. This implies $\lambda^2 = 4$, or $\lambda = \pm 2$. Hence, $\beta = i\alpha$, so we have the two independent solutions

$$Z_+ = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \qquad Z_- = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

with eigenvalues $\lambda_\pm = \pm 2$ respectively.

If $\lambda = 0$, then $Z_{11} = Z_{22}$ and $Z_{12} = -Z_{21}$. Then we have the independent solutions

$$Z_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad Z_{II} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The four matrices Z_+, Z_-, Z_I, Z_{II} form a basis that spans $\mathcal{S}_A$. A real 2×2 matrix U can be written as a linear combination:

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \zeta^+ Z_+ + \zeta^- Z_- + \zeta^I Z_I + \zeta^{II} Z_{II} = \begin{pmatrix} \zeta^+ + \zeta^- + \zeta^I & i(\zeta^- - \zeta^+) - \zeta^{II} \\ i(\zeta^- - \zeta^+) + \zeta^{II} & -\zeta^+ - \zeta^- + \zeta^I \end{pmatrix}. \tag{64}$$

Solving for the coefficients in terms of the $U_{\mu\nu}$:

$$\zeta^+ = \tfrac{1}{4}[U_{11} - U_{22} + i(U_{12} + U_{21})], \qquad \zeta^I = \tfrac{1}{2}(U_{11} + U_{22}),$$
$$\zeta^- = \tfrac{1}{4}[U_{11} - U_{22} - i(U_{12} + U_{21})], \qquad \zeta^{II} = \tfrac{1}{2}(U_{21} - U_{12}).$$

In the case of gravitational waves, $U_{\mu\nu} = u_{\mu\nu}$ and this simplifies to

$$\zeta^+ = \tfrac{1}{2}(u_{11} + iu_{12}) = \tfrac{1}{2}F_-, \qquad \zeta^I = 0,$$
$$\zeta^- = \tfrac{1}{2}(u_{11} - iu_{12}) = \tfrac{1}{2}F_+, \qquad \zeta^{II} = 0.$$

Thus, the decomposition (64) reduces to

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \tfrac{1}{2}(F_- Z_+ + F_+ Z_-).$$

We see that the $\mathcal{S}_A$ basis matrices with eigenvalue ± 2 have coefficients (which are functions of the $u_{\mu\nu}$) with helicity ∓ 2.

Block B: We have

$$Z_{02} = -i\lambda Z_{01}, \qquad Z_{01} = i\lambda Z_{02},$$
$$Z_{23} = -i\lambda Z_{13}, \qquad Z_{13} = i\lambda Z_{23}.$$

Thus $Z_{\mu\nu}(\lambda^2 - 1) = 0$ for every (μ, ν) in Block B. Hence, either all $Z_{\mu\nu} = 0$ (not interesting) or $\lambda = \pm 1$. Therefore, we have the relations:

$$\lambda = +1: \quad Z_{02} = -iZ_{01} \quad Z_{20} = -iZ_{10} \quad Z_{23} = -iZ_{13} \quad Z_{32} = -iZ_{31}$$
$$\lambda = -1: \quad Z_{02} = iZ_{01} \quad Z_{20} = iZ_{10} \quad Z_{23} = iZ_{13} \quad Z_{32} = iZ_{31}.$$

There are 8 matrices forming a basis of $\mathcal{S}_B$, each one with a $(1,\pm i)$ in one of the four regions of Block B and zeros elsewhere:

$$[Z_{a+}]_{ij} = \delta_{0i}\delta_{1j} - i\delta_{0i}\delta_{2j} \qquad [Z_{a-}]_{ij} = \delta_{0i}\delta_{1j} + i\delta_{0i}\delta_{2j}$$
$$[Z_{b+}]_{ij} = \delta_{1i}\delta_{0j} - i\delta_{2i}\delta_{0j} \qquad [Z_{b-}]_{ij} = \delta_{1i}\delta_{0j} + i\delta_{2i}\delta_{0j}$$
$$[Z_{c+}]_{ij} = \delta_{1i}\delta_{3j} - i\delta_{2i}\delta_{3j} \qquad [Z_{c-}]_{ij} = \delta_{1i}\delta_{3j} + i\delta_{2i}\delta_{3j}$$
$$[Z_{d+}]_{ij} = \delta_{3i}\delta_{1j} - i\delta_{3i}\delta_{2j} \qquad [Z_{d-}]_{ij} = \delta_{3i}\delta_{1j} + i\delta_{3i}\delta_{2j}$$

where $\pm$ in the subscript denotes the eigenvalue $\lambda = \pm 1$. Any $U \in \mathcal{S}_B$ is a linear combination:

$$U_{\mu\nu} = \sum_{+,-} \zeta^{a\pm} Z_{a\pm} + \zeta^{b\pm} Z_{b\pm} + \zeta^{c\pm} Z_{c\pm} + \zeta^{d\pm} Z_{d\pm} \tag{65}$$
$$= \begin{pmatrix} 0 & \zeta^{a+} + \zeta^{a-} & -i\zeta^{a+} + i\zeta^{a-} & 0 \\ \zeta^{b+} + \zeta^{b-} & 0 & 0 & \zeta^{c+} + \zeta^{c-} \\ -i\zeta^{b+} + i\zeta^{b-} & 0 & 0 & -i\zeta^{c+} + i\zeta^{c-} \\ 0 & \zeta^{d+} + \zeta^{d-} & -i\zeta^{d+} + i\zeta^{d-} & 0 \end{pmatrix}.$$

Solving for the coefficients in terms of the $U_{\mu\nu}$, we find:

$$\zeta^{a\pm} = \tfrac{1}{2}(U_{01} \pm iU_{02}), \qquad \zeta^{c\pm} = \tfrac{1}{2}(U_{13} \pm iU_{23}),$$
$$\zeta^{b\pm} = \tfrac{1}{2}(U_{10} \pm iU_{20}), \qquad \zeta^{d\pm} = \tfrac{1}{2}(U_{31} \pm iU_{32}).$$

For $U_{\mu\nu} = u_{\mu\nu}$, this simplifies to:

$$\zeta^{a\pm} = \zeta^{b\pm} = -\zeta^{c\pm} = -\zeta^{d\pm} = \tfrac{1}{2}(u_{01} \pm iu_{02}) = \tfrac{1}{2}G_{\mp}$$

and the decomposition (65) reduces to:

$$u_{\mu\nu} = \frac{1}{2}\sum_{+,-} G_{\mp}(Z_{a\pm} + Z_{b\pm} - Z_{c\pm} - Z_{d\pm}).$$

We see that the $\mathcal{S}_B$ basis matrices with eigenvalue ±1 have coefficients with helicity 1.

Block C: We have

$$0 = \lambda Z_{00} = \lambda Z_{03} = \lambda Z_{30} = \lambda Z_{33}$$

Hence, either all $Z_{\mu\nu} = 0$ or $\lambda = 0$. This yields the trivial basis

$$[Z_\alpha]_{ij} = \delta_{0i}\delta_{0j} \qquad [Z_\beta]_{ij} = \delta_{0i}\delta_{3j} \qquad [Z_\gamma]_{ij} = \delta_{3i}\delta_{0j} \qquad [Z_\delta]_{ij} = \delta_{3i}\delta_{3j}$$

and the trivial decomposition:

$$U_{\mu\nu} = U_{00}Z_\alpha + U_{03}Z_\beta + U_{30}Z_\gamma + U_{33}Z_\delta.$$

These results are summarized in Table 1 (omitting Z_I and Z_{II} whose coefficients are zero). Since only u_{11} and u_{12} (the Block A components of $u_{\mu\nu}$) are physically significant, any plane wave $u_{\mu\nu}$ can be decomposed

$$u_{\mu\nu} = \zeta^{+}(Z_{+})_{\mu\nu} + \zeta^{-}(Z_{-})_{\mu\nu}$$

where $Z_{\pm}$ are eigenmatrices of $i\tilde{\boldsymbol{R}}$ with eigenvalues $\lambda = \pm 2$, and the coefficients $\zeta^{\pm}$ have helicity $h = -\lambda$.[15]

Table 1 Relationship among eigenmatrices and eigenvalues of $i\tilde{\boldsymbol{R}}$ and coefficients in the eigenbasis expansion of $u_{\mu\nu}$ for Blocks A, B, C

Block	Eigenmatrices of $i\tilde{\boldsymbol{R}}$	Eigenvalue	Coefficients ζ^{α} in the expansion $u_{\mu\nu} = \zeta^{\alpha} Z_{\alpha}$	Coefficient ζ^{α} helicity
A	$Z_{\pm}$	± 2	$\zeta^{\pm} = \frac{1}{2} F_{\mp}$	2
B	$Z_{a\pm}, Z_{b\pm}$ $Z_{c\pm}, Z_{d\pm}$	± 1	$\zeta^{a\pm} = \zeta^{b\pm} = \frac{1}{2} G_{\mp}$ $\zeta^{c\pm} = \zeta^{d\pm} = -\frac{1}{2} G_{\mp}$	1
C	$Z_{\alpha}, Z_{\beta}, Z_{\gamma}, Z_{\delta}$	0	$\zeta^{\alpha} = u_{00}\,, \zeta^{\beta} = u_{03}$ $\zeta^{\gamma} = u_{30}\,, \zeta^{\delta} = u_{33}$	0

D. Angular momentum in electromagnetic plane waves

We show that a circularly polarized wave packet with frequency ω has spin equal to $\pm 1/\omega$ and $\pm 2/\omega$ times its energy for an electromagnetic wave and a gravitational wave.

§1. Polarization of electromagnetic waves (continued)

For an EM plane wave traveling in the $\hat{\boldsymbol{z}}$ direction, we found in section C §5 that

$$u_{\nu}(\xi) \equiv \frac{dA_{\nu}}{d\xi} = u_1(\xi)\Omega_{\nu}^{(1)} + u_2(\xi)\Omega_{\nu}^{(2)}\,,$$

where $\Omega_{\nu}^{(1)}$ and $\Omega_{\nu}^{(2)}$ are the two independent polarization vectors. We re-examine this case, focusing on the fields $\mathbf{E} = -\boldsymbol{\nabla} A^0 - \partial \mathbf{A}/\partial t$ and $\mathbf{B} = \operatorname{curl} \mathbf{A}$ as a prelude to calculating the energy and angular momentum densities of a wave packet for electromagnetic and gravitational waves.

Consider the monochromatic plane wave

$$A_{\nu} = A\Omega_{\nu} \sin[\omega(t - z)] \tag{66}$$

with real amplitude A and polarization vector Ω_{ν}. We saw in section C §5 that Ω_{ν} is not arbitrary, since putting (66) into the Lorenz condition $A^{\nu}{}_{,\nu} = 0$ gives $\Omega_0 = -\Omega_3$. Therefore, Ω_{ν} is a linear combination of the polarization vectors

[15] This differs from the linear combination $u_{\mu\nu} = u_{11}(\xi)\Omega_{\mu\nu}^{+} + u_{12}(\xi)\Omega_{\mu\nu}^{\times}$ of the two independent polarization states, which reveals information about the wave's effect on the spacetime geometry but not its behavior under a rotation of coordinates.

$$\Omega_\nu^{(1)} = (0,1,0,0) \qquad \Omega_\nu^{(2)} = (0,0,1,0) \qquad \Omega_\nu^{(3)} = (-1,0,0,1)\,.$$

Let $\xi = k_\sigma x^\sigma = \omega(t-z)$. For the wave

$$A_\nu^{(1)} = A\Omega_\nu^{(1)} \sin\xi = A(0,1,0,0)\sin\xi\,,$$

the electric and magnetic fields are:

$$\begin{aligned} E^1 &= F_{10} = A_{1,0} - A_{0,1} = A\omega\cos\xi \qquad & E^2 = E^3 = 0 \\ B^2 &= F_{31} = A_{3,1} - A_{1,3} = A\omega\cos\xi \qquad & B^1 = B^2 = 0 \end{aligned} \tag{67}$$

or

$$\mathbf{E}^{(1)} = E_0 \cos\xi\,\hat{\boldsymbol{x}} \qquad \mathbf{B}^{(1)} = E_0\cos\xi\,\hat{\boldsymbol{y}} \tag{68}$$

where $E_0 = A\omega$. The electric and magnetic fields have equal amplitude and phase and are perpendicular to $\mathbf{k}$ and to each other.

