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คลื่นคิงก์และคลื่นคาบในแบบจำลองน้ำตื้นไม่เชิงเส้นด้วยวิธี IMSSEM และ IGREMM

Kink and Periodic Waves in Nonlinear Shallow Water Models via IMSSEM and IGREMM

จิราพร เสนจันทร^{1*}

Jiraporn Sanjun^{1*}

¹สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยราชภัฏสุราษฎร์ธานี จังหวัดสุราษฎร์ธานี 84100

¹Department of Mathematics, Faculty of Science and Technology, Suratthani Rajabhat University, Surat Thani, 84100, Thailand

บทคัดย่อ

งานวิจัยนี้พิจารณาสมการ simplified modified Camassa–Holm (smCH) ซึ่งเป็นรูปแบบลดรูปของสมการ Camassa–Holm แบบดั้งเดิม เพื่อใช้อธิบายพฤติกรรมของคลื่นไม่เชิงเส้นในน้ำตื้นและระบบทางกายภาพที่เกี่ยวข้อง โดยยังคงคุณลักษณะสำคัญด้านความไม่เชิงเส้นและการกระจายตัวของคลื่นไว้ งานวิจัยนี้นำเสนอการประยุกต์ใช้สองวิธีวิเคราะห์ที่ได้รับการปรับปรุงแล้ว ได้แก่ วิธี improved modified Sardar sub-equation method (IMSSEM) และ วิธี improved generalized Riccati equation mapping method (IGREMM) เพื่อสร้างผลเฉลยแม่นยำตรงเชิงคลื่นเคลื่อนที่ของสมการ smCH ผลลัพธ์ที่ได้แสดงให้เห็นถึงผลเฉลยหลากหลายรูปแบบ ซึ่งสามารถเขียนในรูปฟังก์ชันเอ็กซ์โพเนนเชียล ตรีโกณมิติ และไฮเพอร์โบลิก ครอบคลุมปรากฏการณ์คลื่นไม่เชิงเส้นหลายลักษณะ เช่น คลื่นคาบและคลื่นแบบคิงก์ นอกจากนี้ วิธีการที่นำเสนอยังสามารถขยายไปประยุกต์ใช้กับสมการเชิงอนุพันธ์ย่อยไม่เชิงเส้นอื่น ๆ ได้อย่างมีประสิทธิภาพ สะท้อนถึงความเหมาะสมและประสิทธิผลในการวิเคราะห์สมการเชิงอนุพันธ์ย่อยไม่เชิงเส้นที่มีความซับซ้อน

ABSTRACT

In this work, the simplified modified Camassa–Holm (smCH) equation, a reduced version of the original Camassa–Holm model, is employed to describe nonlinear wave behavior in shallow water and related physical systems, while preserving key nonlinear and dispersive effects. Two recently improved analytical methods, the improved modified Sardar sub-equation method (IMSSEM) and the improved generalized Riccati equation mapping method (IGREMM), are used to construct exact traveling-wave solutions of the smCH equation. These approaches yield various types of solutions expressed in exponential, trigonometric, and hyperbolic forms, capturing diverse nonlinear phenomena, including periodic and kink-type waves. Furthermore, the methodologies demonstrate strong potential for extension to other nonlinear partial differential equations, highlighting their versatility in analyzing complex wave dynamics.

*Corresponding Author, E-mail: jiraporn.san@sru.ac.th

คำสำคัญ: สมการ Simplified Modified Camassa–Holm วิธี Improved Modified Sardar Sub-equation วิธี Improved Generalized Riccati Equation Mapping คลื่นคาบ คลื่นคิงก์

Keywords: Simplified Modified Camassa–Holm, Improved Modified Sardar Sub-equation Method, Improved Generalized Riccati Equation Mapping Method, Periodic Wave, Kink Wave

INTRODUCTION

The simplified modified Camassa–Holm (smCH) equation is a reduced nonlinear evolution equation derived from the Camassa–Holm equation, which describes the propagation of nonlinear waves in shallow water and related physical systems. By simplifying higher-order and small-amplitude terms, this equation preserves the essential nonlinear and dispersive effects of the original model while providing a more tractable form for analytical investigation (Irshad *et al.*, 2012). It is worth noting that the original Camassa–Holm (CH) equation admits peaked soliton (peakon) solutions due to its intrinsic nonlocal structure, which results from the balance between nonlinear steepening and dispersive effects. The simplification leading to the smCH equation modifies this structure by neglecting certain higher-order contributions. Consequently, the classical peakon solutions are generally not preserved in the same form. Instead, the smCH equation supports smooth kink-type and periodic traveling wave solutions, which remain physically relevant in shallow water wave modeling. An important aspect of studying nonlinear evolution equations is the determination of their exact and traveling wave solutions, which are vital for comprehending the underlying physical phenomena. Over the years, many efficient analytical techniques have been developed to construct these solutions, such as the modified exp-function method (Attaullah *et al.*, 2022), the simple equation method (Phoosree *et al.*, 2025; Sanjun and Chankaew, 2022; Sanjun *et al.*, 2025; Sanjun *et al.*, 2024), the planar dynamical system method (Li *et al.*, 2024), the Riccati sub-equation method (Phoosree *et al.*, 2024; Thadee *et al.*, 2023), the improved generalized Riccati equation mapping method (Alhefthi *et al.*, 2024; Zayed *et al.*, 2014), the Sardar sub-equation (Cinar *et al.*, 2022; Rehman *et al.*, 2023), the improved modified Sardar sub-equation method (Alhefthi *et al.*, 2024; Khan *et al.*, 2024), the improved modified extended tanh function method (Alhojilan and Ahmed, 2023), and the general improved Kudryashov method (Hyder and Barakat, 2020), etc.

