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Brahmagupta Triangles in Generalized Fibonacci Sequences

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บทคัดย่อ

สำหรับทุกจำนวนเต็มบวก α และ β นิยามลำดับ $(G_n(\alpha, \beta))$ โดยกำหนดให้ $G_1 = \alpha$, $G_2 = \beta$ และ $G_n = G_{n-1} + G_{n-2}$ สำหรับทุก $n \geq 3$ สามเหลี่ยมพรหมคุปต์เป็นสามเหลี่ยมที่มีความยาวด้านเป็นจำนวนเต็มบวกที่เรียงติดกัน และมีพื้นที่เป็นจำนวนเต็ม ในบทความนี้ เราพิสูจน์ว่าสำหรับ α และ β ใด ๆ จะมีสามเหลี่ยมพรหมคุปต์ที่มีความยาวด้านทั้งหมดอยู่ในลำดับ $(G_n(\alpha, \beta))$ ได้มากที่สุดแค่หนึ่งรูปเท่านั้น ยิ่งไปกว่านั้น เราได้จำแนกเงื่อนไขเชิงลักษณะของค่าเริ่มต้น (α, β) ทั้งหมด ที่ทำให้ลำดับ $(G_n(\alpha, \beta))$ มีสามเหลี่ยมพรหมคุปต์ปรากฏอยู่ และแสดงให้เห็นว่าผลดังกล่าวเกิดขึ้นก็ต่อเมื่อค่าเริ่มต้นอยู่ในสองตระกูลที่สามารถพารามิเตอร์ด้วยสมการเพลล์เท่านั้น

ABSTRACT

For any positive integers α and β , define the sequence $(G_n(\alpha, \beta))$ by $G_1 = \alpha$, $G_2 = \beta$, and $G_n = G_{n-1} + G_{n-2}$ for all $n \geq 3$. A Brahmagupta triangle is a triangle all of whose side lengths are consecutive positive integers with integral area. In this paper, we prove that for any given α and β , there is at most one Brahmagupta triangle all of whose side lengths are elements of $(G_n(\alpha, \beta))$. Moreover, we characterize all initial values (α, β) for which the sequence $(G_n(\alpha, \beta))$ contains a Brahmagupta triangle, and show that this occurs precisely in two Pell-parametrized families.

คำสำคัญ: สามเหลี่ยมพรหมคุปต์ สามเหลี่ยมของเฮรอน ลำดับฟีโบนัชชีทั่วไป

Keywords: Brahmagupta Triangle, Heron Triangle, Generalized Fibonacci Sequence

INTRODUCTION

A Heron triangle (or a Heronian triangle) is an integer triangle (a triangle all of whose side lengths are positive integers) that also has integral area. This kind of triangle connects concepts from number theory to plane geometry and has been classified completely, see e.g. Carmichael (1959). Some authors also studied Heron triangles from various perspectives, see e.g. Bailey and Gosnell (2012), Beaugard and

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Suryanarayan (1998) and Gould (1973). Harborth *et al.* also provided an alternative research direction in which they considered Heron triangles inside the (standard) Fibonacci sequence (F_n) . They proved that Heron triangles of the type (F_k, F_k, F_n) where $1 \leq k < n$ exist only when $k = 5$ and $n = 6$ (Harborth and Kemnitz, 1988). They also proved that Heron triangles of the type (F_{n-k}, F_n, F_n) do not exist when $1 \leq k \leq 5$, see Harborth *et al.* (1996).

A Brahmagupta triangle is a Heron triangle all of whose side lengths are consecutive positive integers. Several authors have studied and characterized all possible Brahmagupta triangles, see e. g. Beauregard and Suryanarayan (1998) and Nelsen (2020). Let α and β be positive integers. A generalized Fibonacci sequence $(G_n(\alpha, \beta))$ is a sequence of positive integers defined by $G_1 = \alpha$, $G_2 = \beta$, and $G_n = G_{n-1} + G_{n-2}$ for all $n \geq 3$.

