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# Dilation and stability of sand in triaxial tests

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**ABSTRACT:** Possible links between certain aspects of pre-failure instabilities of sand (instability line) and plastic dilation are studied. The starting point is experimental results obtained from triaxial investigations, which are approximated separately by analytical formulae for initially contractive and dilative sands. The irreversible strains are deduced from the condition that plastic work (dissipation) should be positive. Then, analytical formulae for plastic dilation are derived and presented in graphical form. In the case of initially contractive sand, a possible link between the instability line and maximum of the plastic dilatancy function is demonstrated. This condition is equivalent to the minimum dilation parameter or maximum of Rowe's function. In the case of initially dilative sand, it is shown that the negative work done by a mean stress on expanding (dilating) soil should not be treated as dissipative because of thermodynamical requirements. Consequently, the plastic dilation function is zero for shearing stress exceeding the instability line.

## 1. INTRODUCTION

The term dilatancy or dilation denotes a change in volume of granular materials due to shearing. Some authors use a narrower definition, e.g. "The term dilatancy is used to describe the increase in volume of a dense sand during shearing", Craig (1987). Dilation is a phenomenon typical for granular materials in contrast to other solid materials, where it can be ignored. Dilation was defined by Reynolds, long ago in 1885, as a ratio between plastic volumetric strain increments and plastic shear (deviatoric) strain increments., and is treated as an internal constraint imposed on these strains, Vardoulakis & Sulem (1995). Rowe (1962) was the first who proposed the so-called stress-dilatancy approach to model the behaviour of granular soils. In such approaches, a mobilized friction is linked to the dilation angle, cf. Bolton (1986), Wood (1990). A notion of dilation still plays an important role in constitutive modelling of geomaterials, but it seems that its potential has not yet been fully explored.

More recent investigations have shown that the pre-failure behaviour of granular soils is much more complex than was recognized some decades ago, partly due to the increasingly advanced experimental devices and methods in use today. For example, it was discovered that the initial state of soil, defined as either contractive or dilative, strongly influences soil behaviour during shearing, and the steady state line is a fundamental object in soil mechanics. It was also discovered that there exists a counterpart fundamental object, designated as the instability line. These results have been obtained within the Steady State School founded by Casagrande and developed by his successors, just to mention Castro (1969), Ibsen (1999), Konrad (1993), Lade (1992, 1999), Lade et al (1987), Skopek et al (1994), Sladen et al (1985) or Verdugo & Ishihara (1996). Most of the above mentioned papers deal with the undrained behaviour of saturated sands, but they are also of some importance to the drained behaviour. In the present paper we shall make use of some of the findings of this School, as well as of some findings of the Cambridge Critical State School, widely known after publication of the famous book by Schofield & Wroth (1968).

The aim of the present paper is to investigate possible links between the unstable behaviour of sand before failure and dilation. The starting point for our investigations is experimental results, presented in analytical form, which allow for simple mathematical manipulations and various interpretations. Thus the present paper is of an empirical character rather than a theoretical one. Initially dilative and contractive sands are analysed separately, as their behaviour during triaxial shearing is described by different functions.

Interesting results have been obtained in the case of initially contractive sand. The experimental strain-stress curves are smooth and have continuous first derivatives. They display no sign of instability, except perhaps for its slope, which is initially small and than increases suddenly. On the other hand, we know that initially contractive sands display pre-failure instabilities, as described in Section 3. These instabilities are related to the instability line. Our investigations show that the instability line may be related to the maximum value of the dilation function, and minimum value of the dilation parameter $\beta$. The increments of plastic work are always positive in the case considered, which means that our interpretation of experimental data is consistent with thermodynamics. A similar conclusion can be drawn from the analysis of Rowe's function. Its maximum may correspond to the instability line.

In the case of initially dilative sands, the instability is built into the equation describing the volumetric changes during shearing. The problem was how to interpret the experimental data in order to extract the plastic work (dissipation) and plastic dilation. Some formal techniques, e.g. direct extraction of elastic strains from total strains, have led to results inconsistent with thermodynamics, meaning that some production of energy has appeared. Therefore, we have applied an analogy between a simple mechanical scheme (a heavy block pushed up along an inclined plane) and dilation. This analogy has shown that negative work during shearing of initially dilative sand is a result of incorrect interpretation of experimental data. Therefore, it was assumed that this negative work is not dissipated but somehow increases potential energy of the system, similarly to the block example. All the work done by compaction due to shearing is dissipated. It is shown that using such an interpretation, the plastic work is always positive, as it should be according to thermodynamics. An interesting result is shape of the plastic dilation function, which monotonically decreases during the first stage of shearing, down to zero at the instability line. During the second stage of shearing, when the non-dimensional deviatoric stress increases to the critical state, the plastic dilation function is zero.

## 2. EXPERIMENTAL INVESTIGATIONS

### 2.1 Basic information and definitions

The study presented in this paper is based on experimental results obtained in the laboratory of the Institute of Hydro-Engineering, see Sawicki & Świdziński (2010 a). The experiments were performed in the triaxial apparatus, manufactured by GDS Instruments, which enables measurement of lateral strains. The experiments were performed on the model "Skarpa" sand, characterized by the following parameters: $D_{50}$ = 420 μm; $C_U$ = 2.5; $G$ = 2.65; $e_{max}$ =

0.677; $e_{\min}$ = 0.432. Angles of internal friction are: $\varphi$ = 34° and 41° for loose and dense sand respectively.

The steady state line (SSL) has been determined from series of both fully drained and undrained experiments, for density index ranging from 0.114 to 0.64, cf. Fig. 2. Its equation is the following:

$$e = 0.746 - 0.0635 \log p', \tag{1}$$

where $e$ denotes the void ratio and $p'$ is the mean effective stress.