The simplified modified Camassa–Holm (smCH) equation is an important extension in the study of nonlinear dispersive wave equations, rooted in the development of the classical Camassa–Holm (CH) equation and its variants, as detailed in the literature (Irshad *et al.*, 2012). The development began with the CH equation, originally derived by Camassa and Holm (Camassa and Holm, 1993) for modeling water waves:

$$\phi_t + 2\alpha\phi_x - \phi_{xxt} + \beta\phi\phi_x = 2\phi_x\phi_{xx} + \phi\phi_{xxx},$$

a modified Camassa–Holm (mCH) equation was studied by Tian and Song (Tian and Song, 2004) through the retention of two terms often neglected in the small-amplitude, shallow-water regime.

$$\phi_t + 2\alpha\phi_x - \phi_{xxt} + \beta\phi^n\phi_x = 2\phi_x\phi_{xx} + \phi\phi_{xxx},$$

Wazwaz (Wazwaz, 2005) investigated a simplified version of the modified Camassa-Holm (mCH) equation, which is derived from the original mCH equation and is expressed as follows:

$$\phi_t + 2\alpha\phi_x - \phi_{xxt} + \beta\phi^n\phi_x = 0.$$

Focusing on the case where the nonlinearity power is $n=2$, leading to the final form of the simplified modified Camassa-Holm (smCH) equation under consideration:

$$\phi_t + 2\alpha\phi_x - \phi_{xxt} + \beta\phi^2\phi_x = 0,$$

where $\beta > 0$, α is a real constant, and $\phi = \phi(x, t)$ denotes the wave profile in shallow water dynamics. In this work, we focus on obtaining novel exact traveling wave solutions for the simplified modified Camassa-Holm (smCH) equation. For this purpose, we employ distinct yet highly effective and recently generalized analytical techniques: the improved modified Sardar sub-equation method (IMSSEM) and the improved generalized Riccati equation mapping method (IGREMM). These methods are chosen for their robustness and their capacity to yield a variety of accurate solitary wave solutions, thereby enriching the current understanding of the physical phenomena described by the smCH equation. To the best of our knowledge, the IMSSEM and the IGREMM have not previously been applied to the smCH equation. Compared with earlier approaches used for this model, these methods provide a systematic balancing procedure that reduces the nonlinear equation to solvable algebraic systems. As a result, they generate a broader class of exact traveling wave solutions with diverse functional structures, demonstrating their robustness and effectiveness. In particular, the IMSSEM yields sixteen solution forms, while the IGREMM yields fourteen distinct forms for the smCH equation. The implementation of these approaches yields several new and diverse exact traveling wave solutions expressed in exponential, trigonometric, and hyperbolic functional forms. Moreover, they offer effective analytical frameworks for investigating other nonlinear models arising in applied sciences and modern physics. For a clearer physical and dynamical interpretation, the obtained solutions can be classified into kink-type and periodic wave structures, whose behaviors are illustrated through 2D, 3D, and contour graphical representations.

MATERIALS AND METHODS

In this part, we introduce two different generalized analytical methods for determining the traveling wave solution to nonlinear equations: the improved modified Sardar sub-equation method (IMSSEM) and the improved generalized Riccati equation mapping method (IGREMM). Assume that the following represents the nonlinear partial equation in two independent variables, x and t :

$$\Psi(\phi, \phi_x, \phi_t, \phi_{xx}, \phi_{xt}, \dots) = 0, \quad (1)$$

where Ψ is generally a polynomial function of $\phi(x, t)$ and its arguments; the partial derivatives are indicated by the subscripts. To begin, consider integrating the independent variables x and t into a single variable, η . Suppose that

$$\phi(x, t) = \phi(\eta), \quad \eta = x - \lambda t. \quad (2)$$

Where λ is the traveling wave's speed and $\lambda \neq 0$. We may transform Eq. (1) into an ordinary differential equation (ODE) for $\phi(x, t) = \phi(\eta)$ using the wave variable (2).

$$\Xi(\phi, \phi', \phi'', \phi''', \dots) = 0, \quad (3)$$

where in $\phi = \phi(\eta), \phi' = \frac{d\phi}{d\eta}, \phi'' = \frac{d^2\phi}{d\eta^2}, \dots$. The nonlinear ODE Eq. (3) has a solution, which we obtained using IMSSEM and IGREMM:

$$\phi(\eta) = c_0 + \sum_{i=1}^M c_i \kappa^i(\eta), \quad c_M \neq 0, \quad (4)$$

at this point, $c_i (i=0, 1, 2, \dots, M)$ are constant. Eq. (4) can be used to calculate the value of the positive integer M by balancing the highest-order derivative term with the highest-order nonlinear term. If the degree of $\frac{d^p \phi}{d\eta^p}$ is classified as (Kaewta *et al.*, 2020):

$$\begin{aligned} \text{Deg} \left[\frac{d^p \phi(\eta)}{d\eta^p} \right] &= M + p, \\ \text{Deg} \left[(\phi(\eta))^q \left(\frac{d^p \phi(\eta)}{d\eta^p} \right)^r \right] &= Mq + r(M + p). \end{aligned} \quad (5)$$

The improved modified Sardar sub-equation method (IMSSEM)

Where $\kappa(\eta)$ in Eq. (4) satisfies the equation:

$$(\kappa'(\eta))^2 = \tau_2 \kappa^4(\eta) + \tau_1 \kappa^2(\eta) + \tau_0, \quad (6)$$

where τ_0, τ_1 and τ_2 are constants. According to IMSSEM (Alhefthi *et al.*, 2024), the general solutions of Eq. (6), in which η_0 denotes the integration constant, are derived as follows:

Type I: for $\tau_0 = \tau_1 = 0$ and $\tau_2 > 0$,

$$\kappa_1^\pm(\eta) = \pm \frac{1}{\sqrt{\tau_2}(\eta + \eta_0)}. \quad (7)$$

Type II: for $\tau_0 = 0$ and $\tau_1 > 0$,

$$\kappa_2^\pm(\eta) = \frac{4\tau_1 e^{\pm\sqrt{\tau_1}(\eta+\eta_0)}}{e^{\pm 2\sqrt{\tau_1}(\eta+\eta_0)} - 4\tau_1 \tau_2}, \quad (8)$$

$$\kappa_3^\pm(\eta) = \pm \frac{4\tau_1 e^{\pm\sqrt{\tau_1}(\eta+\eta_0)}}{1 - 4\tau_1 \tau_2 e^{\pm 2\sqrt{\tau_1}(\eta+\eta_0)}}. \quad (9)$$