Similarly, for the wave

$$A_\nu^{(2)} = A\Omega_\nu^{(2)} \sin\xi = A(0,0,1,0)\sin\xi\,,$$

the fields are

$$\begin{aligned} E^2 &= F_{20} = A_{2,0} - A_{0,2} = A\omega\cos\xi \qquad & E^1 = E^3 = 0 \\ B^1 &= F_{23} = A_{2,3} - A_{3,2} = -A\omega\cos\xi \qquad & B^2 = B^3 = 0 \end{aligned} \tag{69}$$

or

$$\mathbf{E}^{(2)} = E_0 \cos\xi\,\hat{\boldsymbol{y}} \qquad \mathbf{B}^{(2)} = -E_0\cos\xi\,\hat{\boldsymbol{x}}\,. \tag{70}$$

The third wave $A_\nu^{(3)} = A\Omega_\nu^{(3)}\sin\xi$ is not a physical wave, since $F_{\mu\nu} = 0$.

We write the fields as complex functions, e.g., $\mathbf{E}^{(1)} = E_0 e^{-i\xi}\hat{\boldsymbol{x}}$, where it is understood that we take the real part only. Now let

$$\begin{aligned} \mathbf{E}^{(1)} &= E_0 e^{-i\xi}\hat{\boldsymbol{x}} \\ \mathbf{E}^{(2)} &= E_0 e^{-i\xi + i\pi/2}\hat{\boldsymbol{y}} = iE_0 e^{-i\xi}\hat{\boldsymbol{y}} \end{aligned}$$

so that $\mathbf{E}^{(2)}$ differs in phase from $\mathbf{E}^{(1)}$ by $\pi/2$. We define the **circularly polarized** waves

$$\mathbf{E}_\pm \equiv \mathbf{E}^{(1)} \pm \mathbf{E}^{(2)} = E_0 e^{-i\xi}(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})\,. \tag{71}$$

Taking real parts, we have

$$\mathbf{E}_\pm = E_0(\cos\xi\,\hat{\boldsymbol{x}} \pm \sin\xi\,\hat{\boldsymbol{y}})\,. \tag{72}$$

$\mathbf{E}_\pm$ has constant magnitude E_0, and at $z = 0$ makes an angle with the x-axis of $\pm\omega t$. Thus, from the point of view of the wave source, $\mathbf{E}_+$ rotates clockwise and has **positive helicity** (or is right-handed) and $\mathbf{E}_-$ rotates counterclockwise and has **negative helicity** (or is left-handed). We can easily recover $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ from $\mathbf{E}_+$ and $\mathbf{E}_-$.

§2. Momentum density in electromagnetic plane waves

The energy-momentum tensor of the electromagnetic field is:

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{8\pi}(E^2+B^2) & \frac{1}{4\pi}\mathbf{E}\times\mathbf{B} \\ \frac{1}{4\pi}\mathbf{E}\times\mathbf{B} & \boxed{T^{mn}} \end{pmatrix}$$

where the *stress tensor*

$$T^{mn} = (4\pi)^{-1}\left[\tfrac{1}{2}\delta^{mn}(E^2+B^2) - E^m E^n - B^m B^n\right]$$

Therefore, the plane wave (68) has energy density $\mathcal{E}$ and momentum density $\mathbf{p}$ given by:

$$\mathcal{E} = T^{00} = \frac{E_0^2}{4\pi}\cos^2\xi\ , \qquad \mathbf{p} = (T^{01}, T^{02}, T^{03}) = \frac{E_0^2}{4\pi}\cos^2\xi\,\hat{\mathbf{z}}\,.$$

All components of T^{mn} vanish except $T^{33} = (E_0^2/4\pi)\cos^2\xi$. Hence, the wave has momentum flux in the $\hat{\mathbf{z}}$ direction. The energy flux T^{03} is equal to the momentum flux T^{33}, which corresponds to the quantum mechanical view that photons have energy and momentum $E = p$.

The **infinite** plane wave (68) has momentum density

$$\mathbf{p} = (4\pi)^{-1}\mathbf{E}\times\mathbf{B} = (4\pi)^{-1}E^2\hat{\mathbf{z}}\,.$$

Therefore, the angular momentum density $\mathbf{r}\times\mathbf{p}$ at a point on the z-axis is zero.[16] This is so even for a circularly polarized wave.

Now consider a plane wave like (68):

$$\mathbf{E} = E_0(x,y)e^{-i\omega(t-z)}\hat{\boldsymbol{x}} \qquad \mathbf{B} = E_0(x,y)e^{-i\omega(t-z)}\hat{\boldsymbol{y}}$$

of **finite** transverse extent, in the shape of a cylinder of radius R_0 around the z-axis. Suppose $E_0 = E_0(\rho)$ is constant for $\rho < R_0 - \varepsilon$, decreases in the annulus $R_0 - \varepsilon < \rho < R_0$, and vanishes for $\rho \geq R_0$, using cylindrical coordinates with $\rho^2 = x^2 + y^2$.

At $t = 0$, the $\mathbf{E}$ and $\mathbf{B}$ fields point in the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ directions in the plane $z = 0$, and point in the $-\hat{\boldsymbol{x}}$ and $-\hat{\boldsymbol{y}}$ directions in the plane $z = \pi/\omega$. Therefore, the field lines in the planes $z = 0$ and $z = \pi/\omega$ must connect to form closed loops. (Likewise for all pairs of planes $z = z_0$, $z = z_0 + \pi/\omega$.)

Therefore, the $\mathbf{E}$ and $\mathbf{B}$ fields have a z-component in the annulus $R_0 - \varepsilon < \rho < R_0$. $\mathbf{E}$ and $\mathbf{B}$ are not perpendicular in this annulus; they may be nearly parallel, so that $\mathbf{E}\times\mathbf{B}$ may be small. Nevertheless, the momentum density $(4\pi)^{-1}\mathbf{E}\times\mathbf{B}$ in the annulus has a transverse component, so there is a transverse energy flux.

[16]Off the z-axis there will be nonzero orbital angular momentum density, but the total angular momentum within a volume symmetric about the z-axis is, of course, zero.

Now consider the circularly polarized electromagnetic wave in (71)-(72):

$$\mathbf{E}_\pm = E_0 e^{-i\omega(t-z)}(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) \qquad \mathbf{B}_\pm = E_0 e^{-i\omega(t-z)}(\hat{\boldsymbol{y}} \mp i\hat{\boldsymbol{x}})\,. \tag{73}$$

Note that

$$\mathbf{E}_\pm = \pm i\mathbf{B}_\pm \quad \text{or} \quad \mathbf{B}_\pm = \mp i\mathbf{E}_\pm\,. \tag{74}$$

The fields in (73) have a vector potential

$$\mathbf{A}_\pm = -\frac{iE_0}{\omega} e^{-i\omega(t-z)}(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})\,. \tag{75}$$

To represent a wave packet with finite transverse extent, we modify (75):

$$\mathbf{A}_\pm = -\frac{iE_0(x,y)}{\omega} e^{-i\omega(t-z)}(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) \tag{76}$$

where $E_0(x,y)$ is constant throughout the wave packet except near the boundary, where it tapers off to zero. Now $\mathbf{B}_\pm$ picks up a $\hat{\boldsymbol{z}}$ component:

$$B_\pm^3 = \frac{\partial A_\pm^2}{\partial x} - \frac{\partial A_\pm^1}{\partial y} = \frac{1}{\omega}\left[i\frac{\partial E_0(x,y)}{\partial y} \pm \frac{\partial E_0(x,y)}{\partial x}\right] e^{-i\omega(t-z)}\,.$$

By (74), $\mathbf{E}_\pm$ now has a $\hat{\boldsymbol{z}}$ component:

$$E_\pm^3 = \pm iB_\pm^3 = \frac{i}{\omega}\left[\frac{\partial E_0(x,y)}{\partial x} \pm i\frac{\partial E_0(x,y)}{\partial y}\right] e^{-i\omega(t-z)}\,. \tag{77}$$

Therefore, the electric and magnetic fields are:[17]

$$\begin{gathered}\mathbf{E}_\pm = \left(E_0(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) + \frac{i}{\omega}\left[\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right]\hat{\boldsymbol{z}}\right) e^{-i\omega(t-z)} \\ \mathbf{B}_\pm = \mp i\mathbf{E}_\pm\end{gathered} \tag{78}$$

or, taking real parts:

$$\begin{gathered}\mathbf{E}_\pm = E_0 \cos\xi\,\hat{\boldsymbol{x}} \pm E_0 \sin\xi\,\hat{\boldsymbol{y}} + \frac{1}{\omega}\left[\frac{\partial E_0}{\partial x}\sin\xi \mp \frac{\partial E_0}{\partial y}\cos\xi\right]\hat{\boldsymbol{z}} \\ \mathbf{B}_\pm = \mp E_0 \sin\xi\,\hat{\boldsymbol{x}} + E_0 \cos\xi\,\hat{\boldsymbol{y}} + \frac{1}{\omega}\left[\pm\frac{\partial E_0}{\partial x}\cos\xi + \frac{\partial E_0}{\partial y}\sin\xi\right]\hat{\boldsymbol{z}}\,.\end{gathered} \tag{79}$$

§3. Angular momentum of an electromagnetic wave packet

The energy density $\mathcal{E}$ and angular momentum density $\mathbf{M}$ of a circularly polarized wave packet traveling in the $\hat{\boldsymbol{z}}$ direction can be calculated from the energy-momentum tensor $T^{\mu\nu}$. Thus, $M_z = xp_y - yp_x = xT^{02} - yT^{01}$. We will confirm later that $M_x = M_y = 0$.

[17] This is the solution to Problem 7.20 in Jackson (1975), p. 333. [6]

Assume that $E_0(x,y)$ is radially symmetric about the axis of propagation, so $E_0(x,y) = E_0(\rho)$ in polar coordinates, with $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Then:

$$\frac{\partial E_0}{\partial x} = \frac{\partial E_0}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial E_0}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{x}{\rho}\frac{\partial E_0}{\partial \rho}$$

and similarly for $\partial E_0/\partial y$. We also assume that $E_0(\rho)$ tapers off sufficiently gently at the boundary of the wave packet so that

$$\frac{\partial E_0}{\partial \rho} \ll \omega E_0\,. \tag{80}$$

Using $\mathbf{E}_+$ and $\mathbf{B}_+$ from (79):

$$8\pi T^{00} = E^2 + B^2 = 2E_0^2 + \frac{1}{\omega^2}\left[\left(\frac{\partial E_0}{\partial x}\right)^2 + \left(\frac{\partial E_0}{\partial y}\right)^2\right].$$

Since

$$(\partial E_0/\partial x)^2 + (\partial E_0/\partial y)^2 = (\partial E_0/\partial \rho)^2 \ll \omega^2 E_0^2$$

we have $4\pi T^{00} \approx E_0(x,y)^2$, or

$$\mathcal{E} = \frac{1}{4\pi}E_0(x,y)^2\,.$$

We also calculate

$$4\pi T^{01} = E_y B_z - E_z B_y = \frac{E_0}{\omega}\frac{\partial E_0}{\partial y}\,, \qquad 4\pi T^{02} = E_z B_x - E_x B_z = -\frac{E_0}{\omega}\frac{\partial E_0}{\partial x}\,.$$

Therefore,

$$4\pi M_z = xT^{02} - yT^{01} = -\frac{E_0}{\omega}\left(x\frac{\partial E_0}{\partial x} + y\frac{\partial E_0}{\partial y}\right) = -\frac{E_0}{\omega}\rho\frac{\partial E_0}{\partial \rho}\,.$$

Integrating over the x-y plane,

$$4\pi\int M_z\,dA = -\frac{2\pi}{\omega}\int E_0\frac{\partial E_0}{\partial \rho}\rho^2 d\rho = -\frac{\pi}{\omega}\int \frac{dE_0^2}{d\rho}\rho^2 d\rho = \frac{2\pi}{\omega}\int E_0^2\,\rho d\rho = \frac{1}{\omega}\int E_0^2\,dA\,.$$

We will show below that the wave packet has no x- or y-angular momentum. Therefore,

$$\int \mathbf{M}\,dV = \frac{\hat{\mathbf{z}}}{4\pi\omega}\int E_0^2\,dV\,. \tag{81}$$

The total angular momentum in (81) is independent of the coordinate $\mathbf{r}$, so we identify $\mathbf{M}$ as **spin angular momentum density**, denoted by $\mathbf{S}$. Hence, the ratio of z-spin angular momentum to energy of the wave packet is

$$\frac{\int S_z\,dV}{\int \mathcal{E}\,dV} = \frac{(4\pi\omega)^{-1}\int E_0^2\,dV}{(4\pi)^{-1}\int E_0^2\,dV} = \frac{1}{\omega}\,. \tag{82}$$

If we had used $\mathbf{E}_-$ instead of $\mathbf{E}_+$ we would have obtained $-1/\omega$. Eq. (82) reflects the quantum mechanical view that a photon has spin $\pm\hbar$ and energy $\hbar\omega$.[18]

To complete the analysis, we confirm that the wave packet has no x- or y-angular momentum. Since

$$4\pi T^{03} = E_x B_y - E_y B_x = E_0^2$$

we have:

$$4\pi M_x = yT^{03} - zT^{02} = yE_0^2 + z\frac{E_0}{\omega}\frac{\partial E_0}{\partial x} = E_0^2 \rho \sin\phi + z\frac{E_0}{\omega}\frac{\partial E_0}{\partial \rho}\cos\phi \, .$$

Integrating over the x-y plane, remembering that $E_0 = E_0(\rho)$:

$$4\pi \int M_x \, dA = \int \left(E_0^2 \rho \sin\phi + z\frac{E_0}{\omega}\frac{\partial E_0}{\partial \rho}\cos\phi \right) \rho d\rho d\phi = 0 \, .$$

Similarly, $4\pi \int M_y \, dA = 0$. Thus, the angular momentum of the wave packet has only a z-component.