The basic experiments dealt with spherical loading and unloading (path *0A0* in Fig. 1) and with deviatoric loading/unloading (path *ABCBA* in Fig. 1), for different values of the mean effective stress. These basic stress-strain curves were then used to construct a simple incremental model describing the pre-failure behaviour of investigated sand, both drained and undrained, see Sawicki & Świdziński (2010 a, b). In the present paper, these experimental curves will be used to study possible relationships between dilation and instability of sands.

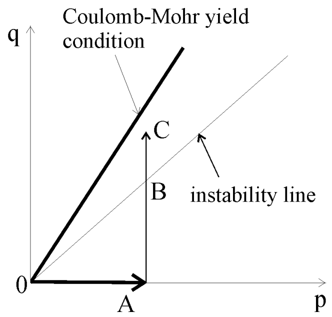


Figure 1 Stress paths for determination of basic stress-strain curves in triaxial conditions

Fig. 1 shows two important objects in the effective stress space. The first one is the Coulomb-Mohr yield condition, which is often identified with the critical state conditions. The second object, designated as "instability line", has some unusual properties, which will be discussed in the present paper, in connection with instability of granular soils and the phenomenon of dilation.

We shall use the following notation for strains and stresses: $\sigma_1$ = vertical stress; $\sigma_3$ = horizontal stress; $p = (\sigma_1 + 2\sigma_3)/3$ = mean stress; $q = \sigma_1 - \sigma_3$ = deviatoric stress; $\varepsilon_1$ = vertical strain; $\varepsilon_3$ = horizontal strain; $\varepsilon_v = \varepsilon_1 + 2\varepsilon_3$ = volumetric strain; $\varepsilon_q = 2(\varepsilon_1 - 2\varepsilon_3)/3$ = deviatoric strain; $\eta = q/p$ = non-dimensional stress ratio; $\eta'$ = stress ratio corresponding to instability line; $\eta''$ = stress ratio corresponding to the critical state (Coulomb-Mohr criterion). The soil mechanics sign convention is applied where a plus sign denotes compression. The behaviour of dry or fully drained sand is analysed, so that the total stresses are equal to the effective ones.

The Coulomb-Mohr failure condition, identified here with the critical state, has the following well known form:

$$q = \frac{6\sin\varphi}{3-\sin\varphi}\, p = \eta'' p \,. \tag{2}$$

## 2.2 Characteristic stress-strain curves

The basic experiments of interest dealt with a pure shearing of sand (path *ABC* in Fig. 1). The response of sand along this path depends on its initial state, defined as either contractive or dilative. Initially contractive sand compacts during shearing as illustrated by the arrow *AB* in Fig. 2. The behaviour of initially dilative sand is more complex, and will be described in more detail later, but the basic trend is illustrated by the arrow *CD* in Fig. 2, which shows the net dilation. Therefore, these two cases will be considered separately.

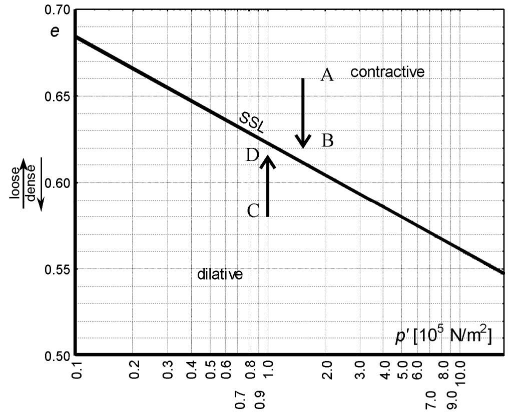


Figure 2 Steady state line for "Skarpa" sand. Path *AB* corresponds to pure shearing, at constant mean stress, of initially contractive sand; path *CD* to initially dilative sand

Fig. 3 shows the volumetric strains that develop during shearing of initially contractive sand, presented in the form of a single curve, in the space $\varepsilon_v/\sqrt{p}, \eta$. Such a single curve corresponds to various experiments, performed at different values of $p$. This single curve has been found using the method of trial and error. It is more convenient in theoretical analyses to deal with a single curve than to use a set of curves, each corresponding to a different value of $p$. In experimental and theoretical analyses, the following units are applied: **stress unit** $10^5\,\mathrm{N/m^2}$ and **strain unit** $10^{-3}$**.** This means that if $p = 200\,\mathrm{kPa} = 2\times10^5\,\mathrm{N/m^2}$, we shall substitute in our equations just a single number 2. If we obtain from some calculations a value of strain, say 0.13, it means that this strain is $0.13\times10^{-3} = 0.00013$. Such a convention is useful in numerical calculations, as we deal with numbers of similar orders of magnitude.

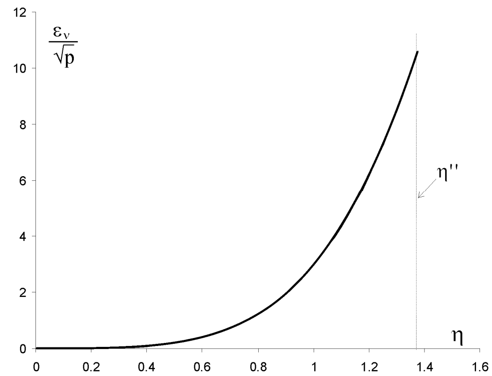


Figure 3 Volumetric strains during shearing of initially contractive sand

Analytical expression for the curve shown in Fig. 3 is the following:

$$\frac{\varepsilon_v}{\sqrt{p}} = c_1 \eta^4 , \tag{3}$$

where $c_1$ is a coefficient. For "Skarpa" sand and using our units, its value is 2.97.

Obviously, in the analysis of experimental data, one can use other analytical approximations. In the case considered, Eq. (3) seems to be a good fit, as there is only a single "material constant". The important problem of a minimal number of "material constants" is beyond the scope of this paper. We deal only with bare experimental data. It should also be added that the number 2.97 is not a fixed "material parameter" as reiteratebility is quite rare in soil mechanics. This number could be equal to 2 or 4 as well, as from the same set of grains we cannot prepare identical samples for laboratory investigations. Some time ago, we performed a simple statistical analysis and found that it is extremely difficult to predict the behaviour of granular soils precisely, see Sawicki & Chybicki (2009).