Type III: for $\tau_0 = 0, \tau_1 > 0$ and $\tau_2 \neq 0$,

$$\kappa_4^\pm(\eta) = \pm \sqrt{-\frac{\tau_1}{\tau_2}} \text{sech}(\sqrt{\tau_1}(\eta + \eta_0)), \quad (10)$$

$$\kappa_5^\pm(\eta) = \pm \sqrt{\frac{\tau_1}{\tau_2}} \text{csch}(\sqrt{\tau_1}(\eta + \eta_0)). \quad (11)$$

Type IV: for $\tau_0 = \frac{(\tau_1)^2}{4\tau_2}$, $\tau_1 < 0$ and $\tau_2 > 0$,

$$\kappa_6^\pm(\eta) = \pm \sqrt{-\frac{\tau_1}{2\tau_2}} \tanh\left(\sqrt{-\frac{\tau_1}{2}}(\eta + \eta_0)\right), \quad (12)$$

$$\kappa_7^\pm(\eta) = \pm \sqrt{-\frac{\tau_1}{2\tau_2}} \coth\left(\sqrt{-\frac{\tau_1}{2}}(\eta + \eta_0)\right), \quad (13)$$

$$\kappa_8^\pm(\eta) = \pm \sqrt{-\frac{\tau_1}{2\tau_2}} \left(\tanh(\sqrt{-2\tau_1}(\eta + \eta_0)) \pm \operatorname{sech}(\sqrt{-2\tau_1}(\eta + \eta_0)) \right), \quad (14)$$

$$\kappa_8^\pm(\eta) = \pm \sqrt{-\frac{\tau_1}{2\tau_2}} \left(\coth(\sqrt{-2\tau_1}(\eta + \eta_0)) + \operatorname{csch}(\sqrt{-2\tau_1}(\eta + \eta_0)) \right), \quad (15)$$

$$\kappa_{10}^\pm(\eta) = \pm \sqrt{-\frac{\tau_1}{8\tau_2}} \left(\tanh\left(\sqrt{-\frac{\tau_1}{8}}(\eta + \eta_0)\right) + \coth\left(\sqrt{-\frac{\tau_1}{8}}(\eta + \eta_0)\right) \right). \quad (16)$$

Type V: for $\tau_0 = 0$, $\tau_1 < 0$ and $\tau_2 \neq 0$,

$$\kappa_{11}^\pm(\eta) = \pm \sqrt{-\frac{\tau_1}{\tau_2}} \sec(\sqrt{-\tau_1}(\eta + \eta_0)), \quad (17)$$

$$\kappa_{12}^\pm(\eta) = \pm \sqrt{-\frac{\tau_1}{\tau_2}} \csc(\sqrt{-\tau_1}(\eta + \eta_0)). \quad (18)$$

Type VI: for $\tau_0 = \frac{(\tau_1)^2}{4\tau_2}$, $\tau_1 > 0$ and $\tau_2 > 0$,

$$\kappa_{13}^\pm(\eta) = \pm \sqrt{\frac{\tau_1}{2\tau_2}} \tan\left(\sqrt{\frac{\tau_1}{2}}(\eta + \eta_0)\right), \quad (19)$$

$$\kappa_{14}^\pm(\eta) = \pm \sqrt{\frac{\tau_1}{2\tau_2}} \cot\left(\sqrt{\frac{\tau_1}{2}}(\eta + \eta_0)\right), \quad (20)$$

$$\kappa_{15}^\pm(\eta) = \pm \sqrt{\frac{\tau_1}{2\tau_2}} \left(\tan(\sqrt{2\tau_1}(\eta + \eta_0)) \pm \sec(\sqrt{2\tau_1}(\eta + \eta_0)) \right), \quad (21)$$

$$\kappa_{16}^\pm(\eta) = \pm \sqrt{\frac{\tau_1}{2\tau_2}} \left(\cot(\sqrt{2\tau_1}(\eta + \eta_0)) \pm \csc(\sqrt{2\tau_1}(\eta + \eta_0)) \right), \quad (22)$$

$$\kappa_{17}^\pm(\eta) = \pm \sqrt{\frac{\tau_1}{8\tau_2}} \left(\tan\left(\sqrt{\frac{\tau_1}{8}}(\eta + \eta_0)\right) - \cot\left(\sqrt{\frac{\tau_1}{8}}(\eta + \eta_0)\right) \right). \quad (23)$$

When Eq. (5) is used, the positive integer M is determined through the balancing procedure. Substituting this value of M into Eq. (4), and then inserting Eqs. (4) and (6) into Eq. (3), we obtain a polynomial in terms of $\kappa(\eta)$. This process allows us to construct an algebraic system for the unknown parameters c_i ($i = 0, 1, 2, \dots, M$), τ_0 , τ_1 , and τ_2 , in which each coefficient of the resulting polynomial is set to zero. The various types of solutions presented in Eqs. (7) - (23) are obtained after substituting these parameter values into Eq. (4). Consequently, several exact and distinct solutions of the NPDE are derived.

The improved generalized Riccati equation mapping method (IGREMM)

Where $\kappa(\eta)$ in Eq. (4) satisfies the equation:

$$\kappa'(\eta) = \delta_2 \kappa^2(\eta) + \delta_1 \kappa(\eta) + \delta_0, \quad (24)$$

where δ_0 , δ_1 and δ_2 are constants. According to IGREMM (Alhefthi *et al.*, 2024), the general solutions of Eq. (6), in which η_0 denotes the integration constant, are derived as follows:

Type I: for $\delta_0 = \delta_1 = 0$ and $\delta_2 \neq 0$,

$$\kappa_1^\pm(\eta) = \pm \frac{1}{\delta_2(\eta + \eta_0)}. \quad (25)$$

Type II: for $\delta_0 = 0$, and $\delta_1 \delta_2 \neq 0$,

$$\kappa_2(\eta) = -\frac{\delta_1 \psi}{\delta_1 \left(e^{-\delta_1(\eta + \eta_0)} + \psi \right)}, \quad (26)$$

$$\kappa_3(\eta) = -\frac{\delta_1 e^{\delta_1(\eta + \eta_0)}}{\delta_2 \left(e^{\delta_1(\eta + \eta_0)} + \psi \right)}. \quad (27)$$