§4. Angular momentum of a gravitational wave packet

Our goal is to derive the ratio

$$\frac{\int S_z \, dV}{\int \mathcal{E} \, dV} = \pm\frac{2}{\omega} \tag{83}$$

for circularly polarized, gravitational plane wave packet.[19] Consider a monochromatic wave packet $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ traveling in the $\hat{\mathbf{z}}$ direction. We work in the transverse-traceless gauge, so $h = \psi = 0$, $h_{\mu\nu} = \psi_{\mu\nu}$ from (17), ${\psi^{\mu\nu}}_{,\nu} = 0$ from (15), and $\psi^{\mu 0} = \psi^{0\nu} = 0$. Let the metric perturbation be

$$\psi^{\mu\nu} = \Omega^{\mu\nu}(x,y)e^{-i\omega(t-z)}$$

where we take the real part as the wave.

For an infinite plane wave, we know from (42) that $\Omega_{\mu\nu}(x,y)$ is a linear combination $a\Omega_{\mu\nu}^{+} + b\Omega_{\mu\nu}^{\times}$ of the polarization tensors (41); and if a and b are complex numbers, then $\psi_{\mu\nu}$ may be elliptically or circularly polarized. However, for a **finite** wave, $\Omega_{\mu\nu}(x,y)$ and $\psi_{\mu\nu}$ are more complicated.

The harmonic coordinates condition (15) gives

$${\psi^{\mu\nu}}_{,\nu} = \left(-i\omega\Omega^{\mu 0} + \frac{\partial \Omega^{\mu 1}}{\partial x} + \frac{\partial \Omega^{\mu 2}}{\partial y} + i\omega\Omega^{\mu 3} \right) e^{-i\omega(t-z)} = 0 \, .$$

---

[18]Eq. (82) can also be obtained by writing $\mathbf{r}\times\mathbf{p} = \mathbf{r}\times(\mathbf{E}\times\mathbf{B}) = \mathbf{r}\times[\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{A})]$ and using vector analysis to isolate the spin angular momentum; see Ohanian (2007) §12.3 [7], Ohanian (1986) [8], and Rohrlich (2007) §4.10 [9]. This approach does not easily extend to the gravitational field; see Barnett (2014) [10].

[19]Adapted from Barker (2017) [11].

Hence, for $m = 1,2,3$:

$$\frac{\partial \Omega^{m1}}{\partial x} + \frac{\partial \Omega^{m2}}{\partial y} + i\omega\Omega^{m3} = 0\,. \tag{84}$$

The general solution to (84) is

$$\Omega^{mn} = \begin{pmatrix} \Omega^{11}(x,y) & \Omega^{12}(x,y) & \dfrac{i}{\omega}\left[\dfrac{\partial \Omega^{11}}{\partial x} + \dfrac{\partial \Omega^{12}}{\partial y}\right] \\ \cdots & \Omega^{22}(x,y) & \dfrac{i}{\omega}\left[\dfrac{\partial \Omega^{12}}{\partial x} + \dfrac{\partial \Omega^{22}}{\partial y}\right] \\ \cdots & \cdots & -\dfrac{1}{\omega^2}\left[\dfrac{\partial^2 \Omega^{11}}{\partial x^2} + 2\dfrac{\partial^2 \Omega^{12}}{\partial x \partial y} + \dfrac{\partial^2 \Omega^{22}}{\partial y^2}\right] \end{pmatrix} \tag{85}$$

where the lower triangle is completed by symmetry. We set

$$\begin{aligned} \Omega^{11}(x,y) &= -\Omega^{22}(x,y) = \Psi(x,y) \\ \Omega^{12}(x,y) &= \Omega^{21}(x,y) = i\Psi(x,y) \end{aligned} \tag{86}$$

where $\Psi(x,y)$ is a real-valued function that is constant in some finite region of the x-y plane and tapers off to zero at the boundary. We choose $\Psi(x,y)$ to be symmetric about the z-axis, so $\Psi(x,y) = \Psi(\rho)$ in polar coordinates, with $x = \rho\cos\phi$ and $y = \rho\sin\phi$. We have:

$$\Psi_{,x} = \frac{\partial \Psi}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial \Psi}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{x}{\rho}\Psi_{,\rho}$$

$$\Psi_{,xx} = \frac{x}{\rho}\frac{\partial \Psi_{,x}}{\partial \rho} = \frac{x}{\rho}\frac{\partial}{\partial \rho}\left(\frac{x}{\rho}\Psi_{,\rho}\right) = \frac{x}{\rho}\frac{\partial(\cos\phi\,\Psi_{,\rho})}{\partial \rho} = \frac{x^2}{\rho^2}\Psi_{,\rho\rho}$$

and similarly for $\Psi_{,y}$, $\Psi_{,yy}$ and $\Psi_{,xy}$. We also choose $\Psi(\rho)$ to taper off sufficiently gently at the boundary of the wave packet so that

$$\frac{\partial \Psi}{\partial \rho} \ll \omega\Psi\,, \qquad \frac{\partial^2 \Psi}{\partial \rho^2} \ll \omega\frac{\partial \Psi}{\partial \rho} \ll \omega^2\Psi\,. \tag{87}$$

Then Ω^{33} is negligible compared to the other Ω^{mn}, and we have

$$\psi^{mn} = \psi_{mn} = \begin{pmatrix} \Psi(x,y) & i\Psi(x,y) & i\omega^{-1}[\Psi_{,x} + i\Psi_{,y}] \\ \cdots & -\Psi(x,y) & i\omega^{-1}[i\Psi_{,x} - \Psi_{,y}] \\ \cdots & \cdots & 0 \end{pmatrix} e^{-i\xi} \tag{88}$$

where $\xi = \omega(t - z)$. Taking real parts:

$$\psi_{mn} = \begin{pmatrix} \Psi(x,y)\cos\xi & \Psi(x,y)\sin\xi & \omega^{-1}[\Psi_{,x}\sin\xi - \Psi_{,y}\cos\xi] \\ \cdots & -\Psi(x,y)\cos\xi & -\omega^{-1}[\Psi_{,x}\cos\xi + \Psi_{,y}\sin\xi] \\ \cdots & \cdots & 0 \end{pmatrix}. \tag{89}$$

Notice that, except near the boundary, we have

$$\psi_{mn} = \left\{\begin{pmatrix} \Psi & 0 & 0 \\ 0 & -\Psi & 0 \\ 0 & 0 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & \Psi & 0 \\ \Psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} e^{-i\xi}$$

or

$$\psi_{\mu\nu} = \Psi e^{-i\omega(t-z)}(\Omega^{+}_{\mu\nu} + i\Omega^{\times}_{\mu\nu})\,,$$

which is a circularly polarized wave analogous to $\mathbf{E}_+$ in (73).

The energy and angular momentum densities of the wave packet can be calculated from the components of an appropriate energy-momentum pseudotensor $t^{\mu\nu}$. As a first approach, we try the Einstein energy-momentum pseudotensor, which for a plane wave has the form:[20]

$$32\pi t^{\mu\nu} = \left(u_{\alpha\beta}u^{\alpha\beta} - \tfrac{1}{2}u^2\right)k^\mu k^\nu\,. \tag{90}$$

Since $h_{\alpha\beta,\sigma} = g_{\alpha\beta,\sigma} = u_{\alpha\beta}k_\sigma$, we have from (16) and (17):

$$-\psi_{,\sigma} = h_{,\sigma} = \eta^{\alpha\beta}h_{\alpha\beta,\sigma} = \eta^{\alpha\beta}u_{\alpha\beta}k_\sigma = uk_\sigma\,,$$

$$u_{\alpha\beta}k_\sigma = h_{\alpha\beta,\sigma} = \psi_{\alpha\beta,\sigma} - \tfrac{1}{2}\eta_{\alpha\beta}\psi_{,\sigma}\,.$$

Therefore:

$$\begin{aligned} 32\pi t_{\mu\nu} &= u_{\alpha\beta}k_\mu u^{\alpha\beta}k_\nu - \tfrac{1}{2}uk_\mu uk_\nu \\ &= \left(\psi_{\alpha\beta,\mu} - \tfrac{1}{2}\eta_{\alpha\beta}\psi_{,\mu}\right)\left(\psi^{\alpha\beta}{}_{,\nu} - \tfrac{1}{2}\eta^{\alpha\beta}\psi_{,\nu}\right) - \tfrac{1}{2}(-\psi_{,\mu})(-\psi_{,\nu}) \\ &= \psi_{\alpha\beta,\mu}\psi^{\alpha\beta}{}_{,\nu} - \tfrac{1}{2}\psi_{,\mu}\psi_{,\nu}\,. \end{aligned}$$

In the TT gauge, this simplifies to

$$32\pi t_{\mu\nu} = \psi_{\rho\sigma,\mu}\psi^{\rho\sigma}{}_{,\nu}\,. \tag{91}$$

Therefore, since $\psi_{\mu 0} = \psi_{0\nu} = 0$:

$$32\pi t_{00} = \psi_{rs,0}\psi_{rs,0} = 2\omega^2\Psi^2(x,y) + 2\Psi^2_{,x} + 2\Psi^2_{,y} \approx 2\omega^2\Psi^2(x,y)$$

on account of (87). Therefore,

$$\mathcal{E} = t^{00} = t_{00} = \frac{\omega^2}{16\pi}\Psi^2(x,y)\,. \tag{92}$$

However, a problem arises when (91) is used to compute the transverse components of the momentum density. For t_{01}, we obtain

[20] Dirac (1975), Eq. (33.9).

$$32\pi t_{01} = \psi_{rs,0}\psi_{rs,1} = \frac{2}{\omega}\left(\Psi_{,y}\Psi_{,xx} - \Psi_{,x}\Psi_{,xy}\right) = \frac{2}{\omega}\left(\frac{yx^2 - x^2y}{\rho^3}\right)\Psi_{,\rho}\Psi_{,\rho\rho} = 0\,.$$

Similarly, $t_{02} = 0$. The problem is that (90)-(91) is the energy-momentum pseudotensor for an **infinite** plane wave, in which there is no momentum component perpendicular to the direction of propagation. For a finite wave packet

$$\psi^{\mu\nu}(\mathbf{r}) = \Omega^{\mu\nu}(x,y)e^{-i\omega(t-z)}$$

with $\Omega^{\mu\nu}$ given by (88), $g^{\mu\nu}(\mathbf{r}) = \eta^{\mu\nu} + \psi^{\mu\nu}(\mathbf{r})$ is not a function of $\xi = \omega(t-z)$, so (90) is not an appropriate starting point.