The other important stress-strain curve is shown in Fig. 4, which illustrates the changes of deviatoric strain during shearing. The shape of this curve is similar for both initially dilative and contractive sands. Previously, see Sawicki & Świdziński (2010 a), we proposed an exponential approximation of experimental data. In this paper, a better approximation is proposed, as it contains just a single parameter instead of two, and it takes into account the asymptotic behaviour at the critical state ( $\eta = \eta''$ ). A good fit to experimental data is the following:

$$\frac{\varepsilon_q}{\sqrt{p}} = \frac{c\eta}{\eta'' - \eta}, \tag{4}$$

where $c$ = 1.9, see previous remarks, and $\eta'' = 1.375$ (for $\varphi = 34°$ ).

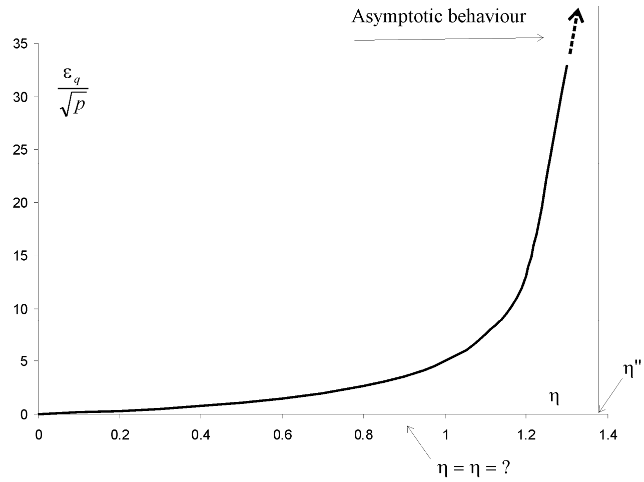


Figure 4 Deviatoric strains developed during shearing of initially contractive sand

In the case of initially dilative sand the deviatoric behaviour is similar to that shown in Fig. 4, and Eq. (4) also applies as a good fit, but for $\eta'' = 1.68$ $(\varphi = 41°)$. Fig. 5 summarizes experimental data describing the volumetric changes of initially dilative sand. The experiments were performed at different values of the mean stress, namely $p$ = 1; 2; 3 and 4 (recall respective units), and their results also fit into a single curve. Note that at the beginning of shearing, some minor compaction takes place, but after reaching some threshold, the rate of volumetric deformation changes sign, and rapid dilation takes place. This threshold corresponds to the instability line, cf. Sawicki & Świdziński (2010 a, b), defined by $\eta'$.

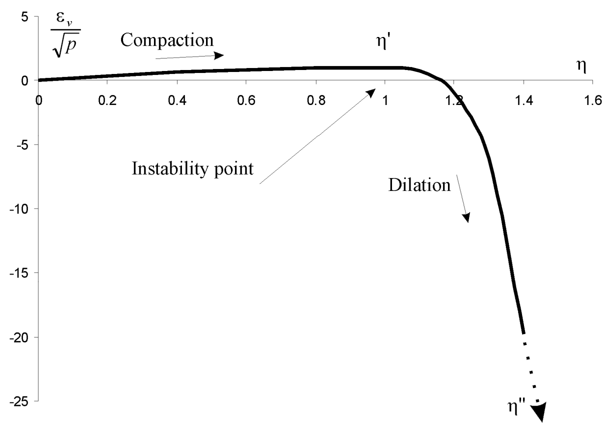


Figure 5 Volumetric strains during shearing of initially dilative sand

The volumetric deformations of initially dilative sand cannot easily be described by a single equation, so it was decided to apply the two following functions:

$$\frac{\varepsilon_v}{\sqrt{p}} = a_1\eta^2 + a_2\eta, \qquad 0 \le \eta \le \eta', \tag{5}$$

$$\frac{\varepsilon_v}{\sqrt{p}} = (a_3\eta + a_4)\exp(a_5\eta), \qquad \eta' \le \eta \le \eta'', \tag{6}$$

where:
$a_1 = -1;\ a_2 = 2;\ a_3 = -0.0106;\ a_4 = 0.0123;\ a_5 = 6.4;\ \eta' = 1$ . Eqs. (5) and (6) have the same value at $\eta' = 1$ , and their first derivatives are zero at this point, so the whole function is continuous and has a continuous first derivative with respect to $\eta$ .

Eqs. (3)–(6) describe the soil behaviour during loading which is defined here by the condition $d\eta > 0$ . These are the total strains which contain both the elastic and plastic parts. The strains that develop during unloading ( $d\eta < 0$ ) are given by the following relations:

$$\frac{\varepsilon_v^{el}}{\sqrt{p}} = a_v\eta, \tag{7}$$

$$\frac{\varepsilon_q^{el}}{\sqrt{p}} = b_q\eta, \tag{8}$$

where typical values of the coefficients appearing in the above equations are the following: $a_v = -0.87$ and –0.39 for initially loose and dense sand respectively; $b_q = 0.76$ and 0.4 also for loose and dense sand. The superscript *el* is used to distinguish these strains, although it is not clear whether in the case of volumetric strains one deals with an elastic response. Note that $a_v < 0$ , which means that compaction takes place during the deviatoric unloading, which is consistent with empirical observations during cyclic loadings, where compaction is caused by both, positive and negative increments of shear stress. Eq. (8) may indeed be identified with the elastic shear strains.

## 3. STABILITY AND DILATION

Some aspects of unstable behaviour of sand in triaxial tests are presented in Sawicki & Świdziński (2010 c). It is interesting to notice that some features of the pre-failure instability are exhibited by initially contractive sands. For example, in the case of spherical unloading of dry/fully drained sand, we observe a sudden collapse of the soil skeleton structure after reaching some threshold. This threshold has been identified with the instability line. Fig. 6 illustrates such a behaviour.