Type III: for $\Lambda = \delta_1^2 - 4\delta_0\delta_2 > 0$, with $\delta_1\delta_2 \neq 0$ or $\delta_0\delta_2 \neq 0$, let G and H be nonzero real constants.

$$\kappa_4(\eta) = -\frac{\sqrt{\Lambda}}{2\delta_2} \tanh\left(\frac{\sqrt{\Lambda}}{2}(\eta + \eta_0)\right) - \frac{\delta_1}{2\delta_2}, \quad (28)$$

$$\kappa_5(\eta) = -\frac{\sqrt{\Lambda}}{2\delta_2} \coth\left(\frac{\sqrt{\Lambda}}{2}(\eta + \eta_0)\right) - \frac{\delta_1}{2\delta_2}, \quad (29)$$

$$\kappa_6^\pm(\eta) = -\frac{\sqrt{\Lambda}}{2\delta_2} \left(\tanh\left(\sqrt{\Lambda}(\eta + \eta_0)\right) \pm \operatorname{sech}\left(\sqrt{\Lambda}(\eta + \eta_0)\right) \right) - \frac{\delta_1}{2\delta_2}, \quad (30)$$

$$\kappa_7^\pm(\eta) = -\frac{\sqrt{\Lambda}}{2\delta_2} \left(\coth\left(\sqrt{\Lambda}(\eta + \eta_0)\right) \pm \operatorname{csch}\left(\sqrt{\Lambda}(\eta + \eta_0)\right) \right) - \frac{\delta_1}{2\delta_2}, \quad (31)$$

$$\kappa_8(\eta) = -\frac{\sqrt{\Lambda}}{4\delta_2} \left(\tanh\left(\frac{\sqrt{\Lambda}}{4}(\eta + \eta_0)\right) + \coth\left(\frac{\sqrt{\Lambda}}{4}(\eta + \eta_0)\right) \right) - \frac{\delta_1}{2\delta_2}, \quad (32)$$

$$\kappa_9^\pm(\eta) = \pm \frac{\sqrt{\Lambda(G^2 + H^2)} - G\sqrt{\Lambda} \cosh\left(\sqrt{\Lambda}(\eta + \eta_0)\right)}{2\delta_2 \left(G \sinh\left(\sqrt{\Lambda}(\eta + \eta_0)\right) + H \right)} - \frac{\delta_1}{2\delta_2}, \quad (33)$$

$$\kappa_{10}(\eta) = \frac{2\delta_0 \cosh\left(\frac{\sqrt{\Lambda}}{2}(\eta + \eta_0)\right)}{\sqrt{\Lambda} \sinh\left(\frac{\sqrt{\Lambda}}{2}(\eta + \eta_0)\right) - \delta_1 \cosh\left(\frac{\sqrt{\Lambda}}{2}(\eta + \eta_0)\right)}, \quad (34)$$

$$\kappa_{11}(\eta) = \frac{2\delta_0 \sinh\left(\frac{\sqrt{\Lambda}}{2}(\eta + \eta_0)\right)}{\sqrt{\Lambda} \cosh\left(\frac{\sqrt{\Lambda}}{2}(\eta + \eta_0)\right) - \delta_1 \sinh\left(\frac{\sqrt{\Lambda}}{2}(\eta + \eta_0)\right)}, \quad (35)$$

$$\kappa_{12}^{\pm}(\eta) = \frac{2\delta_0 \cosh(\sqrt{\Lambda}(\eta + \eta_0))}{\sqrt{\Lambda} \sinh(\sqrt{\Lambda}(\eta + \eta_0)) - \delta_1 \cosh(\sqrt{\Lambda}(\eta + \eta_0)) \pm i\sqrt{\Lambda}}, \quad (36)$$

$$\kappa_{13}^{\pm}(\eta) = \frac{2\delta_0 \sinh(\sqrt{\Lambda}(\eta + \eta_0))}{\sqrt{\Lambda} \cosh(\sqrt{\Lambda}(\eta + \eta_0)) - \delta_1 \sinh(\sqrt{\Lambda}(\eta + \eta_0)) \pm \sqrt{\Lambda}}, \quad (37)$$

$$\kappa_{14}(\eta) = \frac{2\delta_0 \sinh\left(\frac{\sqrt{\Lambda}}{4}(\eta + \eta_0)\right) \cosh\left(\frac{\sqrt{\Lambda}}{4}(\eta + \eta_0)\right)}{2\sqrt{\Lambda} \cosh^2\left(\frac{\sqrt{\Lambda}}{4}(\eta + \eta_0)\right) - 2\delta_1 \sinh\left(\frac{\sqrt{\Lambda}}{4}(\eta + \eta_0)\right) \cosh\left(\frac{\sqrt{\Lambda}}{4}(\eta + \eta_0)\right) - \sqrt{\Lambda}}. \quad (38)$$

Type IV: for $\Lambda = \delta_1^2 - 4\delta_0\delta_1 < 0$, with $\delta_1\delta_2 \neq 0$ or $\delta_0\delta_2 \neq 0$, let G and H be nonzero real constants.

$$\kappa_{15}(\eta) = \frac{\sqrt{-\Lambda}}{2\delta_2} \tan\left(\frac{\sqrt{-\Lambda}}{2}(\eta + \eta_0)\right) - \frac{\delta_1}{2\delta_2}, \quad (39)$$

$$\kappa_{16}(\eta) = -\frac{\sqrt{-\Lambda}}{2\delta_2} \cot\left(\frac{\sqrt{-\Lambda}}{2}(\eta + \eta_0)\right) - \frac{\delta_1}{2\delta_2}, \quad (40)$$

$$\kappa_{17}^{\pm}(\eta) = \frac{\sqrt{-\Lambda}}{2\delta_2} \left(\tan(\sqrt{-\Lambda}(\eta + \eta_0)) \pm \sec(\sqrt{-\Lambda}(\eta + \eta_0)) \right) - \frac{\delta_1}{2\delta_2}, \quad (41)$$