We use the Landau-Lifshitz energy-momentum pseudotensor:[21]

$$\begin{aligned}16\pi t^{\mu\nu} = {} & h^{\mu\nu}{}_{,\rho}h^{\rho\sigma}{}_{,\sigma} - h^{\mu\rho}{}_{,\rho}h^{\nu\sigma}{}_{,\sigma} - h_{\rho\sigma}{}^{,\mu}h^{\nu\rho,\sigma} - h_{\rho\sigma}{}^{,\nu}h^{\mu\rho,\sigma} + \eta_{\rho\sigma}h^{\mu\rho}{}_{,\lambda}h^{\nu\sigma,\lambda} \\ & + \eta^{\mu\nu}\left(\tfrac{1}{2}h_{\lambda\rho,\sigma}h^{\lambda\sigma,\rho} - \tfrac{1}{4}h_{\rho\sigma,\lambda}h^{\rho\sigma,\lambda} + \tfrac{1}{8}h^{\rho}_{\rho,\lambda}h^{\sigma,\lambda}_{\sigma}\right) + \tfrac{1}{2}h^{\rho\sigma,\mu}h_{\rho\sigma}{}^{,\nu} - \tfrac{1}{4}h^{\rho,\mu}_{\rho}h^{\sigma,\nu}_{\sigma}\,.\end{aligned} \tag{93}$$

In the TT gauge we have $h_{\mu\nu} = \psi_{\mu\nu}$, so we may replace $h_{\mu\nu}$ with $\psi_{\mu\nu}$. Since $\psi^{\mu\nu}{}_{,\nu} = 0$ and $\psi = 0$, the 1st, 2nd, 8th and 10th terms vanish, and we are left with:

$$\begin{aligned}16\pi t^{\mu\nu} = {} & -\psi_{\rho\sigma}{}^{,\mu}\psi^{\nu\rho,\sigma} - \psi_{\rho\sigma}{}^{,\nu}\psi^{\mu\rho,\sigma} + \eta_{\rho\sigma}\psi^{\mu\rho}{}_{,\lambda}\psi^{\nu\sigma,\lambda} \\ & + \eta^{\mu\nu}\left(\tfrac{1}{2}\psi_{\lambda\rho,\sigma}\psi^{\lambda\sigma,\rho} - \tfrac{1}{4}\psi_{\rho\sigma,\lambda}\psi^{\rho\sigma,\lambda}\right) + \tfrac{1}{2}\psi^{\rho\sigma,\mu}\psi_{\rho\sigma}{}^{,\nu}\,.\end{aligned} \tag{94}$$

To see how (94) generalizes (91), note that for an *infinite* plane wave in the $\hat{\boldsymbol{z}}$ direction, $\psi_{\rho\sigma,\lambda}$ vanishes unless $\lambda =$ 0 or 3, while $\psi_{\rho\sigma}$ vanishes unless $\rho, \sigma =$ 1 or 2. Hence, in this case the 1st, 2nd and 4th terms of (94) vanish. Additionally, the 3rd term contains the factor

$$\psi^{\mu\rho}{}_{,\lambda}\psi^{\nu\sigma,\lambda} = u^{\mu\rho}k_\lambda u^{\nu\sigma}k^\lambda$$

which vanishes, since $k_\lambda k^\lambda = 0$. Likewise, the 5th term vanishes. Thus, for an infinite plane wave, (94) reduces to (91).

Using (94), we recalculate t^{00} to confirm that the previous expression (92) for $\mathcal{E}$ remains correct. Since $\psi^{0\nu} = 0$, the first three terms vanish, and we are left with

$$16\pi t^{00} = \tfrac{1}{2}\psi_{lr,s}\psi^{ls,r} - \tfrac{1}{4}\psi_{rs,\lambda}\psi^{rs,\lambda} + \tfrac{1}{2}\psi_{rs,0}\psi_{rs,0}\,. \tag{95}$$

We examine the terms in reverse order. In the calculation leading up to (92) we found that $\psi_{rs,0}\psi_{rs,0} \approx 2\omega^2\Psi^2$. Hence, the third term $= \omega^2\Psi^2$.

The second term is proportional to $\psi_{rs,0}\psi_{rs,0} - \psi_{rs,l}\psi_{rs,l}$. Note that $\psi^2_{rs,0} - \psi^2_{rs,3} = 0$, since x^0 and x^3 appear only in the combination $\omega(t-z)$. Thus

$$\psi_{rs,0}\psi_{rs,0} - \psi_{rs,l}\psi_{rs,l} = -\psi_{rs,m}\psi_{rs,m} \ \ (\text{sum on } m = 1,2)\,.$$

[21] Landau and Lifshitz (1975), §96 [12].

Note that

$$\psi_{11,1} = \Psi_{,x} \cos\xi \ll \omega\Psi$$
$$\psi_{12,1} = \Psi_{,x} \sin\xi \ll \omega\Psi$$
$$\psi_{13,1} = \omega^{-1}\left(\Psi_{,xx} \sin\xi - \Psi_{,yx} \cos\xi\right) \ll \omega\Psi$$

and so on. In general, $\psi_{rs,m} \ll \omega\Psi$ for $m = 1, 2$. Hence, the second term in (95) is negligible compared to the third term.

The first term is proportional to $\psi_{lr,s}\psi_{ls,r}$. Considering all combinations of $1 \le l, r, s \le 3$, we find that $\psi_{lr,s}\psi_{ls,r} \ll \omega^2\Psi^2$. Hence, this term is also negligible compared to the third term. Thus, we have confirmed that (92) is correct.

Using (94), we now calculate t^{01}. We have:

$$16\pi t^{01} = \psi_{rs,0}\psi_{1r,s} - \tfrac{1}{2}\psi_{rs,0}\psi_{rs,1}\,. \tag{96}$$

Now we see precisely why (91) fails to detect the transverse components of angular momentum. Eq. (91) gives $32\pi t^{01} = -\psi_{rs,0}\psi_{rs,1}$, which corresponds to the second term in (96) and vanishes; see the calculation following (92). It is the first term that matters, as we now see.

Since $\psi_{rs,0}\psi_{rs,1} = 0$, we have

$$16\pi t^{01} = \psi_{rs,0}\psi_{1r,s}\,. \tag{97}$$

We calculate:

$$\psi_{rs,0}\psi_{1r,s} = 2\omega\Psi\Psi_{,y} + \frac{1}{\omega}\Psi_{,y}\left(\Psi_{,xx}\sin^2\xi + \Psi_{,yy}\cos^2\xi\right) - \frac{1}{\omega}\Psi_{,x}\Psi_{,xy}\cos 2\xi$$
$$+ \frac{1}{2\omega}\left(\Psi_{,x}\Psi_{,xx} - 2\Psi_{,y}\Psi_{,xy} - \Psi_{,x}\Psi_{,yy}\right)\sin 2\xi\,.$$

Time-averaging over multiple cycles, we have

$$\overline{\psi_{rs,0}\psi_{1r,s}} = 2\omega\Psi\Psi_{,y} + \frac{1}{2\omega}\Psi_{,y}\left(\Psi_{,xx} + \Psi_{,yy}\right) = \Psi_{,y}\left[2\omega\Psi + \frac{1}{2\omega}\left(\Psi_{,xx} + \Psi_{,yy}\right)\right]$$
$$= \Psi_{,y}\left(2\omega\Psi + \frac{1}{2\omega}\Psi_{,\rho\rho}\right).$$

Since $\Psi_{,\rho\rho} \ll \omega^2\Psi$, we ignore the second term. Therefore, $\overline{\psi_{rs,0}\psi_{1r,s}} = 2\omega\Psi\Psi_{,y}$, so

$$\overline{t^{01}} = \frac{\omega}{8\pi}\Psi\Psi_{,y}\,. \tag{98}$$

A similar calculation shows that

$$\overline{\psi_{rs,0}\psi_{2r,s}} = -2\omega\Psi\Psi_{,x} - \frac{1}{2\omega}\Psi_{,x}(\Psi_{,xx} + \Psi_{,yy})$$

so

$$\overline{t^{02}} = -\frac{\omega}{8\pi}\Psi\Psi_{,x}\,. \tag{99}$$

Therefore,

$$\overline{M_z(x,y)} = x\overline{t^{02}} - y\overline{t^{01}} = -\frac{\omega}{8\pi}\left(x\Psi\Psi_{,x} + y\Psi\Psi_{,y}\right) = -\frac{\omega}{8\pi}\rho\Psi\Psi_{,\rho}\,.$$

Integrating over the x-y plane:

$$\int \overline{M_z}\,dA = -\frac{\omega}{4}\int \Psi\Psi_{,\rho}\,\rho^2 d\rho = -\frac{\omega}{8}\int \frac{d\Psi^2}{d\rho}\rho^2 d\rho = \frac{\omega}{4}\int \Psi^2\,\rho d\rho = \frac{\omega}{8\pi}\int \Psi^2\,dA\,.$$

We will see below that the wave packet has no x- or y-angular momentum. Therefore,

$$\int \overline{\mathbf{M}}\,dV = \frac{\omega\hat{\mathbf{z}}}{8\pi}\int \Psi^2\,dV\,. \tag{100}$$

The total angular momentum in (100) is independent of the coordinate $\mathbf{r}$, so we identify this as spin angular momentum, denoted by $\mathbf{S}$.

Using (92), the ratio of z-spin angular momentum to energy is

$$\frac{\int \overline{S_z}\,dV}{\int \mathcal{E}\,dV} = \frac{(8\pi)^{-1}\omega\int E_0^2\,dV}{(16\pi)^{-1}\omega^2\int E_0^2\,dV} = \frac{2}{\omega}\,. \tag{101}$$

If we had chosen $\Omega^{12}(x,y) = -i\Psi(x,y)$ in (86), then $\psi^{\mu\nu}$ would be circularly polarized like $\mathbf{E}_-$ in (72) and we would obtain a minus sign in (101). This establishes (83), which reflects the quantum mechanical view that a graviton has spin $\pm 2\hbar$ and energy $\hbar\omega$.

To complete the analysis, we confirm that the wave packet has no x- or y-angular momentum. Using (94) we calculate t^{03}:

$$16\pi t^{03} = \psi_{rs,0}\psi_{3r,s} - \tfrac{1}{2}\psi_{rs,0}\psi_{rs,3}\,. \tag{102}$$

Straightforward calculation shows that

$$\psi_{rs,0}\psi_{3r,s} = -\Psi\Psi_{,xx} - \Psi\Psi_{,yy} - \Psi_{,x}^2 - \Psi_{,y}^2$$

$$-\tfrac{1}{2}\psi_{rs,0}\psi_{rs,3} = \omega^2\Psi^2 + \Psi_{,x}^2 + \Psi_{,y}^2\,.$$

Hence,

$$16\pi t^{03} = \omega^2\Psi^2 - \Psi\Psi_{,xx} - \Psi\Psi_{,yy}$$

Since $\Psi\Psi_{,xx}$ and $\Psi\Psi_{,yy}$ are $\ll \omega^2\Psi^2$, we ignore these terms. So

$$t^{03} = \frac{\omega^2}{16\pi}\Psi^2\,.$$

Therefore,

$$\overline{M_x} = z\overline{t^{02}} - y\overline{t^{03}} = -\frac{z\omega}{8\pi}\Psi\Psi_{,x} - \frac{y\omega^2}{16\pi}\Psi^2 = -\frac{z\omega\cos\phi}{8\pi}\Psi\Psi_{,\rho} - \frac{\rho\omega^2\sin\phi}{16\pi}\Psi^2 .$$

Integrating over the x-y plane:

$$-16\pi\int \overline{M_x}\, dA = \int\left(2z\omega\cos\phi\,\Psi\Psi_{,\rho} + \rho\omega^2\sin\phi\right)\rho d\rho d\phi = 0 .$$

Likewise, $\int \overline{M_y}\, dA = 0$.

E. Radiating systems

An electromagnetic or gravitational plane wave packet can be imagined by considering a (finite) system of charges or masses in accelerated motion, radiating energy. Far from the system, which we take to be located near the origin of our coordinate system, waves traveling in a particular direction $\hat{\mathbf{r}}$ are plane waves, occupying a solid angle $\delta\Omega$ enclosing the ray pointing in the $\hat{\mathbf{r}}$ direction During a time interval Δt, a wave packet is generated traveling in the $\hat{\mathbf{r}}$ direction, enclosed within $\delta\Omega$ with length $c\Delta t$. If the system radiates circularly polarized waves in the $\hat{\mathbf{r}}$ direction, then the wave packet will be circularly polarized.

The energy and angular momentum contained in the wave packet is equal to the energy and angular momentum lost by the system during the interval Δt. We examine whether (83) can be derived from an understanding of the energy and angular momentum of the system.

§1. Electromagnetic dipole and quadrupole radiation

We recall the formulae for EM dipole and quadrupole radiation. The general solution to the wave equation

$$\Box A^\mu(t,\mathbf{r}) = 4\pi J^\mu(t,\mathbf{r})$$

is given by the retarded potential[22]

$$A^\mu(t,\mathbf{r}) = \int \frac{J^\mu(t - |\mathbf{r}-\mathbf{r}'|, \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV' .$$

Suppose the origin lies near the system and $r \equiv |\mathbf{r}|$ is much greater than the size Σ of the system, so that

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\bullet\mathbf{r}'$$

Neglecting terms of $O(1/r^2)$ and retaining only terms of $O(1/r)$, we have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r - \hat{\mathbf{r}}\bullet\mathbf{r}'} = \frac{1}{r} + \frac{\hat{\mathbf{r}}\bullet\mathbf{r}'}{r^2} + \cdots \approx \frac{1}{r}$$

[22]See any standard EM reference; for example, Ohanian (2007), Chap. 13-14.

so we obtain a simplified expression:

$$A^{\mu}(t,\mathbf{r}) = \frac{1}{r}\int J^{\mu}(t - r + \hat{\mathbf{r}}\bullet\mathbf{r}',\mathbf{r}')\,dV'\,. \tag{103}$$

From (103), we can calculate $\mathbf{E} = -\nabla A^0 - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \operatorname{curl}\mathbf{A}$. Keeping terms up to $O(1/r)$, we find:[23]

$$\mathbf{B} = -\hat{\mathbf{r}}\times\frac{\partial\mathbf{A}}{\partial t}\,,\qquad \mathbf{E} = -\hat{\mathbf{r}}\times\mathbf{B}\,.$$

Note that $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to each other and to $\hat{\mathbf{r}}$.