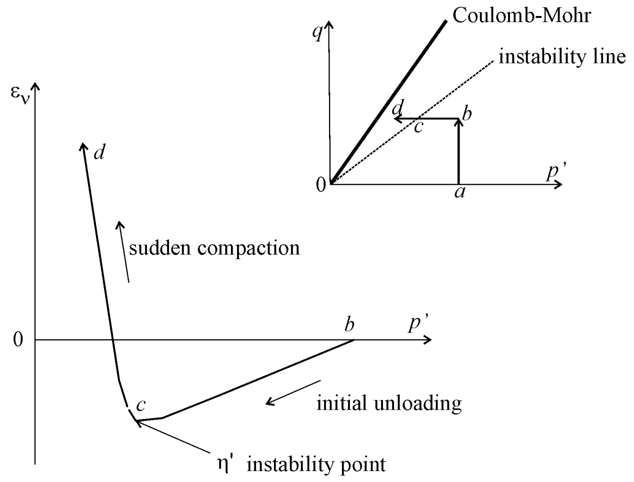


Figure 6 Instability of granular soil during spherical unloading

The experiments illustrated in Fig. 6 started with spherical compaction (path *0a* in the inset to Fig. 6). Then, deviatoric stress was applied (path *ab*). In the third stage of the experiment (path *bcd*), the deviatoric stress was kept constant, and the mean pressure was reduced. In the first stage of unloading (path *bc*), the sand dilates. After reaching the threshold (path *cd*), a sudden collapse (compaction) of the soil skeleton structure is observed. The beginning of this collapse corresponds to the instability line.

The second interesting experiment deals with the undrained behaviour of initially contractive sand. During shearing of such samples the phenomenon of static liquefaction takes place, which is also an unstable behaviour. Fig. 7 shows the effective stress path followed during undrained shearing of initially contractive sand, which is already a well known behaviour.

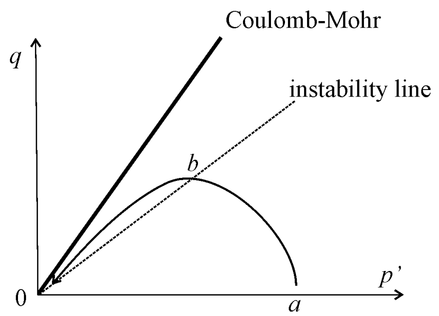


Figure 7 Effective stress path followed during shearing of initially contractive sand – phenomenon of static liquefaction

Along path *ab* the effective mean stress $p'$ decreases, and the stress deviator increases. The point *b* corresponds to the instability line. This point also corresponds to the maximum value of shear stress that can be supported by the soil skeleton. After reaching this point, the shear stress decreases suddenly and the phenomenon of static liquefaction takes place.

Both behaviours can be described within the framework of the incremental model proposed by Sawicki & Świdziński (2010 a, b). Analysis of the above examples using that model leads to analytical determination of the instability line, which is given by the following formula:

$$\eta' = \left(A_v^u / 7c_1\right)^{1/4}, \tag{9}$$

where $c_1$ characterizes the volumetric changes of initially contractive sand during shearing, and $A_v^u$ is a modulus characterizing the elastic volumetric changes during spherical unloading, see Sawicki & Świdziński (2010 c). The shape of Eq. (9) shows that both mechanisms, i.e. volumetric deformations caused by shearing and spherical elastic response, influence the unstable behaviour of sand.

An interesting observation is that the above described instabilities can be observed in initially contractive sand, the behaviour of which is described by monotonically increasing functions, such as those shown in Figs. 3 and 4. Therefore, it would be interesting to see whether the unstable behaviour of sand is related to some other phenomena such as, for example, dilation.

The term dilation, or dilatancy, is used to describe volumetric changes in sand during shearing, see Atkinson (1993), Craig (1987), Vardoulakis & Sulem (1995), Wood (1990). The following definition of dilation is adapted in this paper:

$$D = \frac{d\varepsilon_v}{d\varepsilon_q} = \tan\psi, \tag{10}$$

where $\psi$ denotes the angle of dilation.

In geomechanics, usually the plastic parts of strain increments are taken into account, so dilation may also be defined as follows:

$$D^{pl} = \frac{d\varepsilon_v^{pl}}{d\varepsilon_q^{pl}}, \tag{11}$$

where the superscript *pl* denotes the plastic part of respective quantity.

The reciprocal of dilation is related to the so-called dilatancy parameter $\beta$ , defined as follows:

$$\frac{1}{D} = \tan\beta. \tag{12}$$

The above relations will be used to study the pre-failure behaviour of sand during shearing. The strain increments will be obtained by differentiation of respective approximations of experimental data, given by Eqs. (3)–(6). One advantage of such a method is that one deals with real empirical results, presented in a useful analytical form.

## 4. INITIALLY CONTRACTIVE SAND

### 4.1 Dilation

Differentiation of Eqs. (3) and (4) and substitution to Eq. (10) leads to the following relation:

$$D = \frac{4c_1\eta^3(\eta'' - \eta)^2}{c\eta''}. \tag{13}$$

One can also determine a plastic dilation, which is given by the following formula:

$$D^{pl} = \frac{4c_1\eta^3(\eta'' - \eta)^2}{c\eta'' - (\eta'' - \eta)^2 b_q} . \qquad (14)$$

In the above formula, we have assumed that $d\varepsilon_v^{pl} = d\varepsilon_v$, for the above mentioned reasons. Fig. 8 illustrates the changes in total and plastic dilations during shearing. The shape of both functions is similar, and there are some small differences regarding the peak values of these curves ( $\eta' \approx 0.8$ ). This result may suggest that there is a link between the maximum value of dilation and the instability line. Note that the dilation during shearing of initially contractive sand, both total and plastic, is always positive.