$$\kappa_{18}^{\pm}(\eta) = -\frac{\sqrt{-\Lambda}}{2\delta_2} \left(\cot(\sqrt{-\Lambda}(\eta + \eta_0)) \pm \csc(\sqrt{-\Lambda}(\eta + \eta_0)) \right) - \frac{\delta_1}{2\delta_2}, \quad (42)$$

$$\kappa_{19}(\eta) = \frac{\sqrt{-\Lambda}}{4\delta_2} \left(\tan\left(\frac{\sqrt{-\Lambda}}{4}(\eta + \eta_0)\right) - \cot\left(\frac{\sqrt{-\Lambda}}{4}(\eta + \eta_0)\right) \right) - \frac{\delta_1}{2\delta_2}, \quad (43)$$

$$\kappa_{20}^{\pm}(\eta) = \pm \frac{\sqrt{-\Lambda}(G^2 - H^2) - G\sqrt{-\Lambda} \cos(\sqrt{-\Lambda}(\eta + \eta_0))}{2\delta_2(G \sin(\sqrt{-\Lambda}(\eta + \eta_0)) + H)} - \frac{\delta_1}{2\delta_2}, \quad (44)$$

$$\kappa_{21}(\eta) = -\frac{2\delta_0 \cos\left(\frac{\sqrt{-\Lambda}}{2}(\eta + \eta_0)\right)}{\sqrt{-\Lambda} \sin\left(\frac{\sqrt{-\Lambda}}{2}(\eta + \eta_0)\right) - \delta_1 \cos\left(\frac{\sqrt{-\Lambda}}{2}(\eta + \eta_0)\right)}, \quad (45)$$

$$\kappa_{22}(\eta) = \frac{2\delta_0 \sin\left(\frac{\sqrt{-\Lambda}}{2}(\eta + \eta_0)\right)}{\sqrt{-\Lambda} \cos\left(\frac{\sqrt{-\Lambda}}{2}(\eta + \eta_0)\right) - \delta_1 \sin\left(\frac{\sqrt{-\Lambda}}{2}(\eta + \eta_0)\right)}, \quad (46)$$

$$\kappa_{23}^{\pm}(\eta) = -\frac{2\delta_0 \cos(\sqrt{-\Lambda}(\eta + \eta_0))}{\delta_1 \cos(\sqrt{-\Lambda}(\eta + \eta_0)) + \sqrt{-\Lambda} \sin(\sqrt{-\Lambda}(\eta + \eta_0)) \pm \sqrt{-\Lambda}}, \quad (47)$$

$$\kappa_{24}^{\pm}(\eta) = \frac{2\delta_0 \sin(\sqrt{-\Lambda}(\eta + \eta_0))}{\delta_1 \sin(\sqrt{-\Lambda}(\eta + \eta_0)) - \sqrt{-\Lambda} \cos(\sqrt{-\Lambda}(\eta + \eta_0)) \pm \sqrt{-\Lambda}}, \quad (48)$$

$$\kappa_{25}(\eta) = \frac{4\delta_0 \sin\left(\frac{\sqrt{-\Lambda}}{4}(\eta + \eta_0)\right) \cos\left(\frac{\sqrt{-\Lambda}}{4}(\eta + \eta_0)\right)}{2\sqrt{-\Lambda} \cos^2\left(\frac{\sqrt{-\Lambda}}{4}(\eta + \eta_0)\right) - 2\delta_1 \sin\left(\frac{\sqrt{-\Lambda}}{4}(\eta + \eta_0)\right) \cos\left(\frac{\sqrt{-\Lambda}}{4}(\eta + \eta_0)\right) - \sqrt{-\Lambda}}. \quad (49)$$

When Eq. (5) is used, the positive integer M is determined through the balancing procedure. Substituting this value of M into Eq. (4), and then inserting Eqs. (4) and (24) into Eq. (3), we obtain a polynomial in terms of $\kappa(\eta)$. This process allows us to construct an algebraic system for the unknown parameters $c_i (i=0,1,2,\dots,M)$, δ_0, δ_1 , and δ_2 , in which each coefficient of the resulting polynomial is set to zero. The various types of solutions presented in Eqs. (25) - (49) are obtained after substituting these parameter values into Eq. (4). Consequently, several exact and distinct solutions of the NPDE are derived.

RESULTS AND DISCUSSION

In this part, new exact traveling wave solutions of the simplified modified Camassa–Holm (smCH) equation are obtained using IMSSEM and IGREMM.

$$\phi_t + 2\alpha\phi_x - \phi_{xxt} + \beta\phi^2\phi_x = 0. \quad (50)$$

where α is a real constant and $\beta > 0$. We will use the traveling wave variable $\eta = x - \lambda t$ to reduce it to an ODE. When the transformation is substituted into Eq. (50), the following results:

$$(2\alpha - \lambda)\phi' + \lambda\phi'' + \beta\phi^2\phi' = 0. \quad (51)$$

Integrating Eq. (51) and assuming a zero integration constant gives:

$$(2\alpha - \lambda)\phi + \lambda\phi'' + \frac{\beta\phi^3}{3} = 0, \quad (52)$$

the proposed methods are employed in the following sections to achieve the required outcomes. Next, the balance principle between the highest-order derivative term ϕ'' and the highest-order nonlinear term ϕ^3 in Eq. (52) is applied. Consequently, $M=1$. Hence, the solution of Eq. (52) is obtained as follows:

$$\phi(\eta) = c_0 + c_1\kappa(\eta). \quad (53)$$

Solutions Obtained via the IMSSEM

According to the IMSSEM, since κ satisfies Eq. (6), the expressions for ϕ'' and ϕ^3 are as follows:

$$\phi'' = c_1\tau_1\kappa + 2c_1\tau_2\kappa^3, \quad (54)$$

$$\phi^3 = c_0^3 + 3c_0^2c_1\kappa + 3c_0c_1^2\kappa^2 + c_1^3\kappa^3. \quad (55)$$