The main purpose of this paper is to follow the direction of research in Harborth and Kemnitz (1988) and Harborth *et al.* (1996) for the sequence $(G_n(\alpha, \beta))$. Furthermore, we prove a uniqueness result stating that for any fixed pair of positive integers (α, β) , the sequence $(G_n(\alpha, \beta))$ contains at most one Brahmagupta triangle. Finally, we characterize all initial values (α, β) for which the sequence $(G_n(\alpha, \beta))$ contains a Brahmagupta triangle, and show that this occurs precisely in two Pell-parametrized families.

PRELIMINARIES

Let α and β be positive integers. Define the sequence (G_n) by $G_1 = \alpha$, $G_2 = \beta$ and

$$G_n = G_{n-1} + G_{n-2}$$

for any $n \geq 3$. The sequence (G_n) is called the generalized Fibonacci sequence or the sequence of generalized Fibonacci numbers. We denote (F_n) as the standard Fibonacci sequence, and (L_n) as the Lucas sequence. Note that $F_n = G_n(1, 1)$ and $L_n = G_n(2, 1)$ for all positive integers n . By an integer triangle we mean a triple of positive integers $\Delta = (x, y, z)$ satisfying the triangle inequalities: $x + y > z$, $y + z > x$, and $z + x > y$. Note that if $\Delta = (x, y, z)$ is an integer triangle, then any triple Δ' obtained by a permutation of x , y , and z is again an integer triangle. Moreover, they are also congruent and hence can be considered as the same triangle. For any integer triangle Δ , we denote $[\Delta]$ as the area of Δ . An integer triangle Δ is called Heronian if $[\Delta]$ is a positive integer. A characterization of Heronian triangles can be found, for example, in Carmichael (1959). A Brahmagupta triangle is an integer triangle all of whose side lengths are consecutive positive integers that is also Heronian. We note that throughout the paper, α and β denote the initial values, while i, j, k, n, r denote sequence indices.

We will also need the following well-known results, see e.g. Andreescu *et al.* (2010).

Lemma 2.1. The Pell equation $x^2 - 3y^2 = 1$ has all solutions (x_n, y_n) given by

$$x_n + \sqrt{3}y_n = (2 + \sqrt{3})^n$$

for any $n \in \mathbb{N}$.

Lemma 2.2. The Pell-type equation $x^2 - 3y^2 = -2$ has all solutions (x_n, y_n) given by

$$x_n + \sqrt{3}y_n = (1 + \sqrt{3})(2 + \sqrt{3})^{n-1}$$

for any $n \in \mathbb{N}$.

Lemma 2.3. The triple (x, y, z) is a solution of the Diophantine equation $x^2 + 3y^2 = z^2$ if and only if either one of the following conditions holds.

- (1) There are positive integers t, m , and n with m and n odd, $m > \sqrt{3}n$, and $\gcd(m, 3n) = 1$ such that

$$(x, y, z) = \left(\frac{t}{2}(m^2 - 3n^2), tmn, \frac{t}{2}(m^2 + 3n^2) \right).$$

- (2) There are positive integers t, m , and n with m and n odd, $m < \sqrt{3}n$, and $\gcd(m, 3n) = 1$ such that

$$(x, y, z) = \left(\frac{t}{2}(3n^2 - m^2), tmn, \frac{t}{2}(m^2 + 3n^2) \right).$$

- (3) There are positive integers t, m , and n with m and n having different parity, $m > \sqrt{3}n$, and $\gcd(m, 3n) = 1$ such that

$$(x, y, z) = (t(m^2 - 3n^2), 2tmn, t(m^2 + 3n^2)).$$

- (4) There are positive integers t, m , and n with m and n having different parity, $m < \sqrt{3}n$, and $\gcd(m, 3n) = 1$ such that

$$(x, y, z) = (t(3n^2 - m^2), 2tmn, t(m^2 + 3n^2)).$$

SCALENE INTEGER TRIANGLES IN GENERALIZED FIBONACCI SEQUENCES

In this section, we characterize all possible scalene integer triangles all of whose side lengths are elements of the sequence $(G_n(\alpha, \beta))$. We also characterize all scalene Heron triangles in $(G_n(\alpha, \beta))$. As all Brahmagupta triangles are scalene and Heronian, this will be our main ingredient to characterize Brahmagupta triangles subsequently.

Theorem 3.1. $\Delta = (G_i, G_j, G_k)$ is a scalene integer triangle with $G_i < G_j < G_k$ if and only if $(i, j, k) = (1, 3, 4)$ and $\alpha > \beta$.