The $O(1/r^2)$ terms discarded are of order JV'/r^2, while the $O(1/r)$ terms retained are of order JV'/rT, where T is the "period" of the system, satisfying $|\partial\mathbf{J}/\partial t| \approx |\mathbf{J}/T|$. Therefore, in neglecting the $O(1/r^2)$ terms, we are assuming that $\lambda = T \ll r$, where λ is the wavelength of the emitted radiation. Note $\lambda = T$ in units where $c = 1$.

We make the further assumption that $\Sigma \ll \lambda$, which combined with $\lambda \ll r$, automatically gives us $\Sigma \ll r$.[24] This allows us to approximate $\mathbf{J}$ by its Taylor expansion in the first variable:

$$\mathbf{J}(t - r + \hat{\mathbf{r}}\bullet\mathbf{r}',\mathbf{r}') = \mathbf{J}(t - r,\mathbf{r}') + (\hat{\mathbf{r}}\bullet\mathbf{r}')\frac{\partial\mathbf{J}(t - r,\mathbf{r}')}{\partial t} + \cdots.$$

The first and second terms in the expansion are of order $|\mathbf{J}|$ and $\Sigma|\mathbf{J}|/T$, respectively. If $\Sigma \ll \lambda = T$, then the second term is small and the approximation is valid. Since Σ/T approximates the average velocity of the particles in the system, $\Sigma \ll \lambda$ corresponds to $v \ll 1$. In other words, the motion of the system is assumed to be non-relativistic.

Using this Taylor expansion in (103), we obtain

$$\mathbf{A}(t,\mathbf{r}) = \frac{1}{r}\int [\![\mathbf{J}]\!]\,dV' + \frac{1}{r}\int (\hat{\mathbf{r}}\bullet\mathbf{r}')[\![\partial\mathbf{J}/\partial t]\!]\,dV' + \cdots \tag{104}$$

where $[\![\cdots]\!]$ means the bracketed expression is evaluated at the retarded time $t - r$. The first term in (104) corresponds to **electric dipole radiation**. We rewrite this term:

$$\mathbf{A}_{\text{dip}}(t,\mathbf{r}) = -\frac{1}{r}\int [\![\nabla'\bullet\mathbf{J}]\!]\mathbf{r}'\,dV' = \frac{1}{r}\int [\![\partial\rho/\partial t]\!]\mathbf{r}'\,dV' = \frac{1}{r}\frac{d}{dt}\int [\![\rho]\!]\mathbf{r}'\,dV' = \frac{1}{r}[\![\dot{\boldsymbol{\mu}}]\!]$$

where $\boldsymbol{\mu} = \int \rho\mathbf{r}'\,dV'$ is the dipole moment of the system and a dot $\dot{\square}$ denotes a time derivative. The first step follows from integration by parts. Thus, we find:

$$\mathbf{A}_{\text{dip}}(t,\mathbf{r}) = \frac{[\![\dot{\boldsymbol{\mu}}]\!]}{r}\,,\qquad \mathbf{B}_{\text{dip}}(t,\mathbf{r}) = -\frac{\hat{\mathbf{r}}}{r}\times[\![\ddot{\boldsymbol{\mu}}]\!]\,,\qquad \mathbf{E}_{\text{dip}} = -\hat{\mathbf{r}}\times\mathbf{B}\,.$$

[23]Ibid., §14.1.

[24]The assumption $\Sigma \ll \lambda$ is equivalent to assuming that $\mathbf{J}$ does not vary too rapidly with time. For example, suppose $\mathbf{J}(t',\mathbf{r}') = J_0 \sin\omega t'\,\hat{\mathbf{z}}$. For the Taylor approximation to be valid, we require

$$|(\hat{\mathbf{r}}\bullet\mathbf{r}')\,\partial\mathbf{J}(t - r,\mathbf{r}')/\partial t| \ll |\mathbf{J}(t - r,\mathbf{r}')|$$

or $\Sigma\omega J_0 \ll J_0$; hence, $\omega \ll \Sigma^{-1}$.

The dipole power emitted into a solid angle $d\Omega$ in the $\hat{\mathbf{r}}$ direction is

$$dP_{\text{dip}} = \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})\bullet\hat{\mathbf{r}}\,r^2 d\Omega = \frac{1}{4\pi}|\mathbf{B}|^2 r^2 d\Omega = \frac{1}{4\pi}|\hat{\mathbf{r}}\times[\![\ddot{\boldsymbol{\mu}}]\!]|^2 d\Omega$$

or

$$\frac{dP_{\text{dip}}}{d\Omega} = \frac{1}{4\pi}[\![\ddot{\boldsymbol{\mu}}]\!]^2 \sin^2\Theta \tag{105}$$

where Θ is the angle between $\hat{\mathbf{r}}$ and $[\![\ddot{\boldsymbol{\mu}}]\!]$. The total power emitted from the system is[25]

$$P_{\text{dip}} = \frac{2}{3}[\![\ddot{\boldsymbol{\mu}}]\!]^2\,. \tag{106}$$

The second term in (104) can be written[26]

$$\frac{\hat{\mathbf{r}}}{r}\times\frac{d}{dt}\int\frac{[\![\mathbf{J}]\!]}{2}\times\mathbf{r}'\,dV' + \frac{1}{6r}\frac{d^2}{dt^2}\int[3(\hat{\mathbf{r}}\bullet\mathbf{r}')\mathbf{r}' - r'^2\hat{\mathbf{r}}][\![\rho]\!]\,dV'\,.$$

These two terms correspond to **magnetic dipole radiation** *and* **electric quadrupole radiation**, respectively. We will not be concerned with magnetic dipole radiation.

Define the vector $\mathbf{q} = \mathbf{q}(\hat{\mathbf{r}})$ by

$$\mathbf{q} \equiv \int[3(\hat{\mathbf{r}}\bullet\mathbf{r}')\mathbf{r}' - r'^2\hat{\mathbf{r}}]\rho\,dV'$$

and the **electric quadrupole tensor**

$$Q_{kl} = \int(3x'_k x'_l - \delta_{kl}r'^2)\rho\,dV'\,.$$

Then

$$\mathbf{q} = \mathrm{Q}\hat{\mathbf{r}} \qquad \text{or} \qquad q_k = Q_{kl}n_l$$

where Q denotes the matrix Q_{kl} and $\hat{\mathbf{r}} = (n_1, n_2, n_3)$. The magnetic field is then

$$\mathbf{B} = -\hat{\mathbf{r}}\times\frac{\partial\mathbf{A}}{\partial t} = -\frac{\hat{\mathbf{r}}}{6r}\times[\![\dddot{\mathbf{q}}]\!]\,.$$

The power emitted into a solid angle $d\Omega$ in the $\hat{\mathbf{r}}$ direction is:

$$dP_{\text{quad}} = \frac{1}{4\pi}|(\mathbf{E}\times\mathbf{B})\bullet\hat{\mathbf{r}}|r^2 d\Omega = \frac{1}{4\pi}|\mathbf{B}|^2 r^2 d\Omega = \frac{1}{144\pi}|\hat{\mathbf{r}}\times[\![\dddot{\mathbf{q}}]\!]|^2 d\Omega\,.$$

[25]Note that $P_{\text{dip}} = (2/3c^3)[\![\ddot{\boldsymbol{\mu}}]\!]^2$ in conventional units. This is easily inferred by dimensional analysis: the units of power are [charge]^2[distance]$^{-1}$[time]$^{-1}$ and the units of $|\ddot{\boldsymbol{\mu}}|^2$ are [charge]^2[distance]^2[time]$^{-4}$. Hence, $[\![\ddot{\boldsymbol{\mu}}]\!]^2$ must be multiplied by c^{-3} [distance]$^{-3}$[time]^3. For a single charged particle, $\ddot{\boldsymbol{\mu}} = q\ddot{\mathbf{r}}'$ and we obtain the *Larmor formula* $P = 2q^2a^2/3c^2$, where $a = |\ddot{\mathbf{r}}|^2$ is the acceleration of the particle.

[26]Ohanian (2007), §14.4-14.5.

From Lagrange's identity $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\bullet\mathbf{b})^2$, we have

$$|\hat{\mathbf{r}}\times[\![\ddot{\mathbf{q}}]\!]|^2 = \ddot{\mathbf{q}}^2 - (\ddot{\mathbf{q}}\bullet\hat{\mathbf{r}})^2 \,.$$

Therefore,

$$\frac{dP_{\text{quad}}}{d\Omega} = \frac{1}{144\pi}[\![\dddot{Q}_{kl}\dddot{Q}_{km}n_l n_m - \dddot{Q}_{kl}\dddot{Q}_{mr}n_k n_l n_m n_r]\!] \,. \tag{107}$$

This angular distribution is rather complicated. Integrating over all space, we obtain the total power emitted by the system:

$$P_{\text{quad}} = \frac{1}{144\pi}\left([\![\dddot{Q}_{kl}\dddot{Q}_{km}]\!]\int n_l n_m \, d\Omega - [\![\dddot{Q}_{kl}\dddot{Q}_{mr}]\!]\int n_k n_l n_m n_r \, d\Omega\right).$$

Direct computation using $n_x = \sin\theta\cos\phi$, $n_y = \sin\theta\sin\phi$, $n_z = \cos\theta$ shows that[27]

$$\int n_k n_l \, d\Omega = \frac{4\pi}{3}\delta_{kl} \,, \qquad \int n_k n_l n_m n_r \, d\Omega = \frac{4\pi}{15}(\delta_{kl}\delta_{mr} + \delta_{km}\delta_{lr} + \delta_{kr}\delta_{lm}) \,. \tag{108}$$

Plugging in the values of these integrals, we find:

$$P_{\text{quad}} = \frac{1}{180}[\![\dddot{Q}_{kl}\dddot{Q}_{kl}]\!] = \frac{1}{180}[\![\dddot{\mathbf{Q}}_k \bullet \dddot{\mathbf{Q}}_k]\!] \tag{109}$$

where $\mathbf{Q}_k \equiv (Q_{kx}, Q_{ky}, Q_{kz})$. Note that $P_{\text{quad}} = O(1/c^5)$, while $P_{\text{dip}} = O(1/c^3)$; see footnotes 25 and 33.

§2. Example: a charge in uniform circular motion

We examine a system that generates circularly polarized waves along the z-axis. Consider a charge q in uniform circular motion in the x-y plane with

$$\mathbf{r}' = r_0\cos\omega t\,\hat{\boldsymbol{x}} + r_0\sin\omega t\,\hat{\boldsymbol{y}} \,.$$

We have $\ddot{\boldsymbol{\mu}} = q\ddot{\mathbf{r}}' = -q\omega^2\mathbf{r}'$. Then

$$\frac{dP_{\text{dip}}}{d\Omega} = \frac{1}{4\pi}|\hat{\mathbf{r}}\times[\![\ddot{\boldsymbol{\mu}}]\!]|^2 = \frac{q^2\omega^4}{4\pi}|\hat{\mathbf{r}}\times[\![\mathbf{r}']\!]|^2$$
$$= \frac{q^2\omega^4 r_0^2}{4\pi}[\![n_3^2 + n_1^2\sin^2\omega t + n_2^2\cos^2\omega t - n_1 n_2\sin 2\omega t]\!] \,.$$

Taking the time average over many cycles:

$$\overline{\frac{dP_{\text{dip}}}{d\Omega}} = \frac{q^2\omega^4 r_0^2}{4\pi}\left(n_3^2 + \frac{n_1^2 + n_2^2}{2}\right).$$

In spherical coordinates, we have $n_1 = \sin\theta\cos\phi$, $n_2 = \sin\theta\sin\phi$, $n_3 = \cos\theta$, so

[27] These integrals also follow from symmetry arguments; see Landau and Lifshitz (1975), p. 189.

$$\overline{\frac{dP_{\text{dip}}}{d\Omega}} = \frac{q^2\omega^4 r_0^2}{4\pi}\left(1 - \tfrac{1}{2}\sin^2\theta\right).$$

The angular distribution of electric dipole radiation intensity is shown in Figure 1.