Substitution of Eqs. (53) - (55) into Eq. (52) yields the following result:

$$\left((2\alpha - \lambda)c_0 + \frac{\beta}{3}c_0^3\right) + \left((2\alpha - \lambda)c_1 + \lambda c_1\tau_1 + \beta c_0^2c_1\right)\kappa + \left(\beta c_0c_1^2\right)\kappa^2 + \left(2\lambda c_1\tau_2 + \frac{\beta}{3}c_1^3\right)\kappa^3 = 0, \quad (56)$$

grouping terms with equal powers of κ^i , where $i = 0, 1, 2, 3$, and equating the coefficients to zero gives the following system:

$$\kappa^0 : (2\alpha - \lambda)c_0 + \frac{\beta}{3}c_0^3 = 0, \quad (57)$$

$$\kappa^1 : (2\alpha - \lambda)c_1 + \lambda c_1\tau_1 + \beta c_0^2c_1 = 0, \quad (58)$$

$$\kappa^2 : \beta c_0c_1^2 = 0, \quad (59)$$

$$\kappa^3 : 2\lambda c_1\tau_2 + \frac{\beta}{3}c_1^3 = 0. \quad (60)$$

After this system has been solved, we get

$$c_0 = 0, c_1 = c_1, \tau_1 = \frac{(\lambda - 2\alpha)}{\lambda} \text{ and } \tau_2 = -\frac{\beta}{6\lambda} c_1^2. \quad (61)$$

Combining Eq. (61) with Eqs. (7) - (23), the following results are derived.

Type II: for $\tau_0 = 0$ and $\tau_1 > 0$,

$$\phi_1^\pm(x, t) = \frac{4c_1(\lambda - 2\alpha)e^{\pm\sqrt{\frac{\lambda-2\alpha}{\lambda}}(\eta+\eta_0)}}{\lambda \left(e^{\pm 2\sqrt{\frac{\lambda-2\alpha}{\lambda}}(\eta+\eta_0)} + \frac{2(\lambda - 2\alpha)\beta c_1^2}{3\lambda^2} \right)}, \quad (62)$$

$$\phi_2^\pm(x, t) = \frac{4c_1(\lambda - 2\alpha)e^{\pm\sqrt{\frac{\lambda-2\alpha}{\lambda}}(\eta+\eta_0)}}{\lambda \left(1 + \frac{2(\lambda - 2\alpha)\beta c_1^2 e^{\pm 2\sqrt{\frac{\lambda-2\alpha}{\lambda}}(\eta+\eta_0)}}{3\lambda^2} \right)}, \quad (63)$$

Type III: for $\tau_0 = 0, \tau_1 > 0$ and $\tau_2 \neq 0$,

$$\phi_3^\pm(\eta) = \pm \sqrt{\frac{6(\lambda - 2\alpha)}{\beta}} \operatorname{sech} \left(\sqrt{\frac{(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right), \quad (64)$$

$$\phi_4^\pm(\eta) = \pm \sqrt{-\frac{6(\lambda - 2\alpha)}{\beta}} \operatorname{csch} \left(\sqrt{\frac{(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right). \quad (65)$$

Type IV: for $\tau_0 = \frac{(\tau_1)^2}{4\tau_2}, \tau_1 < 0$ and $\tau_2 > 0$,

$$\phi_5^\pm(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \tanh \left(\sqrt{-\frac{(\lambda - 2\alpha)}{2\lambda}}(\eta + \eta_0) \right), \quad (66)$$

$$\phi_6^\pm(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \coth \left(\sqrt{-\frac{(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right), \quad (67)$$

$$\phi_7^\pm(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \left(\tanh \left(\sqrt{\frac{-2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) \pm i \operatorname{sech} \left(\sqrt{\frac{-2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) \right), \quad (68)$$

$$\phi_8^\pm(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \left(\coth \left(\sqrt{\frac{-2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) \pm \operatorname{csch} \left(\sqrt{\frac{-2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) \right), \quad (69)$$

$$\phi_9^\pm(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{4\beta}} \left(\tanh \left(\sqrt{-\frac{(\lambda - 2\alpha)}{8\lambda}}(\eta + \eta_0) \right) \pm \coth \left(\sqrt{-\frac{(\lambda - 2\alpha)}{8\lambda}}(\eta + \eta_0) \right) \right). \quad (70)$$

Type V: for $\tau_0 = 0, \tau_1 < 0$ and $\tau_2 \neq 0$,

$$\phi_{10}^\pm(x, t) = \pm \sqrt{\frac{6(\lambda - 2\alpha)}{\beta}} \sec \left(\sqrt{\frac{(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right), \quad (71)$$

$$\phi_{11}^\pm(x, t) = \pm \sqrt{\frac{6(\lambda - 2\alpha)}{\beta}} \csc \left(\sqrt{\frac{(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right). \quad (72)$$

Type VI: for $\tau_0 = \frac{(\tau_1)^2}{4\tau_2}$, $\tau_1 > 0$ and $\tau_2 > 0$,

$$\phi_{12}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \tan \left(\sqrt{\frac{(\lambda - 2\alpha)}{2\lambda}} (\eta + \eta_0) \right), \quad (73)$$

$$\phi_{13}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \cot \left(\sqrt{\frac{(\lambda - 2\alpha)}{2\lambda}} (\eta + \eta_0) \right), \quad (74)$$

$$\phi_{14}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \left(\tan \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}} (\eta + \eta_0) \right) \pm \sec \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}} (\eta + \eta_0) \right) \right), \quad (75)$$

$$\phi_{15}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \left(\cot \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}} (\eta + \eta_0) \right) \pm \csc \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}} (\eta + \eta_0) \right) \right), \quad (76)$$

$$\phi_{16}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{4\beta}} \left(\tan \left(\sqrt{\frac{(\lambda - 2\alpha)}{8\lambda}} (\eta + \eta_0) \right) - \cot \left(\sqrt{\frac{(\lambda - 2\alpha)}{8\lambda}} (\eta + \eta_0) \right) \right). \quad (77)$$

Solutions Obtained via the IGREMM

According to the IGREMM, since κ satisfies Eq. (24), the expressions for ϕ'' and ϕ^3 are as follows:

$$\phi'' = c_1 \delta_0 \delta_1 + 2c_1 \delta_0 \delta_1 \kappa + 3c_1 \delta_1 \delta_2 \kappa^2 + 2c_1 \delta_2^2 \kappa^3, \quad (78)$$

$$\phi^3 = c_0^3 + 3c_0^2 c_1 \kappa + 3c_0 c_1^2 \kappa^2 + c_1^3 \kappa^3. \quad (79)$$

