# CREATION MODEL OF TIME-DEPENDENT LOWERING OPERATOR AND RAISING OPERATOR UNDER TIME-DEPENDENT EXTERNAL DAMPING FORCE SIMPLE HARMONIC OSCILLATOR

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**Abstract**

In this research paper, we developed a theory and model of the forced simple harmonic oscillators in quantum mechanics system in a framework of a general approach to the Heisenberg picture. We can be used the time-dependent Hamiltonian for the forced simple harmonic oscillators. We use first-order ordinary linear differential equation to solve the time-dependent lowering operator and raising operator. We evaluate the lowering operator and raising operator as the function of time for particle mass $m$ depend on the parameter frequency of oscillation $(f)$, the damping coefficient $(\beta)$ of particle bound in the simple harmonic oscillator potential, the initial force $(Q_0)$.

**Keywords:** Lowering operator, Raising operator, Heisenberg equation

## INTRODUCTION

The harmonic oscillators are widely known and many examples of a quantum mechanics system. It is a solvable system and allows the exploration of quantum dynamics in detail as well as the study of quantum states with classical properties. The harmonic oscillator is a system where the classical description suggests clearly the definition of the quantum mechanics system. Classically a harmonic oscillator is described by the position $x(t)$ of a particle of mass $m$ and its momentum $p(t)$. The quantum mechanics system is easily defined. The operator method is an excellent means of illustrating the power and the utility of operators in solving problems in quantum mechanics.

In this note, we shall show that the fundamental commutator can be derived from the Heisenberg equation of motion (Dirac ,1981). Carruthers, & Michael (1965) is shown that the forced quantum mechanics harmonics oscillator in term coherent states. Andrews (2010) is shown that the Heisenberg's operator equations of motion for the position and momentum relative to the centroid are independent of any uniform forces. Andrey (2010) studies contribution dynamics of quantum mechanics harmonics oscillator under external impulse force. Chew et al. (2016) studies the equation of motion for quantum electromagnetics resemble those of classical electromagnetics. We can be used the time-dependent lowering operator and raising operator evaluation of the position operator and the momentum operator of particle bound under harmonic oscillator potential. The time-dependent lowering operator and raising operator is important for find the eigenvalue of the Hamiltonian of a one-dimensional harmonic oscillator and anharmonic oscillator potential.

The purpose of this paper, we will calculate the time-dependent lowering operator and raising operator (Chew et al., 2019) under the time-dependent external damping force simple harmonic oscillator. The scheme of the article is as follows. In section materials and method, we illustrate evaluation of the time-dependent lowering operator and raising operator under external damping force simple harmonic oscillator (Ricardo et al., 2009). In section results, we can be plot graph relation between lowering operator, raising operator and time. The last section contains our conclusions.

## MATERIALS AND METHODS

### Evaluation of time-dependent lowering operator and raising operator under external damping force simple harmonic oscillator

We consider a particle of mass $m$ moving on the x-axis in a time-independent potential $V(x) = (1/2)k\,x^2$ , where $k$ is the spring constant. The Hamiltonian for external damping force simple harmonic oscillator is

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 + \hat{x}Q(t) + \hat{p}G(t). \tag{1}$$

As before, we introduce the classical angular frequency $\omega = \sqrt{k/m} = 2\pi f$ . To this end we introduce two non-hermitian operators, called lowering operator and raising operator respectively

$$\hat{a}(t) = \sqrt{\frac{1}{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\sqrt{\frac{1}{m\hbar\omega}}\hat{p}\right), \tag{2}$$

$$\hat{a}^{\dagger}(t) = \sqrt{\frac{1}{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\sqrt{\frac{1}{m\hbar\omega}}\hat{p}\right). \tag{3}$$

$\hat{x}$ is the position operator, $\hat{p}$ is the linear momentum operator. $Q(t)$ and $G(t)$ is the time-dependent external damping force. Subtract Equation (3) from Equation (2) to yield

$$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}^{\dagger} - \hat{a}\right). \tag{4}$$