A simple analytical method shows that *D* attains its maximum at $\eta = 0.6\eta'' = 0.825$. This is an interesting result, which shows again a possible link between dilation and the instability line. It is also interesting that the maximum value of *D* depends solely on the single parameter $\eta''$, which is related to the critical state. No other material coefficient characterizing the deformability of sand influences the maximum value of *D.*

Fig. 9 shows the dilation parameter as a function of $\eta$ . The minimum value of $\beta$ roughly corresponds to the instability line $(\eta' = 0.8)$ . This relation was determined for the plastic strain increments. It should be noticed that the shape of the respective function is different from that suggested in some classical soil mechanics textbooks, see Fig. 8.2b on page 227 in Wood (1990). At the critical state $(\eta = \eta'')$ there is $\beta = 90°$, as the volumetric strain increments are zero in this case.

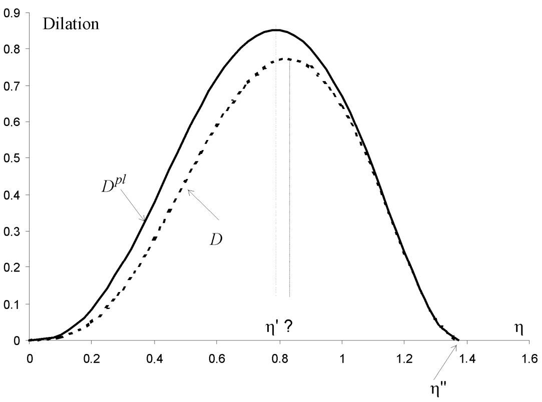


Figure 8 Dilation of contractive sand during shearing. The differences between the total dilation and plastic are small in practice

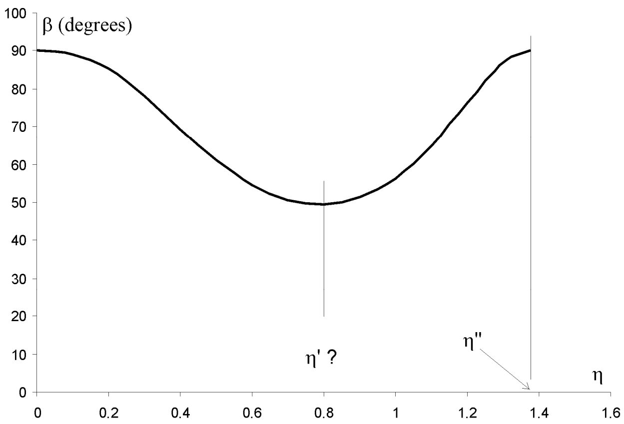


Figure 9 Dilation parameter as function of non-dimensional stress ratio

## 4.2. Strains

It is interesting to study how the strains develop during shearing of initially contractive sand. Fig. 10 illustrates the development of plastic strains, in the space of volumetric and deviatoric strains, which are divided by the square root of the mean stress. This figure was drawn on the basis of Eqs. (3) and (4). We can observe the two almost linear, basic trends. The first one deals with the shearing of sand up to a value of $\eta$ between 1.1 and 1.2. The second trend is characterized by a steeper slope. Note that a value of $\eta$ , corresponding to the instability line, is close to the origin of co-ordinates. Therefore, a change of respective trends cannot be explained in terms of soil stability. It should be added that Fig. 10 does not illustrate some extreme trends simply due to scale: the first one near the origin of co-ordinates, and the second near the critical state conditions. Fig. 11 shows how these small strains develop. Note that the strains corresponding to $\eta = 0.8$ (instability line ?) are located at the beginning of the first trend. The end of trend 2 is asymptotic as the deviatoric strains increase indefinitely when the critical state $(\eta = \eta'')$ is approached. The strains shown in Fig. 10 correspond to the variation of dilation coefficient shown in Fig. 9.

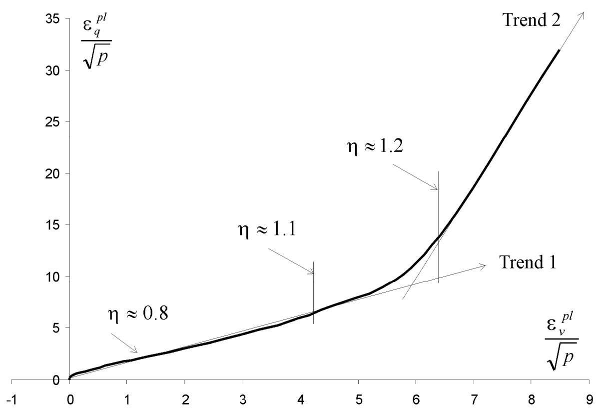


Figure 10 Development of plastic strains during shearing of initially contractive sand

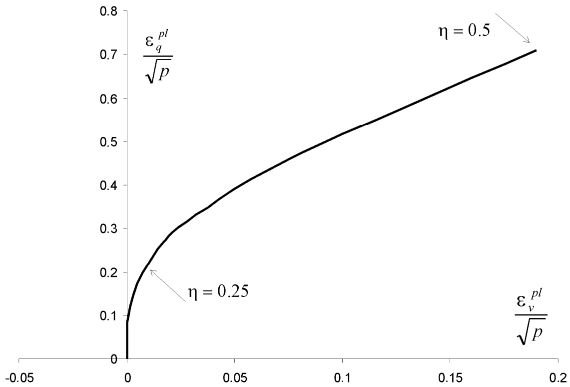


Figure 11 Development of plastic strains at the beginning of shearing, cf. Fig. 10

### 4.3 Plastic work

The plastic work increment, in the case of triaxial tests, is given by the following formula:

$$dW^{pl} = pd\varepsilon_v^{pl} + qd\varepsilon_q^{pl}. \tag{15}$$

Respective strain increments can be obtained by differentiation of Eqs. (3), (4) and (8). In the case of volumetric strains it is assumed that the plastic strains are equal to the total ones. After some re-arrangement, Eq. (15) takes the following form:

$$\frac{dW^{pl}}{p^{3/2}d\eta} = \left\{4c_1\eta^3 + \left[\frac{c\eta''}{(\eta''-\eta)^2} - b_q\right]\eta\right\}. \tag{16}$$

Fig. 12 illustrates Eq. (16) for the data corresponding to "Skarpa" sand. Recall that $p$ = const. Note that the function (16) is always positive. Its slope rapidly increases for $\eta$ from the interval 0.8–1 which may be related to a "hidden" instability line.