To calculate the angular distribution of quadrupole radiation, we need the quadrupole tensor:

$$Q_{kl} = qr_0^2 \begin{pmatrix} 3\cos^2\omega t - 1 & 3\sin\omega t\cos\omega t & 0 \\ 3\sin\omega t\cos\omega t & 3\sin^2\omega t - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{110}$$

and its third derivatives

$$\dddot{Q}_{kl} = 12qr_0^2\omega^3 \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

After some algebra, and taking time averages:

$$\overline{\left(\frac{dP_{\text{quad}}}{d\Omega}\right)} = \frac{1}{144\pi}\overline{[\![\dddot{Q}_{kl}\dddot{Q}_{km}n_l n_m - \dddot{Q}_{kl}\dddot{Q}_{mr}n_k n_l n_m n_r]\!]} = \frac{q^2\omega^6 r_0^4}{\pi}\left(\sin^2\theta - \tfrac{1}{2}\sin^4\theta\right). \tag{111}$$

This angular distribution is shown in Figure 1. Note that the quadrupole power vanishes in the $\hat{\boldsymbol{z}}$ direction.

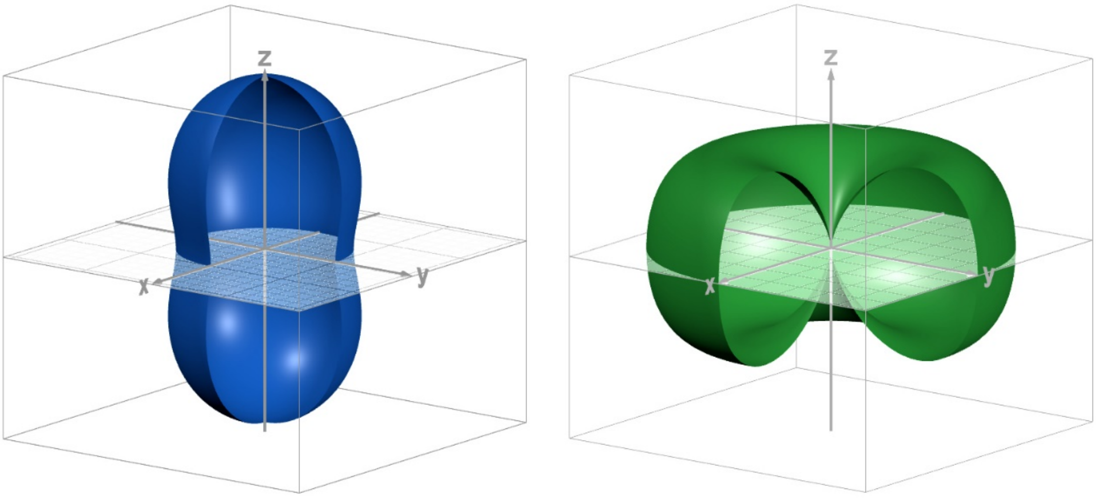


Figure 1 Angular distribution of electromagnetic dipole and quadrupole radiation for a charge in uniform circular motion in the x-y plane

§3. Radiation damping forces

For each charge q in the system, define the vector:

$$\mathbf{f}_q \equiv \frac{2q}{3}\dddot{\boldsymbol{\mu}}_q\,. \tag{112}$$

Consider $\mathbf{f}_q$ as a hypothetical force exerted on the particle q. Henceforth, we omit the subscript q in $\mathbf{r}'_q$, $\boldsymbol{\mu}_q$, $\mathbf{f}_q$, etc. The work done per unit time by this force on the system is

$$\frac{dW}{dt}=\sum_q \mathbf{f}\bullet\frac{d\mathbf{r}'}{dt}=\sum_q \mathbf{f}\bullet\mathbf{v}'=\frac{2}{3}\sum_q q\dddot{\boldsymbol{\mu}}\bullet\mathbf{v}'=\frac{2}{3}\sum_q \dddot{\boldsymbol{\mu}}\bullet\dot{\boldsymbol{\mu}}=\frac{2}{3}\sum_q\left(\frac{d}{dt}(\dot{\boldsymbol{\mu}}\bullet\ddot{\boldsymbol{\mu}})-|\ddot{\boldsymbol{\mu}}|^2\right).$$

The time average of $(d/dt)(\dot{\boldsymbol{\mu}}\bullet\ddot{\boldsymbol{\mu}})$ is

$$\overline{\frac{d}{dt}(\dot{\boldsymbol{\mu}}\bullet\ddot{\boldsymbol{\mu}})}=\lim_{T\to\infty}\frac{1}{T}\int_\tau^{\tau+T}\frac{d}{dt}(\dot{\boldsymbol{\mu}}\bullet\ddot{\boldsymbol{\mu}})dt=\lim_{T\to\infty}\frac{1}{T}(\dot{\boldsymbol{\mu}}\bullet\ddot{\boldsymbol{\mu}})\Big|_\tau^{\tau+T}=\lim_{T\to\infty}\frac{1}{T}(q\mathbf{v}\bullet q\dot{\mathbf{v}})\Big|_\tau^{\tau+T}=0$$

providing that the particle velocities and accelerations are bounded. Therefore,

$$\overline{dW/dt}=-\frac{2}{3}\overline{|\ddot{\boldsymbol{\mu}}|^2}\,. \tag{113}$$

Thus, $\overline{dW/dt}<0$; from (106) we see that the time-average rate of work done by $\mathbf{f}$ on the system is precisely $-P_{\text{dip}}$. Thus, each charge does work $\mathbf{f}\bullet d\mathbf{r}'$ on the electric field and loses a corresponding amount of energy through dipole radiation, so $-\mathbf{f}$ is the **dipole radiation damping force** on q.

§4. Angular momentum lost through dipole radiation

We now calculate the angular momentum lost by the system through dipole radiation. For each charged particle in the system, we have

$$\dot{\mathbf{M}}^\odot=\dot{\mathbf{r}}'\times\mathbf{p}'+\mathbf{r}\times\dot{\mathbf{p}}'=\mathbf{r}'\times\mathbf{f}\,.$$

Therefore:

$$\dot{\mathbf{M}}^\odot=\sum_q \mathbf{r}'\times\frac{2q}{3}\dddot{\boldsymbol{\mu}}=\frac{2}{3}\sum_q \boldsymbol{\mu}\times\dddot{\boldsymbol{\mu}}\,.$$

We use $\mathcal{E}^\odot$ and $\mathbf{M}^\odot$ to denote the total energy and angular momentum of the system.[28] Since $\boldsymbol{\mu}\times\dddot{\boldsymbol{\mu}}=(d/dt)(\boldsymbol{\mu}\times\ddot{\boldsymbol{\mu}})-\dot{\boldsymbol{\mu}}\times\ddot{\boldsymbol{\mu}}$, the time-average rate of change of angular momentum of the system due to dipole radiation is

$$\overline{\dot{\mathbf{M}}^\odot_{\text{dip}}}=-\frac{2}{3}\overline{\dot{\boldsymbol{\mu}}\times\ddot{\boldsymbol{\mu}}}\,.$$

Therefore, including (106), we have the formulae:

$$\dot{\mathcal{E}}^\odot_{\text{dip}}=-\frac{2}{3}|\ddot{\boldsymbol{\mu}}|^2\qquad\overline{\dot{\mathbf{M}}^\odot_{\text{dip}}}=-\frac{2}{3}\overline{\dot{\boldsymbol{\mu}}\times\ddot{\boldsymbol{\mu}}}\,. \tag{114}$$

[28] Recall that $\mathcal{E}$ and $\mathbf{M}$ were used to denote the energy and angular momentum densities of a plane wave. We will find that the ratios $\dot{\mathbf{M}}^\odot/\dot{\mathcal{E}}^\odot$ and $\mathbf{M}/\mathcal{E}$ are equal. Our notation avoids potential confusion.

§5. Angular momentum lost through quadrupole radiation

We repeat the analysis for electric quadrupole radiation, using (109) instead of (106) for the radiated power. Then the radiation damping force $\mathbf{f}$ satisfies the equation

$$\frac{dW}{dt} = \sum_q \mathbf{f} \bullet \mathbf{v}' = -\frac{1}{180} \dddot{Q}_{kl} \dddot{Q}_{kl} \,.$$

Note that

$$\dddot{Q}_{kl} \dddot{Q}_{kl} = \frac{d}{dt}\left(\ddot{Q}_{kl} \dddot{Q}_{kl}\right) - \ddot{Q}_{kl} Q^{(4)}_{kl} = \frac{d}{dt}\left(\ddot{Q}_{kl} \dddot{Q}_{kl}\right) - \frac{d}{dt}\left[\dot{Q}_{kl} Q^{(4)}_{kl}\right] + \dot{Q}_{kl} Q^{(5)}_{kl} \,.$$

Taking time averages, we have

$$\overline{\dddot{Q}_{kl} \dddot{Q}_{kl}} = \overline{\dot{Q}_{kl} Q^{(5)}_{kl}}$$

so

$$\overline{\sum_q \mathbf{f} \bullet \mathbf{v}'} = -\frac{1}{180} \overline{\dot{Q}_{kl} Q^{(5)}_{kl}} \,.$$

We can express $\dot{Q}_{kl}$ this way:

$$\dot{Q}_{kl} = \frac{d}{dt} \sum_q q(3x'_k x'_l - \delta_{kl} r'^2) = \sum_q q(3x'_k v'_l + 3v'_k x'_l - 2\delta_{kl} \mathbf{r}' \bullet \mathbf{v}')$$

so

$$\overline{\sum_q \mathbf{f} \bullet \mathbf{v}'} = -\frac{1}{180} \overline{\sum_q q(6x'_k v'_l - 2\delta_{kl} \mathbf{r}' \bullet \mathbf{v}') Q^{(5)}_{kl}} = -\frac{1}{30} \overline{\sum_q q x'_l v'_k Q^{(5)}_{kl}} \,.$$

In the last step we used $\delta_{kl} Q^{(5)}_{kl} = Q^{(5)}_{kk} = 0$, since Q_{kl} is traceless.[29] Hence,

$$\overline{\sum_q \mathbf{f} \bullet \mathbf{v}'} = \overline{\sum_q f_k v'_k} = -\frac{1}{30} \overline{\sum_q q Q^{(5)}_{kl} x'_l v'_k} \,.$$

Thus, the radiation damping force $\mathbf{f}$ on q is:

$$f_k = -\frac{q}{30} Q^{(5)}_{kl} x'_l \,.$$

We can now calculate $\dot{\mathbf{M}}^{\odot} = \sum_q \mathbf{r}' \times \mathbf{f}$. We have

$$\dot{M}^{\odot}_i = \sum_q \varepsilon_{ijk} x'_j f_k = -\frac{\varepsilon_{ijk}}{30} \sum_q q x'_j x'_l Q^{(5)}_{kl} = -\frac{\varepsilon_{ijk}}{30} \sum_q \frac{q}{3}(3x'_j x'_l - \delta_{jl} r'^2) Q^{(5)}_{kl} = -\frac{\varepsilon_{ijk}}{90} \sum_q Q_{jl} Q^{(5)}_{kl}$$

---

[29] This is so because

$$Q_{kk} = \int (3x'_k x'_k - \delta_{kk} r'^2) \rho \, dV' = \int (3x'_k x'_k - 3r'^2) \rho \, dV' = 0 \,.$$

where in the last step we used the fact that

$$\varepsilon_{ijk}\delta_{jl}Q_{kl}^{(5)} = \varepsilon_{ijk}Q_{jk}^{(5)} = 0$$

Observe that

$$Q_{jl}\,Q_{kl}^{(5)} = \frac{d}{dt}\Big[Q_{jl}\,Q_{kl}^{(4)} - \dot{Q}_{jl}\dddot{Q}_{kl}\Big] + \ddot{Q}_{jl}\dddot{Q}_{kl}\,.$$

Taking time averages:

$$\overline{Q_{jl}\,Q_{kl}^{(5)}} = \overline{\dddot{Q}_{jl}\,\dddot{Q}_{kl}}\,.$$

Therefore, including (109), we have the formulae:

$$\dot{\mathcal{E}}^{\odot}_{\text{quad}} = -\frac{1}{180}\dddot{Q}_{ij}\dddot{Q}_{ij} \qquad \overline{(\dot{M}_i^{\odot})_{\text{quad}}} = -\frac{1}{90}\varepsilon_{ijk}\overline{\ddot{Q}_{jl}\dddot{Q}_{kl}}\,, \tag{115}$$

or, using $\mathbf{Q}_k \equiv (Q_{kx}, Q_{ky}, Q_{kz})$:

$$\dot{\mathcal{E}}^{\odot}_{\text{quad}} = -\frac{1}{180}\dddot{\mathbf{Q}}_k \bullet \dddot{\mathbf{Q}}_k \qquad \overline{\dot{\mathbf{M}}^{\odot}_{\text{quad}}} = -\frac{1}{90}\overline{\ddot{\mathbf{Q}}_k \times \dddot{\mathbf{Q}}_k}\,. \tag{116}$$