Proof. Let $\Delta = (G_i, G_j, G_k)$ be such that $G_i < G_j < G_k$. Suppose $i \geq 2$. Then $2 \leq i < j < k$ and hence $j \geq i + 1$ and $k \geq j + 1$. This implies

$$G_k \geq G_{j+1} = G_j + G_{j-1} \geq G_j + G_i$$

which contradicts the triangle inequality. Thus $i = 1$. For $j = 2$, we have $k \geq 3$ and $G_3 = G_2 + G_1 > G_k \geq G_3$ which is a contradiction. Next, suppose $j \geq 4$. Then $G_{j-1} > G_1$ and, using the triangle inequality,

$$G_1 + G_j > G_k \geq G_{j+1} = G_j + G_{j-1} > G_j + G_1$$

which is a contradiction. Thus $j = 3$. Finally, suppose $k \geq 5$. Then

$$G_k = G_{k-1} + G_{k-2} > G_3 + G_1 = G_j + G_i$$

which is impossible. In conclusion, we have $(i, j, k) = (1, 3, 4)$. The triangle inequality implies that

$G_1 + G_3 > G_4$ which is equivalent to $\alpha > \beta$. The converse is obvious by triangle inequalities.

Corollary 3.2. The sequence (F_n) contains no scalene integer triangles. The sequence (L_n) contains exactly one scalene integer triangle, namely, $\Delta = (2, 3, 4)$.

Proof. We note that $(F_1, F_3, F_4) = (1, 2, 3)$ which is a degenerate triangle. Also, we have that $(L_1, L_3, L_4) = (2, 3, 4)$ which is an integer triangle.

Corollary 3.3. For any positive integers α and β , the sequence $(G_n(\alpha, \beta))$ contains at most one Brahmagupta triangle.

Proof. This follows directly from theorem 3.1 since there is only one scalene integer triangle in $(G_n(\alpha, \beta))$.

Next, we characterize all possible scalene Heron triangles in the sequence $G_n(\alpha, \beta)$. This can be done only in terms of the initial values α and β .

Theorem 3.4. $\Delta = (G_1, G_3, G_4)$ is a Heron triangle if and only if either one of the following conditions holds.

- (1) $(\alpha, \beta) = \left(\frac{t}{2}(m^2 + 9n^2), \frac{t}{2}(m^2 - 3n^2)\right)$ where t, m, n are positive integers with $m > \sqrt{3}n$, m and n odd, and $\gcd(m, 3n) = 1$. In this case, $[\Delta] = \frac{3}{2}mn(m^2 + 3n^2)t^2$.
- (2) $(\alpha, \beta) = \left(\frac{t}{2}(3m^2 + 3n^2), \frac{t}{2}(3n^2 - m^2)\right)$ where t, m, n are positive integers with $m < \sqrt{3}n$, m and n odd, and $\gcd(m, 3n) = 1$. In this case, $[\Delta] = \frac{3}{2}mn(m^2 + 3n^2)t^2$.
- (3) $(\alpha, \beta) = (t(m^2 + 9n^2), t(m^2 - 3n^2))$ where t, m, n are positive integers with $m > \sqrt{3}n$, $\gcd(m, 3n) = 1$, and m and n having different parity. In this case, $[\Delta] = 6mn(m^2 + 3n^2)t^2$.
- (4) $(\alpha, \beta) = (t(3m^2 + 3n^2), t(3n^2 - m^2))$ where t, m, n are positive integers with $m < \sqrt{3}n$, $\gcd(m, 3n) = 1$, and m and n having different parity. In this case, $[\Delta] = 6mn(m^2 + 3n^2)t^2$.