According to Hill's principle, the behaviour of initially contractive sand during shearing is stable, as the second order work is positive in this case, i.e. $d^2W = dqd\varepsilon_q > 0$. It can also be easily shown that Drucker's classical postulate of stability is also satisfied, i.e. $d^2W^{pl} = dqd\varepsilon_q^{pl} > 0$. Recall that $dp = 0$ in the case considered.

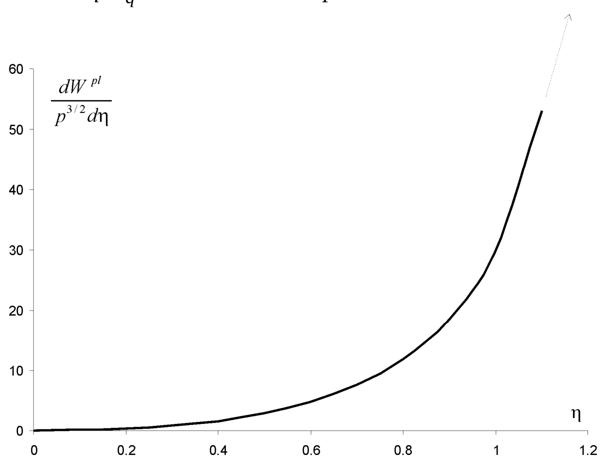


Figure 12 Illustration of Eq. (16)

### 4.4 Rowe's stress-dilatancy relation

It is also interesting to see how our experimental data fit into the original stress-dilatancy theory proposed half a century ago by Rowe, which is still applied in soil mechanics. We shall make use of the following formulation of this relation, quoted by Jefferies & Been (2006) on page 39:

$$\frac{\sigma_1}{\sigma_3} = K\left(1 - \frac{d\varepsilon_v}{d\varepsilon_1}\right), \tag{17}$$

where $K$ was originally considered as constant, but is rather a function of applied shearing stress. Eq. (17) can be re-arranged to the following form:

$$K = \left(\frac{3+2\eta}{3-\eta}\right)\left(\frac{1/3d\varepsilon_v + d\varepsilon_q}{-2/3d\varepsilon_v + d\varepsilon_q}\right). \tag{18}$$

In the above equation the total strain increments are introduced. Fig. 13 illustrates the relation (18) for the data corresponding to initially contractive "Skarpa"sand. Eq. (18) can be re-arranged again to show a link between the Rowe's function and dilation:

$$K = \left(\frac{3+2\eta}{3-\eta}\right)\left(\frac{3+D}{3-2D}\right). \tag{19}$$

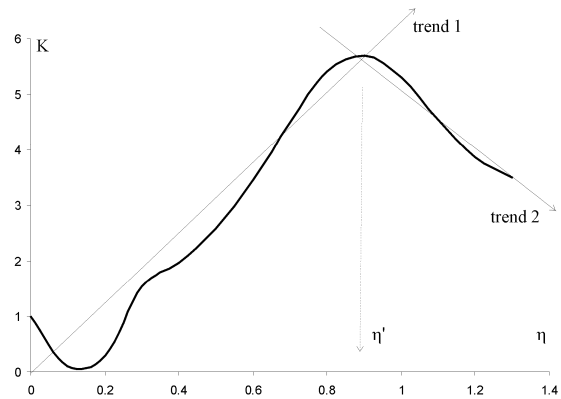


Figure 13 Rowe's function for initially contractive sand

The shape of function $K = K(\eta)$ is irregular, but one can identify the two basic linear trends. The first trend is characterized by increasing values of $K$, whilst the second is marked by decreasing values of this function. These trends meet at the point $\eta \approx 0.9$, which suggests that there could possibly be a link between the instability line and maximum value of the Rowe's function.

### 4.5 Summary of results for initially contractive sand

a. Previous studies have shown that initially contractive sand displays some pre-failure instabilities, see examples presented in Section 3 (spherical unloading of dry sand; static liquefaction).
b. This unstable behaviour can be predicted from the incremental model constructed on the basis of experimental data. Experimentally determined stress-strain curves, see Figs. 3 and 4, are smooth and monotonically increasing functions of non-dimensional stress ratio. Their shape does not exhibit any sign of unstable behaviour, but using these curves instability can

be predicted for different loading paths. The question is then, whether this "hidden" instability can be detected using other methods.

c. It was shown that commonly accepted stability criteria, such as Drucker's and Hill's postulates, confirm a stable behaviour of initially contractive sand during pure shearing.

d. A natural method for investigating a "hidden" instability was to study dilation. It was shown that maximum dilation may correspond to the instability line, see Fig. 8. This criterion is equivalent to the minimum of dilation parameter $\beta$, see Fig. 9. These findings may be treated, at present, merely as some hypotheses.

e. The other interesting observation deals with the slope of stress-strain curves (Figs. 3 and 4). At the beginning of shearing we deal with small strains, and the slopes of these curves are also small. After reaching a "hidden" instability line, this slope increases drastically, and the strains become large. This experimental finding could be a basis for some theoretical hypotheses.

f. Finally, we have applied Rowe's stress-dilatancy theory to see how this approach is related to our empirical results. It was shown that the maximum value of Rowe's function may correspond to a "hidden" instability line.

g. One of the basic questions deals with a decomposition of the strain tensor onto the elastic and plastic parts. In the case of deviatoric behaviour, the answer is almost obvious, but in the case of volumetric strains things are not so straightforward. The basic question is whether dilation has a reversible aspect or not? In this paper it was assumed that the total volumetric strains are equal to the plastic ones. For further explanation, see the above text.