§6. Gravitational waves

We perform the same analysis for gravitational waves. Consider the radiation emitted by a system of particles, making the same assumption that $\Sigma \ll \lambda \ll r$, recalling that $\Sigma \ll \lambda$ is equivalent to $v \ll 1$. As in the EM case, the wave equation with source term,

$$\Box\psi^{\mu\nu} = -16\pi T^{\mu\nu}$$

is solved by the retarded potential

$$\psi^{\mu\nu}(t,\mathbf{r}) = -4\int \frac{T^{\mu\nu}(t-|\mathbf{r}-\mathbf{r}'|,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'\,.$$

As before, we take the origin near the system and assume $\Sigma \ll r$, so $|\mathbf{r}-\mathbf{r}'|^{-1} \approx r^{-1}$. The assumption $\Sigma \ll \lambda$ allows us to approximate $T^{\mu\nu}$ by its Taylor expansion:

$$T^{\mu\nu}(t-r+\hat{\mathbf{r}}\bullet\mathbf{r}',\mathbf{r}') = T^{\mu\nu}(t-r,\mathbf{r}') + (\hat{\mathbf{r}}\bullet\mathbf{r}')\frac{\partial T^{\mu\nu}(t-r,\mathbf{r}')}{\partial t} + \cdots\,. \tag{117}$$

The second term in the expansion is of order $\Sigma T^{\mu\nu}/\lambda \ll T^{\mu\nu}$; therefore, the expansion is valid. Then, keeping only the first term, we have

$$\psi^{\mu\nu}(t,\mathbf{r}) = -\frac{4}{r}\int [\![T^{\mu\nu}]\!]\,dV'\,. \tag{118}$$

To obtain a solution to (118), we use the identity

$$\int T_{mn}\,dV = \frac{1}{2}\frac{\partial^2}{\partial t^2}\int_{\text{system}} T_{00}x_m x_n\,dV\,. \tag{119}$$

To establish (119), we separate the time and space components of ν in the equation $T^{\mu\nu}{}_{,\nu} = 0$:[30]

$$T^{\mu 0}{}_{,0} = -T^{\mu n}{}_{,n}\,. \tag{120}$$

This yields the equations:[31]

$$\int T^{mn}\,dV = \frac{1}{2}\frac{\partial}{\partial t}\int (T^{m0}x^n + T^{n0}x^m)\,dV\,,$$

$$\int (T^{m0}x^n + T^{n0}x^m)\,dV = \frac{\partial}{\partial t}\int T^{00}x^m x^n\,dV\,.$$

Combining these expressions gives (119) after lowering indices.

In a non-relativistic system, the energy density T_{00} is approximately the rest mass density ρ_0, so (118) and (119) give

$$\psi_{mn}(t,\mathbf{r}) = -\frac{2}{r}\frac{\partial^2}{\partial t^2}\int [\![\rho_0(\mathbf{r}')]\!]x'_m x'_n\,dV'\,.$$

This can be written

$$\psi_{mn}(t,\mathbf{r}) = -\frac{2}{3r}\left([\![\ddot{Q}_{mn}]\!] + \delta_{mn}\frac{\partial^2}{\partial t^2}\int r'^2[\![\rho_0(\mathbf{r}')]\!]\,dV'\right)$$

where Q_{mn} is the **mass quadrupole tensor**

$$Q_{mn} = \int (3x'_m x'_n - \delta_{mn}r'^2)\rho_0\,dV'\,.$$

When r is large, ψ_{mn} is a plane wave, whose only polarization states are Ω^{+}_{mn} and $\Omega^{\times}_{mn}$, neither of which is proportional to δ_{mn}. Therefore, the second term on the right side does not represent a disturbance in the spacetime geometry. Omitting this term, we have[32]

$$\psi_{mn}(t,\mathbf{r}) = -\frac{2}{3r}[\![\ddot{Q}_{mn}]\!]\,. \tag{121}$$

[30] In a general curved spacetime, the conservation of matter is expressed by the vanishing of the covariant divergence $T^{\mu\nu}{}_{;\nu} = 0$. In the weak field approximation, this reduces to $T^{\mu\nu}{}_{,\nu} = 0$.

[31] To obtain the first equation, use (120) and integrate by parts:

$$\frac{\partial}{\partial t}\int (T^{m0}x^n + T^{n0}x^m)\,dV = -\int (T^{mk}{}_{,k}x^n + T^{nk}{}_{,k}x^m)\,dV = \int (T^{mk}\delta^n_k + T^{nk}\delta^m_k)\,dV = \int 2T^{mn}\,dV\,.$$

To obtain the second equation, set $\mu = 0$ in (120) and integrate by parts:

$$\frac{\partial}{\partial t}\int T^{00}x^m x^n\,dV = -\int T^{0k}{}_{,k}x^m x^n\,dV = \int T^{0k}(x^m\delta^n_k + x^n\delta^m_k)\,dV = \int (T^{0n}x^m + T^{0m}x^n)\,dV\,.$$

[32] The reader may wonder why the first term in the expansion (117) gives rise to quadrupole radiation — what happened to the dipole term analogous to $P_{\text{dip}} = (2/3)[\![\ddot{\boldsymbol{\mu}}]\!]^2$ in electromagnetism? For gravity, the dipole moment $\ddot{\boldsymbol{\mu}} = \sum m\ddot{\mathbf{r}}'$ equals the net external force on the system, which vanishes for an isolated system, so there is no dipole radiation. This implies that an isolated system of electric charges with the same charge-to-mass ratio does not emit dipole radiation.

The energy flux in the $\hat{\mathbf{r}} = (n_1, n_2, n_3)$ direction can be calculated using an appropriate energy-momentum pseudotensor $t^{\mu\nu}$; then the energy flux is

$$(t^{01}, t^{02}, t^{03}) \bullet \hat{\mathbf{r}} = t^{0s} n_s$$

For large r, we can use the pseudotensor (90) for an infinite plane wave. In the derivation leading to (91), we found:

$$32\pi t_{\mu\nu} = \psi_{\kappa\lambda,\mu}\psi^{\kappa\lambda}{}_{,\nu} - \tfrac{1}{2}\psi_{,\mu}\psi_{,\nu} \,. \qquad (122)$$

We cannot work in the TT gauge here, since we are dealing with radiation from an omnidirectional source, and there is no coordinate system in which $\psi_{\mu\nu}$ is traceless. This is clear by contracting the wave equation

$$\Box\psi = -16\pi T^{\mu}_{\mu}$$

which shows that $\psi \neq 0$ when there are sources. Also, $\psi^{\mu\nu}$ is given by the retarded potential, which is not traceless. However, we continue to work in harmonic coordinates, where $\psi^{\mu\nu}{}_{,\nu} = 0$.

Therefore, we have

$$64\pi t^{0s} = -64\pi t_{0s} = -2\psi_{\kappa\lambda,0}\psi^{\kappa\lambda}{}_{,s} + \psi_{,0}\psi_{,s} \,. \qquad (123)$$

From (121) we suspect that a useful expression may be obtained if each factor in (123) can be expressed in terms of $\psi_{kl,0} \propto \dddot{Q}_{kl}/r$. To this end, we separate the time and space components of $\psi_{\kappa\lambda}$:

$$64\pi t^{0s} = -2\psi_{kl,0}\psi_{kl,s} + 4\psi_{k0,0}\psi_{k0,s} - 2\psi_{00,0}\psi_{00,s} + \psi_{,0}\psi_{,s} \,.$$

Since the mass quadrupole tensor is traceless, we have $\psi_{kk} = 0$ by (121). Therefore, we have

$$\psi_{,s} = \eta^{\mu\nu}\psi_{\mu\nu,s} = \psi_{00,s} - \psi_{kk,s} = \psi_{00,s}$$

so that

$$64\pi t^{0s} = -2\psi_{kl,0}\psi_{kl,s} + 4\psi_{k0,0}\psi_{k0,s} - \psi_{00,0}\psi_{00,s} \,. \qquad (124)$$

From (118), we have

$$\psi_{\mu\nu,s}(t,\mathbf{r}) = \frac{x^s}{r^3}\int [\![T_{\mu\nu}]\!]\, dV' - \frac{4}{r}\frac{\partial}{\partial x^s}\int [\![T_{\mu\nu}]\!]\, dV' \,.$$

Since we assume r is large, we can ignore the first term, which is of order $1/r^2$. So, when differentiating $\psi_{\mu\nu}$, the only dependence on r, hence on x^s, occurs in the retarded time $t - r$ within $[\![T_{\mu\nu}]\!] = T_{\mu\nu}(t - r, \mathbf{r}')$. Hence, we may write

$$\psi_{\mu\nu,s} = \frac{\partial\psi_{\mu\nu}}{\partial r}\frac{\partial r}{\partial x^s} = -\psi_{\mu\nu,0} n_s \,. \qquad (125)$$

We use (125) together with the harmonic gauge condition

$$\psi^{\mu 0}{}_{,0} + \psi^{\mu n}{}_{,n} = 0 \qquad \text{or} \qquad \psi_{\mu 0,0} - \psi_{\mu n,n} = 0$$

to manipulate the terms appearing in (124). Thus, we have

$$\begin{aligned}
\psi_{kl,s} &= -\psi_{kl,0} n_s \\
\psi_{k0,0} &= \psi_{kl,l} = -\psi_{kl,0} n_l \\
\psi_{k0,s} &= -\psi_{k0,0} n_s = \psi_{kl,0} n_l n_s \\
\psi_{00,0} &= \psi_{0k,k} = \psi_{kl,0} n_k n_l \\
\psi_{00,s} &= -\psi_{00,0} n_s = -\psi_{kl,0} n_k n_l n_s \,.
\end{aligned} \tag{126}$$

Plugging (126) into (124), we obtain

$$64\pi t^{0s} = \left(2\psi_{kl,0}\psi_{kl,0} - 4\psi_{kl,0}\psi_{km,0} n_l n_m + \psi_{kl,0}\psi_{mr,0} n_k n_l n_m n_r\right) n_s \,. \tag{127}$$

Using $n_s n_s = 1$, we have

$$64\pi t^{0s} n_s = 2\psi_{kl,0}\psi_{kl,0} - 4\psi_{kl,0}\psi_{km,0} n_l n_m + \psi_{kl,0}\psi_{mr,0} n_k n_l n_m n_r \,.$$

Plugging in (121) gives

$$t^{0s} n_s = \frac{1}{144\pi r^2} [\![2\dddot{Q}_{kl}\dddot{Q}_{kl} - 4\dddot{Q}_{kl}\dddot{Q}_{km} n_l n_m + \dddot{Q}_{kl}\dddot{Q}_{mr} n_k n_l n_m n_r]\!] \,.$$

So, the power per unit solid angle in the $\hat{\mathbf{r}}$ direction is

$$\frac{dP}{d\Omega} = t^{0s} n_s r^2 = \frac{1}{144\pi} [\![2\dddot{Q}_{kl}\dddot{Q}_{kl} - 4\dddot{Q}_{kl}\dddot{Q}_{km} n_l n_m + \dddot{Q}_{kl}\dddot{Q}_{mr} n_k n_l n_m n_r]\!] \,. \tag{128}$$

Note how this differs from the angular distribution of electric quadrupole intensity (107).