Proof. Let $\Delta = (G_1, G_3, G_4)$. Suppose Δ is Heronian. Note that $\Delta = (\alpha, \alpha + \beta, \alpha + 2\beta)$. Heron's formula gives

$$[\Delta]^2 = 3 \left(\frac{\alpha - \beta}{2}\right) \left(\frac{\alpha + 3\beta}{2}\right) \left(\frac{\alpha + \beta}{2}\right)^2.$$

If $\alpha - \beta$ is odd, then $[\Delta]^2$ is not an integer, a contradiction. Thus $\alpha - \beta$ is even. This implies $\frac{\alpha - \beta}{2}$, $\frac{\alpha + \beta}{2}$, and $\frac{\alpha + 3\beta}{2}$ are all integers. Since $[\Delta]$ is a positive integer, we have

$$\left(\frac{\alpha - \beta}{2}\right) \left(\frac{\alpha + 3\beta}{2}\right) = 3\gamma^2 \quad (3.1)$$

for some positive integer γ . Note that $\frac{\alpha + 3\beta}{2} = \frac{\alpha - \beta}{2} + 2\beta$. Thus (3.1) is equivalent to

$$\begin{aligned} \left(\frac{\alpha - \beta}{2}\right) \left(\frac{\alpha - \beta}{2} + 2\beta\right) &= 3\gamma^2 \\ \left(\frac{\alpha - \beta}{2}\right)^2 + 2\beta \left(\frac{\alpha - \beta}{2}\right) &= 3\gamma^2 \\ \left(\frac{\alpha - \beta}{2}\right)^2 + 2\beta \left(\frac{\alpha - \beta}{2}\right) + \beta^2 &= 3\gamma^2 + \beta^2 \\ \left(\frac{\alpha - \beta}{2} + \beta\right)^2 &= \beta^2 + 3\gamma^2 \\ \left(\frac{\alpha + \beta}{2}\right)^2 &= \beta^2 + 3\gamma^2 \end{aligned} \quad (3.2)$$

By Lemma 2.3, we set $\beta = x$, $\gamma = y$, $\frac{\alpha + \beta}{2} = z$. Then (3.2) becomes $x^2 + 3y^2 = z^2$ and we have the following 4 possibilities.

- (1) $\frac{\alpha+\beta}{2} = \frac{t}{2}(m^2 + 3n^2)$, $\beta = \frac{t}{2}(m^2 - 3n^2)$, and $\gamma = tmn$ where t , m , and n are positive integers with m and n odd, $m > \sqrt{3}n$, and $\gcd(m, 3n) = 1$. This implies

$$(\alpha, \beta, \gamma) = \left(\frac{t(m^2 + 9n^2)}{2}, \frac{t(m^2 - 3n^2)}{2}, tmn \right).$$

Thus $\Delta = \left(\frac{t(m^2 + 9n^2)}{2}, \frac{t(2m^2 + 6n^2)}{2}, \frac{t(3m^2 + 3n^2)}{2} \right)$. By Heron's formula, we have

$$[\Delta] = \frac{3}{2}mn(m^2 + 3n^2)t^2.$$

- (2) $\frac{\alpha+\beta}{2} = \frac{t}{2}(m^2 + 3n^2)$, $\beta = \frac{t}{2}(3n^2 - m^2)$, and $\gamma = tmn$ where t , m , and n are positive integers with m and n odd, $m < \sqrt{3}n$, and $\gcd(m, 3n) = 1$. This implies

$$(\alpha, \beta, \gamma) = \left(\frac{t(3m^2 + 3n^2)}{2}, \frac{t(3n^2 - m^2)}{2}, tmn \right).$$

Thus $\Delta = \left(\frac{t(3m^2 + 3n^2)}{2}, \frac{t(2m^2 + 6n^2)}{2}, \frac{t(m^2 + 9n^2)}{2} \right)$. By Heron's formula, we have

$$[\Delta] = \frac{3}{2}mn(m^2 + 3n^2)t^2.$$

- (3) $\frac{\alpha+\beta}{2} = t(m^2 + 3n^2)$, $\beta = t(m^2 - 3n^2)$, and $\gamma = 2tmn$ where t , m , and n are positive integers with $m > \sqrt{3}n$, $\gcd(m, n) = 1$, and m and n having different parity. This implies

$$(\alpha, \beta, \gamma) = (t(m^2 + 9n^2), t(m^2 - 3n^2), 2tmn).$$

Thus $\Delta = (t(m^2 + 9n^2), t(2m^2 + 6n^2), t(3m^2 + 3n^2))$. By Heron's formula, we have

$$[\Delta] = 6mn(m^2 + 3n^2)t^2.$$

- (4) $\frac{\alpha+\beta}{2} = t(m^2 + 3n^2)$, $\beta = t(3n^2 - m^2)$, and $\gamma = 2tmn$ where t , m , and n are positive integers with $m < \sqrt{3}n$, $\gcd(m, n) = 1$, and m and n having different parity. This implies