## 5. INITIALLY DILATIVE SAND

### 5.1 Plastic strains and dissipation function

The basic behaviour of initially dilative sand is described by functions (4), (5) and (6), and the elastic deviatoric strain given by Eq. (8). The basic stress-strain curves are presented in Figs. 4 and 5. Recall that these stress-strain relations summarise experimental results, so they should be treated as relations which describe the real behaviour of initially dilative sand during shearing. A similar procedure to that applied in the case of initially contractive sand will be employed.

Fig. 14 illustrates an initial part of the total strain path followed during shearing along the stress path *ABC* in Fig. 1, cf. Figs. 4 and 5. A similar curve is shown by Vardoulakis & Sulem (1995) on page 242 of their book. At the beginning of shearing the volumetric strain increases, according to the initial compaction shown in Fig. 5, and after reaching the instability line, the process of dilation begins.

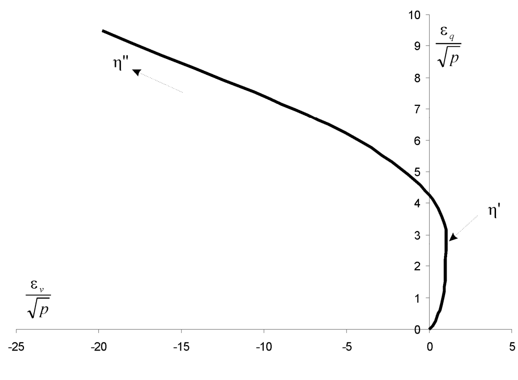


Figure 14 Total strain path followed during shearing of initially dilative sand

An important problem is how to isolate the plastic (irreversible) strains. A formal procedure of direct extraction of elastic strains from both the volumetric and deviatoric parts of the total strain tensor does not work in this case, as it leads to violation of basic thermodynamic principles. For example, the dissipation function (15) becomes negative during advanced shearing. Analysis of a simple mechanical system suggested by Atkinson (1993), cf. Fig. 10.10 b on page 130 of his book, as an analogue to dilation, leads to some useful hints. Consider a heavy block resting on an inclined plane, as illustrated in Fig. 15. In order to push upwards the block, the force *P* should resist the friction force $T = \mu G \cos\alpha$ and component of the block's own weight *G*, parallel to the sliding plane, i.e. $G\sin\alpha$ The following symbols are used: $\mu$ = coefficient of friction between the block and sliding plane; $\alpha$ = inclination of sliding plane. The work done by force *P* on displacement *u* is the following:

$$L = Pu = \mu uG\cos\alpha + uG\sin\alpha\ . \tag{20}$$

The first member on the RHS of Eq. (20) represents a dissipated (irreversible) part of work due to friction. The second is the part of the work used for lifting the center of gravity of the block by $u\sin\alpha$. This part of the work is reversible, as the potential energy of the block has increased.

The above analogy suggests that the negative work done by *p* on dilative volumetric strain increments $(\text{for } \eta' \le \eta < \eta'')$ should be treated in a similar way, i.e. certainly not as a dissipative part. Positive work done by *p* on compaction, during the first stage of shearing, $(0 \le \eta \le \eta')$ is dissipated, so the total volumetric strains are equal to plastic ones in this case. The above assumptions lead to a new interpretation of experimental data, which does not violate thermodynamic principles. Fig. 16 shows the path of plastic strains followed during shearing of initially dilative sand, obtained on the basis of the above interpretation. During the first part of shearing $(0 \le \eta \le \eta')$, both strain components increase monotonically, and after reaching the instability line, only the deviatoric strains increase, whilst the volumetric strains do not change.

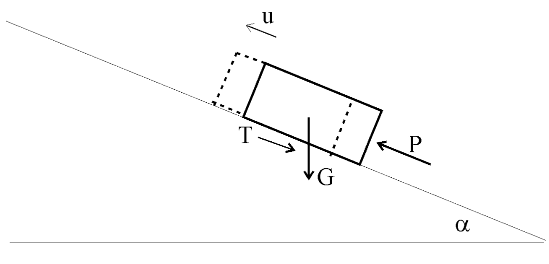


Figure 15 A heavy block pushed along an inclined plane

Fig. 17 shows the dissipation function (plastic work, cf. Eq. 15) for the plastic strains presented in Fig. 16. It is always positive, as *p > 0* and $d\eta > 0$, as it should be according to thermodynamics

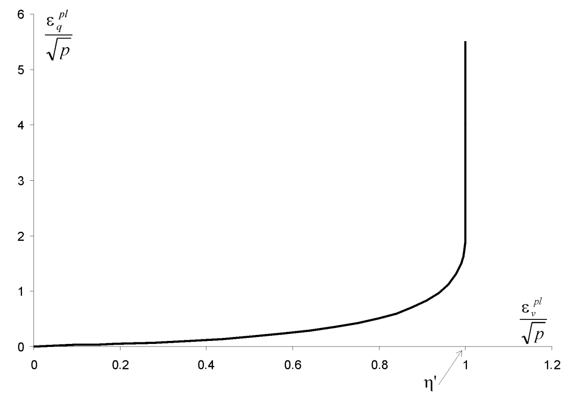


Figure 16 Plastic strains that develop during shearing of initially dilative sand