Substituting Eqs. (53) and (78) - (79) into Eq. (52) yields the following result: equal powers of κ^i , where $i = 0, 1, 2, 3$,

$$\left((2\alpha - \lambda)c_0 + \lambda c_1 \delta_0 \delta_1 + \frac{\beta c_0^3}{3} \right) + \left((2\alpha - \lambda)c_1 + 2\lambda c_1 \delta_0 \delta_1 + \lambda c_1 \delta_1^2 + \beta c_0^2 c_1 \right) \kappa + \left(3\lambda c_1 \delta_1 \delta_2 + \beta c_0 c_1^2 \right) \kappa^2 + \left(2\lambda c_1 \delta_2^2 + \frac{\beta c_1^3}{3} \right) \kappa^3 = 0 \quad (80)$$

grouping terms with equal powers of κ^i , where $i = 0, 1, 2, 3$, and equating the coefficients to zero gives the following system:

$$\kappa^0: (2\alpha - \lambda)c_0 + \lambda c_1 \delta_0 \delta_1 + \frac{\beta c_0^3}{3} = 0, \quad (81)$$

$$\kappa^1: (2\alpha - \lambda)c_1 + 2\lambda c_1 \delta_0 \delta_1 + \lambda c_1 \delta_1^2 + \beta c_0^2 c_1 = 0, \quad (82)$$

$$\kappa^2: 3\lambda c_1 \delta_1 \delta_2 + \beta c_0 c_1^2 = 0, \quad (83)$$

$$\kappa^3: 2\lambda c_1 \delta_2^2 + \frac{\beta c_1^3}{3} = 0. \quad (84)$$

After this system has been solved, we get

$$c_0 = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}}, c_1 = \pm \delta_2 \sqrt{-\frac{6\lambda}{\beta}}, \delta_0 = 0, \delta_1 = \pm \sqrt{\frac{2(2\alpha - \lambda)}{\lambda}} \text{ and } \delta_2 = \delta_2. \quad (85)$$

Type II: for $\delta_0 = 0$, and $\delta_1 \delta_2 \neq 0$,

$$\phi_{17}^{\pm}(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} - \delta_2 \sqrt{\frac{-6\lambda}{\beta}} \left(\frac{\psi}{e^{\pm \sqrt{\frac{2(\alpha - \lambda)}{2\lambda}} (\eta + \eta_0)} + \psi} \right), \quad (86)$$

$$\phi_{18}^{\pm}(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \left(1 - \frac{2e^{\pm \sqrt{\frac{2\alpha - \lambda}{2\lambda}}(\eta + \eta_0)}}{\left(e^{\pm \sqrt{\frac{2\alpha - \lambda}{2\lambda}}(\eta + \eta_0)} + \psi \right)} \right). \quad (87)$$

Type III: for $\Lambda = \delta_1^2 - 4\delta_0\delta_2 > 0$, with $\delta_1\delta_2 \neq 0$ or $\delta_0\delta_2 \neq 0$, let G and H be nonzero real constants.

$$\phi_{19}^{\pm}(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \tanh \left(\sqrt{\frac{2\alpha - \lambda}{2\lambda}}(\eta + \eta_0) \right), \quad (88)$$

$$\phi_{20}^{\pm}(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \coth \left(\sqrt{\frac{2\alpha - \lambda}{2\lambda}}(\eta + \eta_0) \right), \quad (89)$$

$$\phi_{21}^{\pm}(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \left(\tanh \left(\sqrt{\frac{2(2\alpha - \lambda)}{\lambda}}(\eta + \eta_0) \right) \pm \operatorname{sech} \left(\sqrt{\frac{2(2\alpha - \lambda)}{\lambda}}(\eta + \eta_0) \right) \right), \quad (90)$$

$$\phi_{22}^{\pm}(x, t) = \pm \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \left(\coth \left(\sqrt{\frac{2(2\alpha - \lambda)}{\lambda}}(\eta + \eta_0) \right) \pm \operatorname{csch} \left(\sqrt{\frac{2(2\alpha - \lambda)}{\lambda}}(\eta + \eta_0) \right) \right), \quad (91)$$

$$\phi_{23}^{\pm}(x, t) = \pm \frac{1}{2} \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \left(\tanh \left(\sqrt{\frac{2\alpha - \lambda}{8\lambda}}(\eta + \eta_0) \right) + \coth \left(\sqrt{\frac{2\alpha - \lambda}{8\lambda}}(\eta + \eta_0) \right) \right), \quad (92)$$

$$\phi_{24}^{\pm}(x, t) = \sqrt{\frac{3(\lambda - 2\alpha)}{\beta}} \left(\frac{\sqrt{G^2 + H^2} - G \cosh \left(\sqrt{\frac{2(2\alpha - \lambda)}{\lambda}}(\eta + \eta_0) \right)}{G \sinh \left(\sqrt{\frac{2(2\alpha - \lambda)}{\lambda}}(\eta + \eta_0) \right) + H} \right). \quad (93)$$

Type IV: for $\Lambda = \delta_1^2 - 4\delta_0\delta_2 < 0$, with $\delta_1\delta_2 \neq 0$ or $\delta_0\delta_2 \neq 0$, let G and H be nonzero real constants.