The total radiated power is found by integrating over all Ω:

$$P = \frac{1}{144\pi} \left[\!\!\left[2\dddot{Q}_{kl}\dddot{Q}_{kl} \int d\Omega - 4\dddot{Q}_{kl}\dddot{Q}_{km} \int n_l n_m \, d\Omega + \dddot{Q}_{kl}\dddot{Q}_{mr} \int n_k n_l n_m n_r \, d\Omega \right]\!\!\right] = \frac{1}{45} [\![\dddot{Q}_{kl}\dddot{Q}_{kl}]\!]$$

using (108).[33] Therefore, the rate of change of the energy of the system is

$$\dot{\mathcal{E}}^{\odot} = -P = -\frac{1}{45} \dddot{Q}_{kl}\dddot{Q}_{kl} \,. \tag{129}$$

To calculate the rate of change of angular momentum, as before, we need to determine the damping force $\mathbf{f}$ which satisfies

$$\sum_m \mathbf{f} \bullet \mathbf{v}' = \dot{\mathcal{E}}^{\odot} = -\frac{1}{45} \dddot{Q}_{kl}\dddot{Q}_{kl}$$

and calculate $\dot{\mathbf{M}} = \sum \mathbf{r}' \times \mathbf{f}$. But we already solved this problem in §5 with $-1/180$ in place of $-1/45$ and q in place of m. So we have:

[33]We may convert this formula to conventional units by making the changes $d/dt \to d/d(ct)$ and $\rho \to \rho G/c^2$ to obtain $P = (G/45c^5)[\![\dddot{Q}_{kl}\dddot{Q}_{kl}]\!]$. The right side contributes two factors of G/c^5, while the left side contributes one. The factor $G/c^5 = 2.75 \times 10^{-53}\ \mathrm{m^{-2}s^3kg^{-1}}$ explains why gravitational radiation is so weak.

$$\dot{\mathcal{E}}^{\odot} = -\frac{1}{45}\dddot{Q}_{kl}\dddot{Q}_{kl} \qquad \overline{\dot{M}_i}^{\odot} = -\frac{2}{45}\varepsilon_{ijk}\overline{\ddot{Q}_{jl}\dddot{Q}_{kl}} \,, \tag{130}$$

or, using $\mathbf{Q}_k \equiv (Q_{kx}, Q_{ky}, Q_{kz})$:

$$\dot{\mathcal{E}}^{\odot} = -\frac{1}{45}\dddot{\mathbf{Q}}_k \bullet \dddot{\mathbf{Q}}_k \qquad \overline{\dot{\mathbf{M}}}^{\odot} = -\frac{2}{45}\overline{\ddot{\mathbf{Q}}_k \times \dddot{\mathbf{Q}}_k} \,.$$

§7. Angular momentum in circularly polarized gravitational waves

We repeat the example of §2 for a mass m in uniform circular motion in the x-y plane with

$$\mathbf{r}' = r_0 \cos\omega t\,\hat{\boldsymbol{x}} + r_0 \sin\omega t\,\hat{\boldsymbol{y}} \,.$$

The quadrupole tensor is the same as (110) with q replaced by m:

$$\ddot{Q}_{kl} = 6mr_0^2\omega^2 \begin{pmatrix} -\cos 2\omega t & -\sin 2\omega t & 0 \\ -\sin 2\omega t & \cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \dddot{Q}_{kl} = 12mr_0^2\omega^3 \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore:

$$\dot{\mathcal{E}}^{\odot} = -\frac{1}{45}\dddot{Q}_{kl}\dddot{Q}_{kl} = -\frac{1}{45}\left(\dddot{Q}_{xx}^2 + \dddot{Q}_{yy}^2 + 2\dddot{Q}_{xy}^2\right) = -\frac{32m^2r_0^4\omega^6}{5} \,. \tag{131}$$

We also have:

$$\begin{aligned} \ddot{\mathbf{Q}}_x &= -6mr_0^2\omega^2\hat{\mathbf{r}}' \,, & \ddot{\mathbf{Q}}_y &= 6mr_0^2\omega^2\hat{\mathbf{v}}' \,, & \ddot{\mathbf{Q}}_z &= \mathbf{0} \,, \\ \dddot{\mathbf{Q}}_x &= -12mr_0^2\omega^3\hat{\mathbf{v}}' \,, & \dddot{\mathbf{Q}}_y &= -12mr_0^2\omega^3\hat{\mathbf{r}}' \,, & \dddot{\mathbf{Q}}_z &= \mathbf{0} \,. \end{aligned}$$

The cross products $\ddot{\mathbf{Q}}_k \times \dddot{\mathbf{Q}}_k$ are all time-independent and point in the $\hat{\boldsymbol{z}}$ direction. Therefore,

$$\dot{\mathbf{M}}^{\odot} = -\frac{2}{45}\ddot{\mathbf{Q}}_k \times \dddot{\mathbf{Q}}_k = -\frac{2}{45}\left(\ddot{\mathbf{Q}}_x \times \dddot{\mathbf{Q}}_x + \ddot{\mathbf{Q}}_y \times \dddot{\mathbf{Q}}_y\right) = -\frac{32m^2r_0^4\omega^5}{5}\hat{\boldsymbol{z}} \,. \tag{132}$$

Interestingly, we find that $\dot{M}_z^{\odot}/\dot{\mathcal{E}}^{\odot} = 1/\omega$. Now, waves propagating in the $\hat{\boldsymbol{z}}$ direction are circularly polarized, while waves propagating in the x-y plane are linearly polarized (no angular momentum). Hence, for waves in the $\hat{\boldsymbol{z}}$ direction, we expect $\dot{M}_z/\dot{\mathcal{E}} > 1/\omega$. We will see that (101) implies $\dot{M}_z/\dot{\mathcal{E}} = 2/\omega$ for waves in the $\hat{\boldsymbol{z}}$ direction.

The angular distribution of power is given by (128). Repeating a similar calculation to (111), we find

$$\frac{dP}{d\Omega} = \frac{m^2r_0^4\omega^6}{\pi}\left[4(1-\sin^2\theta) + \tfrac{1}{2}\sin^4\theta\right] . \tag{133}$$

This distribution is shown in Figure 2. Notice that the shape of gravitational quadrupole radiation does not resemble electromagnetic quadrupole radiation at all; indeed, it looks more like the peanut-shaped distribution of electromagnetic dipole radiation.

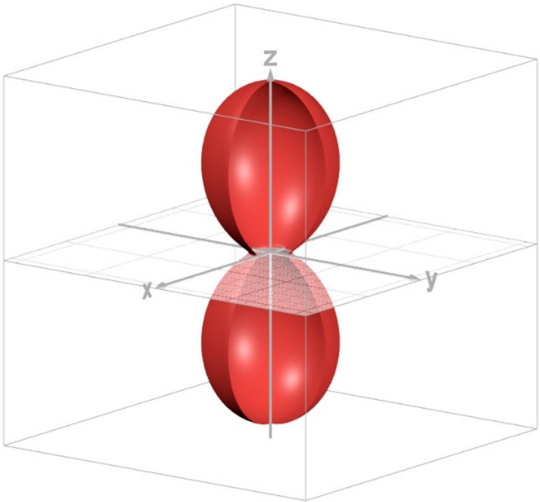


Figure 2 Angular distribution of gravitational quadrupole radiation for a mass in uniform circular motion in the x-y plane

We wish to compare $dP/d\Omega$ in the $\hat{\boldsymbol{z}}$ direction to the average over all directions. From (129):

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{\text{ave}} = \frac{1}{4\pi}\left(\frac{1}{45}[\![\dddot{Q}_{kl}\dddot{Q}_{kl}]\!]\right) = \frac{1}{180\pi}[\![\dddot{Q}_{kl}\dddot{Q}_{kl}]\!].$$

Setting $\hat{\mathbf{r}} = \hat{\boldsymbol{z}} = (0,0,1)$ in (128), we have

$$\left(\frac{dP}{d\Omega}\right)_{\hat{\boldsymbol{z}}} = \frac{1}{144\pi}[\![2\dddot{Q}_{kl}\dddot{Q}_{kl} - 4\dddot{Q}_{kz}\dddot{Q}_{kz} + \dddot{Q}_{zz}\dddot{Q}_{zz}]\!] = \frac{1}{72\pi}[\![\dddot{Q}_{kl}\dddot{Q}_{kl}]\!].$$

Therefore, the power radiated in the $\hat{\boldsymbol{z}}$ direction is 5/2 times the power averaged over all directions:

$$\left(\frac{dP}{d\Omega}\right)_{\hat{\boldsymbol{z}}} = \frac{5}{2}\left\langle \frac{dP}{d\Omega} \right\rangle_{\text{ave}}. \tag{134}$$

Unfortunately, the angular momentum flow in the $\hat{\mathbf{r}}$ direction cannot be determined solely from the properties of the system, as we did for the energy flow in (128). Eq. (127) shows that the momentum density points in the $\hat{\mathbf{r}} = (n_1, n_2, n_3)$ direction, so the angular momentum density $\mathbf{M} = \mathbf{r} \times \mathbf{p}$ vanishes. This is to be expected, since we observed in section D §3-4 that the existence of angular momentum in a wave packet is essentially a boundary phenomenon.

However, for the system studied in §7 consisting of a mass in uniform circular motion in the x-y plane, we can ascertain the angular momentum flow in the $\hat{\boldsymbol{z}}$ direction. Let $\Phi_{\mathcal{E}}(\hat{\mathbf{r}})$ and $\Phi_M(\hat{\mathbf{r}})$ be the energy flux and z-angular momentum flux in the $\hat{\mathbf{r}}$ direction. Then the energy and z-angular momentum flowing through the solid angle $\delta\Omega$ at a distance r from the origin during a time interval δt is $\Phi_{\mathcal{E}}(\hat{\mathbf{r}})r^2\delta\Omega\delta t$ and $\Phi_M(\hat{\mathbf{r}})r^2\delta\Omega\delta t$.

Within a solid angle $\delta\Omega$ around the z-axis, during a time interval δt, the system generates a wave packet of length $c\delta t$. Close to the system the radiation pattern is complicated, but far away the wave packet is a circularly polarized plane wave. Within this wave packet, by (101):

$$\frac{\Phi_M(\hat{\mathbf{z}})}{\Phi_{\mathcal{E}}(\hat{\mathbf{z}})} = \frac{\int M\,dV}{\int \mathcal{E}\,dV} = \frac{2}{\omega}.$$

From (134), the energy flow $\Phi_{\mathcal{E}}(\hat{\mathbf{z}})$ is 5/2 times the average over all directions. Hence, $\Phi_M(\hat{\mathbf{z}})$ must be 5 times the average. More precisely:

$$\Phi_M(\hat{\mathbf{z}}) = \frac{\Phi_M(\hat{\mathbf{z}})}{\Phi_{\mathcal{E}}(\hat{\mathbf{z}})}\Phi_{\mathcal{E}}(\hat{\mathbf{z}}) = \frac{2}{\omega}\cdot\frac{5}{2}\langle\Phi_{\mathcal{E}}(\hat{\mathbf{z}})\rangle_{\text{ave}} = \frac{5}{\omega}\left[\frac{1}{4\pi}\left(-\frac{32m^2r_0^4\omega^6}{5}\right)\right] = 5\langle\Phi_M(\hat{\mathbf{z}})\rangle_{\text{ave}}$$

where we used (131) and (132) in the last two steps.

Alas, this deduction appears to be possible only for the $\hat{\mathbf{z}}$ direction.

Conclusion

The main results of this work are:

1. For gravitational plane waves, the eigentensors Z_i of $i\widetilde{\boldsymbol{R}}$ have eigenvalues ±2 (section C §7, Table 1).[34] For a plane wave $u_{\mu\nu}$ expressed in the eigenbasis

$$\sum \zeta^i Z_i$$

the coefficients ζ^i have helicity $-\lambda$, where λ is the eigenvalue of Z_i. This establishes a connection between the eigenvalues of $i\widetilde{\boldsymbol{R}}$ and helicity.

2. The ratio of spin angular momentum to energy for a circularly polarized, gravitational wave packet with frequency ω is $\pm 2/\omega$, compared with $\pm 1/\omega$ for an EM wave (sections D §3-4).

3. Information about the energy and spin of a gravitational wave packet should be inferable from the distribution of energy and angular momentum in the radiation field of a system of masses. In the case of a system emitting circularly polarized, gravitational waves in the $\hat{\mathbf{z}}$ direction, we can determine the flow of angular momentum in the $\hat{\mathbf{z}}$ direction, using (83) from section D. The problem of determining the general distribution of angular momentum radiated by a system remains unsolved, whether in the case of EM or gravitational radiation.

The method of section E, regardless of which energy-momentum tensor (or pseudotensor) is used, will always yield an angular momentum density of zero. Hence, it seems unlikely that a formula for $\Phi_M(\hat{\mathbf{r}})$, analogous to (107) and (127), can be found to derive the ratios

$$\frac{dS_z}{d\mathcal{E}} = \pm\frac{1}{\omega}\ (\text{electromagnetism}) \qquad \frac{dS_z}{d\mathcal{E}} = \pm\frac{2}{\omega}\ (\text{gravity}) \tag{135}$$

by considering a system that radiates circularly polarized waves. Again, this is due to the angular momentum in a wave packet being a boundary phenomenon.

[34]There are eigentensors with eigenvalues ±1 and 0, but these are unphysical, coordinate waves.

Nevertheless, it is not inconceivable that (135) can be derived from the radiation properties of a system of charges or masses. Compare the formulae for electric dipole radiation (114) and gravitational quadrupole radiation (130). Notice that the formula for $\dot{\mathbf{M}}^{\odot}$ picks up an extra factor of 2 in the gravitational case. It is true that the same factor of 2 appears for electric quadrupole radiation (115), which suggests that this factor of 2 may simply reflect the nature of quadrupole vs. dipole radiation, rather than gravitational vs. EM radiation. On the other hand, notice that:

1. The leading term in EM radiation is dipole; in gravitational radiation it is quadrupole.

2. The gravitational quadrupole field shown in Figure 2 closely resembles the **electric dipole field** — not the quadrupole field in Figure 1!

3. In the case of a particle in circular motion in the x-y plane, EM quadrupole radiation **vanishes** in the $\hat{\boldsymbol{z}}$ direction, where EM dipole and gravitational quadrupole radiation attain their **maximum intensity.** This clue suggests that the factor of 2 in (130) vs. (114) reflects something more than just quadrupole vs. dipole radiation.

This remains an open question for further study.

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