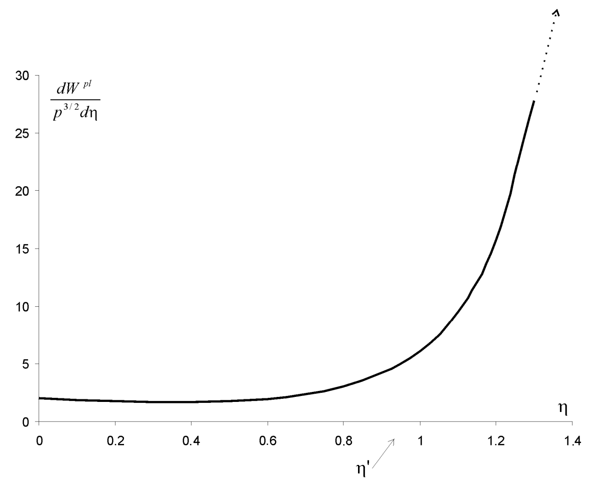


Figure 17 Dissipation function for the plastic strains from Fig. 16

## 5.2 Dilation and Rowe's function

In the case of dilative sand, the instability has been built into Eq. (5) and (6), see also Fig. 5, in contrast to the relations describing the pre-failure behaviour of initially contractive sands, where the respective stress-strain curves are smooth, monotonically increasing and having a monotonically increasing first derivative. Fig. 18 shows the dilation function (Eq. 11) for the case considered. $D^{pl}$ is a monotonically decreasing function of $\eta$ during the first stage of shearing, and during the second stage $(\eta > \eta')$ its value is zero, which corresponds to the dilation parameter $\beta = 90°$.

Fig. 19 shows the respective Rowe's function, determined from Eq. (19) after substituting $D = D^{pl}$. This function has a singularity corresponding to $D^{pl} = 3/2$ (denominator of Eq. 19 tends to zero), which can hardly be explained on the basis of physical considerations. For $\eta > 0.4$, this function is almost constant, similarly to an original concept of Rowe.

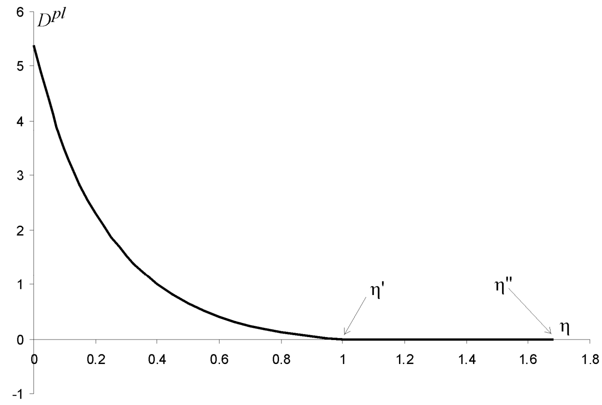


Figure 18 Plastic dilation during shearing of initially dilative sand

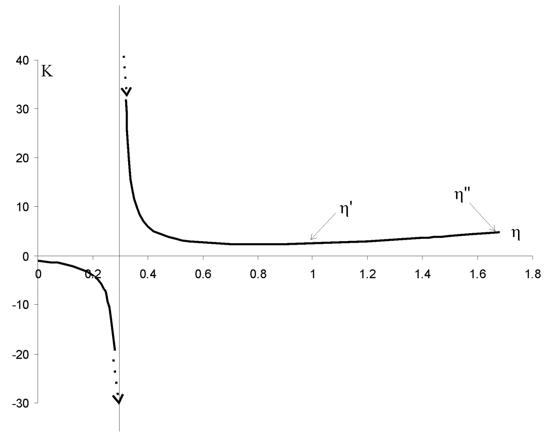


Figure 19 Rowe's function for initially dilative sand

## 5.3 Summary of results for initially dilative soil

a. The basic difference between initially contractive and dilative sands is that the character of volumetric strains differs during shearing, cf. Figs. 3 and 5. In the case of initially dilative sand, compaction takes place during the first stage of shearing, and after reaching the instability line the process of dilation takes place. These observations have been taken into account in analytical approximation of experimental data, cf. Eqs. (5) and (6).

b. Analysis of the plastic dissipation function shows that the negative work done by *p* on expanding sand during the stage of dilation should not be included into the plastic work relationship. This conclusion is supported by a simple mechanical model shown in Fig. 15.

c. The work done by *p* on initial compaction is irreversible, and therefore should be included into the plastic work relationship. The plastic work of deviatoric stress on respective strains is obtained in a traditional manner, simply by extracting the reversible part of work from the total. The above assumptions assure that the function of plastic work is always positive, as is required by thermodynamics.

d. An interesting shape of the plastic dilation function (Fig. 18) is a consequence of the above interpretation of experimental data. It is interesting to note that after reaching the instability line, the plastic dilation is equal to zero.

## 6. DISCUSSION AND CONCLUSIONS

The basic results obtained in the present paper are summarized in Sections 4.5 and 5.3 for the initially contractive and dilative sands respectively. It is believed that these are original results, as they show possible links between some aspects of pre-failure instability of sand (instability line) and plastic dilation. Recall that the analysis presented is based on bare experimental data, approximated by analytical formulae, so we deal with the real behaviour of sand. Interpretation of these experimental data was performed under the single theoretical assumption that the basic thermodynamical principles should not be violated. Experimental curves, presented in analytical form, can easily be used by other researchers for their purposes.

The parameters appearing in equations approximating the experimental data should be treated as some averages for the model "Skarpa" sand and for a limited range of void ratios. Loose samples, characterized by $I_D$ ranging from 0.22 to 0.36, and dense samples ( $I_D = 0.713 - 0.859$ ) were investigated. As already mentioned in Section 2.2, it is very difficult in laboratory investigations to obtain reiteratebility of results. We have analyzed this problem for isotropic compression along path 0A in Fig. 1. In such a case, the volumetric deformations can nicely be approximated by the following formula: $\varepsilon_v = A_v\sqrt{p'}$ , where $A_v$ is a coefficient. For initially loose sand ( $I_D = 0.016 - 0.445$ , 23 experiments), we have obtained the values of $A_v$ ranging from 3.73 to 7.32. For dense sand ( $I_D = 0.71 - 0.86$ , 26 experiments), this coefficient varied from 2.23 to 4.54. The above data provide some view regarding the values of material parameters.

Recall that it is not the aim of this paper to discuss such problems, which have already been presented in the quoted literature, Sawicki & Chybicki (2009), Sawicki & Świdziński (2010 a,b), but to find possible links between dilation and instability of granular soils.

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