Combination Equation (3) from Equation (2), we have

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^{\dagger} + \hat{a}\right). \tag{5}$$

From the above definitions, the first and second terms right-hand side of Equation (1) can be written as

$$\frac{\hat{p}^2}{2m} + \frac{1}{2}kx^2 = \frac{\hbar\omega}{4}\left(\left(\hat{a}^{\dagger}\right)^2 + \hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger} + \left(\hat{a}\right)^2 - \left(\hat{a}^{\dagger}\right)^2 + \hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger} - \left(\hat{a}\right)^2\right),$$

$$\frac{\hat{p}^2}{2m} + \frac{1}{2}kx^2 = \frac{\hbar\omega}{2}\left(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}\right). \tag{6}$$

From the above definitions of Equation (4), Equation (5), the two last terms right-hand side of Equation (1) can be rewritten as

$$\begin{aligned}\hat{x}Q(t) + \hat{p}G(t) &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^{\dagger} + \hat{a}\right)Q(t) + i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}^{\dagger} - \hat{a}\right)G(t) \\ &= \left(\sqrt{\frac{\hbar}{2m\omega}}Q(t) + i\sqrt{\frac{m\hbar\omega}{2}}G(t)\right)\hat{a}^{\dagger} + \left(\sqrt{\frac{\hbar}{2m\omega}}Q(t) - i\sqrt{\frac{m\hbar\omega}{2}}G(t)\right)\hat{a}. \end{aligned} \tag{7}$$

We now define the parameter of force $g(t) \equiv \left(\sqrt{\frac{\hbar}{2m\omega}}Q(t) + i\sqrt{\frac{m\hbar\omega}{2}}G(t)\right)$ and we can be rewritten of Equation (7) as

$$\hat{x}Q(t) + \hat{p}G(t) = g(t)\hat{a}^{\dagger} + g^{*}(t)\hat{a}. \tag{8}$$

We can rewrite the Hamiltonian for the time-dependent external damping force simple harmonic oscillator in terms of lowering operator and raising operator to give

$$\hat{H}(t) = \frac{\hbar\omega}{2}\left(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}\right) + g(t)\hat{a}^{\dagger} + g^{*}(t)\hat{a} \tag{9}$$

To illustrate this Heisenberg equation of motion (Dirar et al., 2013) approach we consider once more the time-dependent external damping force simple harmonic oscillator in Equation (9). In the Heisenberg equation of motion (Moya-Cessa et al. 2008) the case of the lowering operator are

$$i\hbar\frac{d\hat{a}(t)}{dt}=\left[\hat{a},\hat{H}(t)\right]$$

$$=\left[\hat{a},\left(\frac{\hbar\omega}{2}\left(\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger}\right)+g(t)\hat{a}^{\dagger}+g^{*}(t)\hat{a}\right)\right],$$

$$i\hbar\frac{d\hat{a}(t)}{dt}=\frac{\hbar\omega}{2}(2\hat{a})+g(t)$$

or

$$\frac{d\hat{a}(t)}{dt}+i\omega\hat{a}(t)=\frac{g(t)}{i\hbar} \qquad (10)$$

These are the famous first-order linear ordinary differential equation (Tikjha et al., 2018).

In the Heisenberg equation of motion the case of the raising operator are

$$i\hbar\frac{d\hat{a}^{\dagger}(t)}{dt}=\left[\hat{a}^{\dagger},\hat{H}(t)\right]$$

$$=\left[\hat{a}^{\dagger},\left(\frac{\hbar\omega}{2}\left(\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger}\right)+g(t)\hat{a}^{\dagger}+g^{*}(t)\hat{a}\right)\right],$$

$$i\hbar\frac{d\hat{a}^{\dagger}(t)}{dt}=\frac{\hbar\omega}{2}\left(-2\hat{a}^{\dagger}\right)+-g^{*}(t)$$

or

$$\frac{d\hat{a}^{\dagger}(t)}{dt}-i\omega\hat{a}^{\dagger}(t)=-\frac{g^{*}(t)}{i\hbar} \qquad (11)$$