$$(\alpha, \beta, \gamma) = (t(3m^2 + 3n^2), t(3n^2 - m^2), 2tmn).$$

Thus $\Delta = (t(3m^2 + 3n^2), t(2m^2 + 6n^2), t(m^2 + 9n^2))$. By Heron's formula, we have

$$[\Delta] = 6mn(m^2 + 3n^2)t^2.$$

For the converse, we can check directly, using Heron's formula, that the triangle obtained in each case is a Heron triangle.

BRAHMAGUPTA TRIANGLES IN GENERALIZED FIBONACCI SEQUENCES

In this section, we further characterize all values of $\alpha = G_1$ (with $\beta = G_2 = 1$) such that the generalized Fibonacci sequence $(G_n(\alpha, \beta))$ contains exactly one Brahmagupta triangle.

Theorem 4.1. The generalized Fibonacci sequence $(G_n(\alpha, 1))$ contains a Brahmagupta triangle if and only if α belongs to one of the following two Pell-parametrized families; in that case, the triangle is unique and equals (G_1, G_3, G_4) .

- (1) $\Delta = (6n_r^2 - 3, 6n_r^2 - 2, 6n_r^2 - 1)$ for some positive integer n_r where n_r is chosen from the r^{th} solution (m_r, n_r) of the Pell-type equation $m^2 - 3n^2 = -2$. In this case, the generalized Fibonacci sequence is given by $(\alpha, \beta) = (6n_r^2 - 3, 1)$ and $[\Delta] = 3m_r n_r (3n_r^2 - 1)$.
- (2) $\Delta = (12n_r^2 + 1, 12n_r^2 + 2, 12n_r^2 + 3)$ for some positive integer n_r where n_r is chosen from the r^{th} solution (m_r, n_r) of the Pell equation $m^2 - 3n^2 = 1$. In this case, the generalized Fibonacci sequence is given by $(\alpha, \beta) = (12n_r^2 + 1, 1)$ and $[\Delta] = 6m_r n_r (6n_r^2 + 1)$.

Proof. Suppose $\Delta = (G_i, G_j, G_k)$ is a Brahmagupta triangle with $G_i < G_j < G_k$. By Theorem 3.1, we have $(i, j, k) = (1, 3, 4)$ and $\Delta = (\alpha, \alpha + \beta, \alpha + 2\beta)$. Since a Brahmagupta triangle has consecutive side lengths, the difference between adjacent sides must be 1. For $(G_1, G_3, G_4) = (\alpha, \alpha + \beta, \alpha + 2\beta)$, this forces $\beta = 1$.

We consider 4 cases.

Case 1 $\beta = t \left(\frac{m^2 - 3n^2}{2} \right)$ for some positive integers t, m , and n with m, n odd, $m > \sqrt{3}n$, and $\gcd(m, 3n) = 1$. Then we have $t = 1$ and $m^2 - 3n^2 = 2$ because $\beta = 1$. Since 2 is a quadratic nonresidue modulo 3, this case cannot happen.

Case 2 $\beta = t \left(\frac{3n^2 - m^2}{2} \right)$ for some positive integers t, m , and n with m, n odd, $m < \sqrt{3}n$, and $\gcd(m, 3n) = 1$. Then we have $t = 1$ and $m^2 - 3n^2 = -2$ because $\beta = 1$. Let (m_r, n_r) be a general solution of the equation $m^2 - 3n^2 = -2$. Then Lemma 2.2 implies that (m_r, n_r) can be computed using the equation

$$m_r + \sqrt{3}n_r = (1 + \sqrt{3})(2 + \sqrt{3})^{r-1}.$$

Thus $(\alpha, \beta) = \left(\frac{3}{2}(m_r^2 + n_r^2), 1 \right) = (6n_r^2 - 3, 1)$ for some positive integer r . This implies $\Delta = (6n_r^2 - 3, 6n_r^2 - 2, 6n_r^2 - 1)$. Heron's formula shows that $[\Delta] = 3m_r n_r (3n_r^2 - 1)$.