The solution given by Equation (10) and Equation (11) may be written in a different form as follows using integrating factors $e^{i\omega t}$ and $e^{-i\omega t}$ respectively to give

$$\hat{a}(t)=\frac{-i}{\hbar}e^{-i\omega t}\int_{0}^{t}e^{i\omega t}\,g(t)\,dt+\hat{a}(0)e^{-i\omega t} \qquad (12)$$

$$\hat{a}^{\dagger}(t)=\frac{i}{\hbar}e^{i\omega t}\int_{0}^{t}e^{-i\omega t}\,g^{*}(t)\,dt+\hat{a}^{\dagger}(0)e^{i\omega t}, \qquad (13)$$

where $\hat{a}=\hat{a}(0)$, $\hat{a}^{\dagger}=\hat{a}^{\dagger}(0)$. We now define of the time-dependent external damping force simple harmonic oscillator to give

$$Q(t)=G(t)=Q_0\,e^{-\beta t}\sin(\sigma t). \qquad (14)$$

Substituting Equation (14) into $g(t) \equiv \left( \sqrt{\frac{\hbar}{2m\omega}} Q(t) + i\sqrt{\frac{m\hbar\omega}{2}} G(t) \right)$, we can rewrite $g(t)$ as

$$g(t) = \left( \sqrt{\frac{\hbar}{2m\omega}} + i\sqrt{\frac{m\hbar\omega}{2}} \right) Q_0\, e^{-\beta t} \sin(\sigma t), \quad (15)$$

where $Q_0$ is intitial force, $\beta$ is the damping coefficient. With this definition the parameter becomes

$$\eta \equiv \sqrt{\frac{\hbar}{2m\omega}} + i\sqrt{\frac{m\hbar\omega}{2}}, \qquad \eta^* = \sqrt{\frac{\hbar}{2m\omega}} - i\sqrt{\frac{m\hbar\omega}{2}}. \quad (16)$$

We can rewrite the time-dependent lowering operator Equation (12) as

$$\hat{a}(t) = \frac{-i}{\hbar} e^{-i\omega t}\, Q_0 \eta \int_0^t e^{i\omega t}\, e^{-\beta t} \sin(\sigma t)\, dt + \hat{a}(0) e^{-i\omega t},$$

$$\hat{a}(t) = \frac{i\eta Q_0 e^{-\beta t}\left( (i\omega - \beta)\sin(\sigma t) - \sigma\cos(\sigma t) \right)}{(i\omega-\beta)^2 + \sigma^2} + \frac{i\eta Q_0 \sigma e^{-i\omega t}}{(i\omega-\beta)^2+\sigma^2} + \hat{a}(0) e^{-i\omega t}. \quad (17)$$

We can rewrite the time-dependent raising operator Equation (13) as

$$\hat{a}^\dagger(t) = \frac{i}{\hbar} e^{i\omega t}\, Q_0 \eta^* \int_0^t e^{-i\omega t}\, e^{-\beta t} \sin(\sigma t)\, dt + \hat{a}^\dagger(0) e^{i\omega t}$$

$$\hat{a}^\dagger(t) = \frac{i\eta^* Q_0 e^{-\beta t}\left( (-i\omega - \beta)\sin(\sigma t) - \sigma\cos(\sigma t) \right)}{(i\omega+\beta)^2 + \sigma^2} + \frac{i\eta^* Q_0 \sigma e^{i\omega t}}{(i\omega+\beta)^2+\sigma^2} + \hat{a}^\dagger(0) e^{i\omega t}. \quad (18)$$

From Equation (17) and Equation (18) is the time-dependent lowering operator and raising operator show in program. Putting this into program mathematica for plot graph.

Case 1: The time-dependent lowering operator.

The frequency parameter $(f_1 = 0.12,\ f_2 = 0.14,\ f_3 = 0.16,\ f_4 = 0.18)$ of lowering operator is the independent variable. The parameter $\sigma = 0.12,\ Q_0 = 10,\ m = \hbar = 1,$ $\sigma = 0.12,$ and $\beta = 0.01$ of the lowering operator is the control variable. In Figure 1(a) we illustrate schematically the time-dependent lowering operator for a particle bound the harmonic oscillator potential by change the frequency parameter. Next, we can be used the damping coefficient parameter $\beta_1 = 0.02,\ \beta_2 = 0.04,\ \beta_3 = 0.06,\ \beta_4 = 0.08,$ (independent variable) evaluation the lowering operator in Figure 1(b). The parameter $\sigma = 0.12,\ Q_0 = 10,\ m = \hbar = 1,\ \sigma = 0.12,$ and $f = 0.15$ of lowering operator is the control variable.

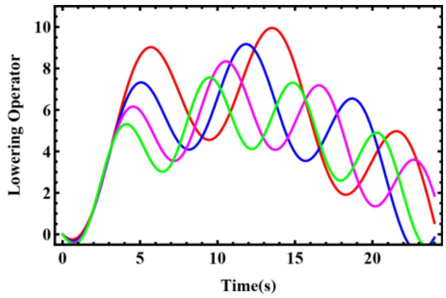
Case 2: The time-dependent raising operator.

The frequency parameter $(f_1 = 0.12,\ f_2 = 0.14,\ f_3 = 0.16,\ f_4 = 0.18)$ of raising operator is the independent variable. The parameter $\sigma = 0.12,\ Q_0 = 10,\ m = \hbar = 1,$

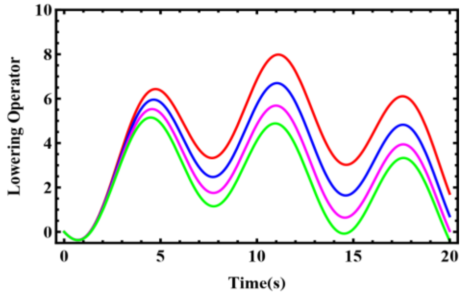
$\sigma = 0.12$, and $\beta = 0.01$ of the raising operator for image part is the control variable. In Figure 2(a) we illustrate schematically the time-dependent raising operator of image part for a particle bound the harmonic oscillator potential by change the frequency parameter. Next, we can be used the damping coefficient parameter $\beta_1 = 0.02$, $\beta_2 = 0.04$, $\beta_3 = 0.06$, $\beta_4 = 0.08$, (independent variable) evaluation the raising operator for image part in Figure 2(b). The parameter $\sigma = 0.12$, $Q_0 = 10$, $m = \hbar = 1$, $\sigma = 0.12$, and $f = 0.15$ of raising operator is the control variable.

## RESULTS

We can explain of numerical and result in Equation (17) and Equation (18). The time-dependent lowering operator can be plot graph in Figure 1(a) and Figure 1(b). The time-dependent raising operator can be plot graph in Figure 2(a) and Figure 2(b). The time-dependent lowering operator for image part are plotted in Figure 1.



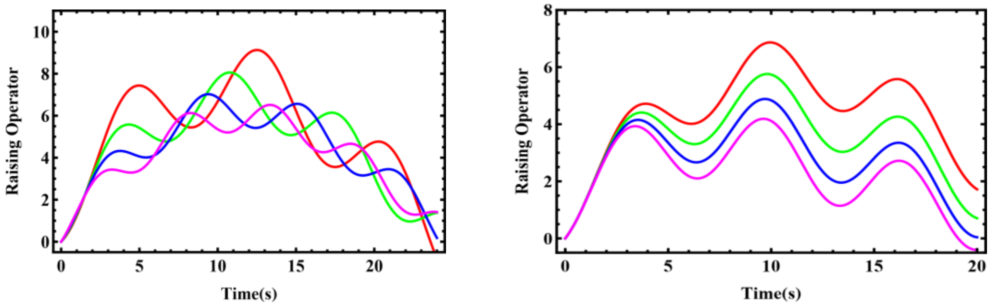

(a)