Case 3 $\beta = t(m^2 - 3n^2)$ for some positive integers t, m , and n with $m > \sqrt{3}n$, $\gcd(m, 3n) = 1$, and m, n having different parity. Then $t = 1$ and $m^2 - 3n^2 = 1$ because $\beta = 1$. Let (m_r, n_r) be a general solution of the equation $m^2 - 3n^2 = 1$. Then Lemma 2.1 implies that (m_r, n_r) can be computed using the equation

$$m_r + \sqrt{3}n_r = (2 + \sqrt{3})^r.$$

Thus $(\alpha, \beta) = (m_r^2 + 9n_r^2, 1) = (12n_r^2 + 1, 1)$ for some positive integer r . This implies $\Delta = (12n_r^2 + 1, 12n_r^2 + 2, 12n_r^2 + 3)$. Heron's formula shows that $[\Delta] = 6m_r n_r (6n_r^2 + 1)$.

Case 4 $\beta = t(3n^2 - m^2)$ for some positive integers t, m , and n with $m < \sqrt{3}n$, $\gcd(m, 3n) = 1$ and m, n having different parity. Then $t = 1$ and $3n^2 - m^2 = 1$ because $\beta = 1$. Since -1 is a quadratic nonresidue modulo 3, this case is impossible.

Conversely, in each of the two families above, Heron's formula shows that (G_1, G_3, G_4) is a Brahmagupta triangle. By Corollary 3.3, this triangle is the unique Brahmagupta triangle in the sequence.

Remark 4.2. Theorem 4.1 shows that all possible Brahmagupta triangles can be put in some generalized Fibonacci sequence with specific initial values. Notice that the solutions obtained in Theorem 4.1 coincide with the results of Gould (1973, Theorem 1) and Nelsen (2020, p. 29), which classify Brahmagupta triangles into two Pell-type families. The new aspect of Theorem 4.1 lies in this representation in a generalized Fibonacci sequence. In addition, we observe that, within a given generalized Fibonacci sequence, there is at most one Brahmagupta triangle, and it occurs only for certain initial values (α, β) . We also observe that the complete list of Brahmagupta triangles obtained in Gould (1973) and Nelsen (2020) consists of case 1 and case 2 in Theorem 4.1 alternately. Please check the following examples for more details.

Example 4.3. Examples of the first 4 Brahmagupta triangles in each case of Theorem 4.1.

Table 1 Case (1)

r	(m_r, n_r)	$(\alpha, \beta) = (6n_r^2 - 3, 1)$	$\Delta = (6n_r^2 - 3, 6n_r^2 - 2, 6n_r^2 - 1)$
1	(1, 1)	(3, 1)	(3, 4, 5)
2	(5, 3)	(51, 1)	(51, 52, 53)
3	(19, 11)	(723, 1)	(723, 724, 725)
4	(71, 41)	(10083, 1)	(10083, 10084, 10085)

Table 2 Case (2)

r	(m_r, n_r)	$(\alpha, \beta) = (12n_r^2 + 1, 1)$	$\Delta = (12n_r^2 + 1, 12n_r^2 + 2, 12n_r^2 + 3)$
1	(1, 1)	(13, 1)	(13, 14, 15)
2	(7, 4)	(193, 1)	(193, 194, 195)
3	(26, 15)	(2701, 1)	(2701, 2702, 2703)
4	(97, 56)	(37633, 1)	(37633, 37634, 37635)

Example 4.4. List of the first 8 Brahmagupta triangles mentioned in Gould (1973) and Nelsen (2020).

Table 3 List of the 8 Brahmagupta triangles

r	Δ
1	(3, 4, 5)
2	(13, 14, 15)
3	(51, 52, 53)
4	(193, 194, 195)
5	(723, 724, 725)
6	(2701, 2702, 2703)
7	(10083, 10084, 10085)
8	(37633, 37634, 37635)

Corollary 4.5. The classical Fibonacci sequence (F_n) and the Lucas sequence (L_n) contain no Brahmagupta triangle.

Proof. Since $(F_n) = (G_n(1, 1))$ and $(L_n) = (G_n(2, 1))$, there is no Brahmagupta triangle appearing in (F_n) and (L_n) as their initial values $\alpha = 1$ and $\alpha = 2$ do not lie in either Pell family from Theorem 4.1.

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