(b)

**Figure1** Plots of $\hat{a}(t)$, which represents the right-hand sides of Equation (17), as a function of the time $t$ (a) for change of the frequency parameters (The red hard thing line is $f_1 = 0.12\ \mathrm{Hz}$. The blue hard thing line is $f_2 = 0.14\ \mathrm{Hz}$. The pink hard thing line is $f_3 = 0.16\ \mathrm{Hz}$. The green hard thing line is $f_4 = 0.18\ \mathrm{Hz}$.) and (b) for change of the parameters $\beta$ (The red hard thing line is $\beta_1 = 0.02$. The blue hard thing line is $\beta_2 = 0.04$. The pink hard thing line is $\beta_3 = 0.06$. The green hard thing line is $\beta_4 = 0.08$.)

The time-dependent raising operator for conception of part are plotted in Figure 2.




(a) (b)

**Figure 2** Plots of $\hat{a}^{\dagger}(t)$, which represents the right-hand sides of Equation (18), as a function of the time $t$ (a) for change of the frequency parameters (The red hard thing line is $f_1 = 0.12\ \mathrm{Hz}$. The blue hard thing line is $f_2 = 0.14\ \mathrm{Hz}$. The pink hard thing line is $f_3 = 0.16\ \mathrm{Hz}$. The green hard thing line is $f_4 = 0.18\ \mathrm{Hz}$.) and (b) for change of the parameters $\beta$ (The red hard thing line is $\beta_1 = 0.02$. The blue hard thing line is $\beta_2 = 0.04$. The pink hard thing line is $\beta_3 = 0.06$. The green hard thing line is $\beta_4 = 0.08$.)

## DISCUSSION

The time-dependent lowering operator for fantasy part are plotted in Figure 1. From Figure 1(a), if higher the frequency $(f)$ affect decreasing value amplitude and wavelength of the time-dependent lowering operator. Thus the effect of the interaction Heisenberg picture is that the product of the frequency parameter and lowering operator is function of time and denote the change for particle harmonic oscillator under influence of the time-dependent external damping force simple harmonic oscillator ( $g(t)$ ). Such a system of time-dependent lowering operator in Figure 1(a) is said to be moderately damped.

From Figure 1(b), if higher the damping coefficient $(\beta)$ affect decreasing value amplitude of the time-dependent lowering operator. Such a system of time-dependent lowering operator in Figure 1(b) is said to be lightly damped. The time-dependent raising operator for meditate part are plotted in Figure 2.

From Figure 2(a), if extra the frequency $(f)$ impinge relieving value amplitude and wavelength of the time-dependent raising operator. Thus the effect of the interaction

Heisenberg picture is that the product of the frequency parameter and raising operator is function of time and denote the change for particle harmonic oscillator under influence of the time-dependent external damping force simple harmonic oscillator ( $g(t)$ ). From Figure 2(b), if superior the damping coefficient $(\beta)$ influence mitigating value amplitude of the time-dependent raising operator.

## CONCLUSIONS

We can be calculated the time-dependent lowering operator and raising operator of particle mass $m$ lean on the time-dependent external damping force simple harmonic oscillator $(Q(t), G(t))$, the parameter frequency of oscillation $(f)$, the damping coefficient $(\beta)$, the initial force $(Q_0)$ of particle bound in the simple harmonic oscillator potential (Liang et al., 2018). The behavior of the time-dependent lowering operator and raising operator for imagine part is wave group. We can apply the lowering and raising operator result to the particular case of a particle of mass moving in the Gaussian potential and other potential. Finally, we can be used to calculation of the position operator and momentum operator of a particle in vibration bound the harmonic oscillator potential.

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