$$\phi_{25}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \tan \left(\sqrt{\frac{\lambda - 2\alpha}{2\lambda}}(\eta + \eta_0) \right), \quad (94)$$

$$\phi_{26}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \cot \left(\sqrt{\frac{\lambda - 2\alpha}{2\lambda}}(\eta + \eta_0) \right), \quad (95)$$

$$\phi_{27}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \left(\tan \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) \pm \sec \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) \right), \quad (96)$$

$$\phi_{28}^{\pm}(x, t) = \pm \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \left(\cot \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) \pm \csc \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) \right), \quad (97)$$

$$\phi_{29}^{\pm}(x, t) = \pm \frac{1}{2} \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \left(\tan \left(\sqrt{\frac{\lambda - 2\alpha}{8\lambda}}(\eta + \eta_0) \right) - \cot \left(\sqrt{\frac{\lambda - 2\alpha}{8\lambda}}(\eta + \eta_0) \right) \right), \quad (98)$$

$$\phi_{30}^{\pm}(x, t) = \sqrt{\frac{3(2\alpha - \lambda)}{\beta}} \left(\frac{\sqrt{G^2 - H^2} - G \cos \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right)}{G \sin \left(\sqrt{\frac{2(\lambda - 2\alpha)}{\lambda}}(\eta + \eta_0) \right) + H} \right). \quad (99)$$

Graphical representation of the smCH equation

In order to visualize the wave structures resulting from the smCH equation, Figures 1 - 6 display the solutions obtained via the IMSSEM, and the parameter values employed in these simulations are listed in Table 1. Figures 7 - 11 present the corresponding results derived from the IGREMM, with the parameter values specified in Table 2.

Table 1. Parameter settings for Eqs. (62), (66), (69), (71), (73), and (77).

Eqs.	Parameters	Figures	Wave effects
(62)	$\alpha = -5, \beta = 3/5, c_1 = 1, \lambda = 3/10, \eta = 0, -10 \leq x, t \leq 10$	1	kink
(66)	$\alpha = -5, \beta = 3/5, \lambda = -3/-5, \eta_0 = 0, -10 \leq x, t \leq 10$	2	kink
(69)	$\alpha = -5, \beta = 3/5, \lambda = -3/-5, \eta_0 = 0, -10 \leq x, t \leq 10$	3	kink
(71)	$\alpha = -5, \beta = 3/5, \lambda = -3/-5, \eta_0 = 0, -10 \leq x, t \leq 10$	4	periodic
(73)	$\alpha = -5, \beta = 3/5, \lambda = -3/-5, \eta_0 = 0, -10 \leq x, t \leq 10$	5	periodic
(77)	$\alpha = -5, \beta = 3/5, \lambda = -3/-5, \eta_0 = 0, -10 \leq x, t \leq 10$	6	periodic

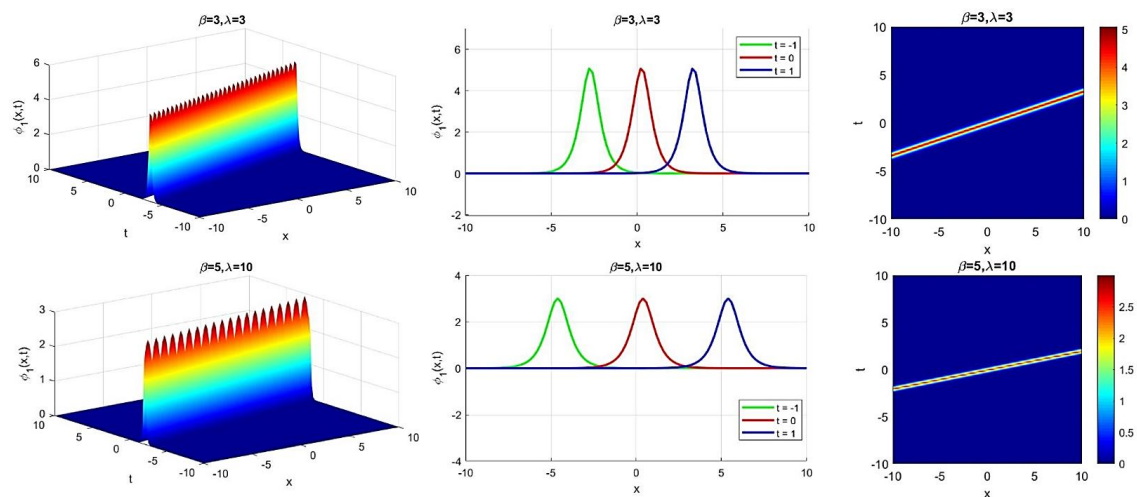


Figure 1 3D, 2D, and contour representations of the solution (62) show a kink wave.

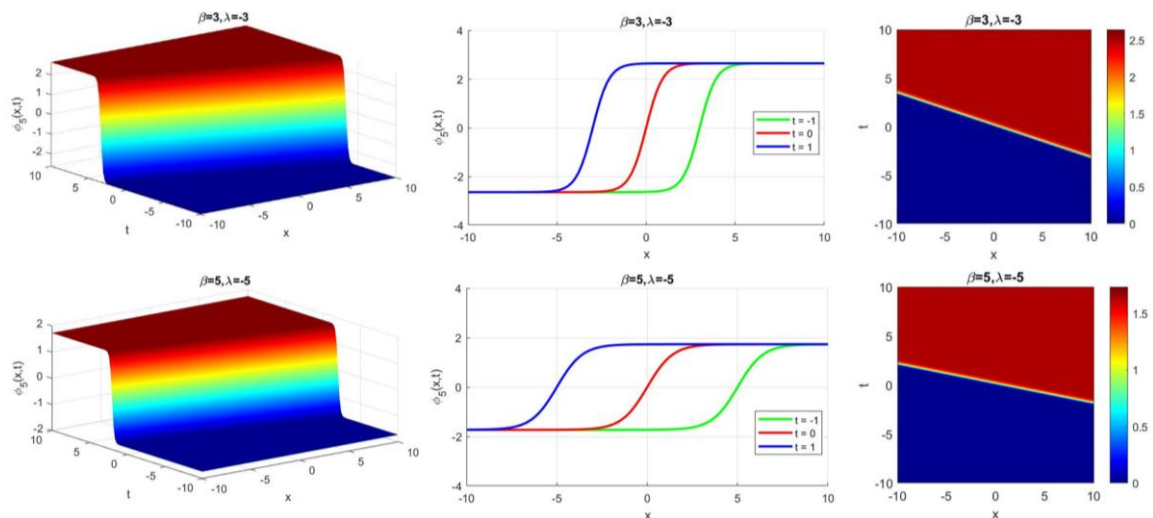


Figure 2 3D, 2D, and contour representations of the solution (66) show a kink wave.

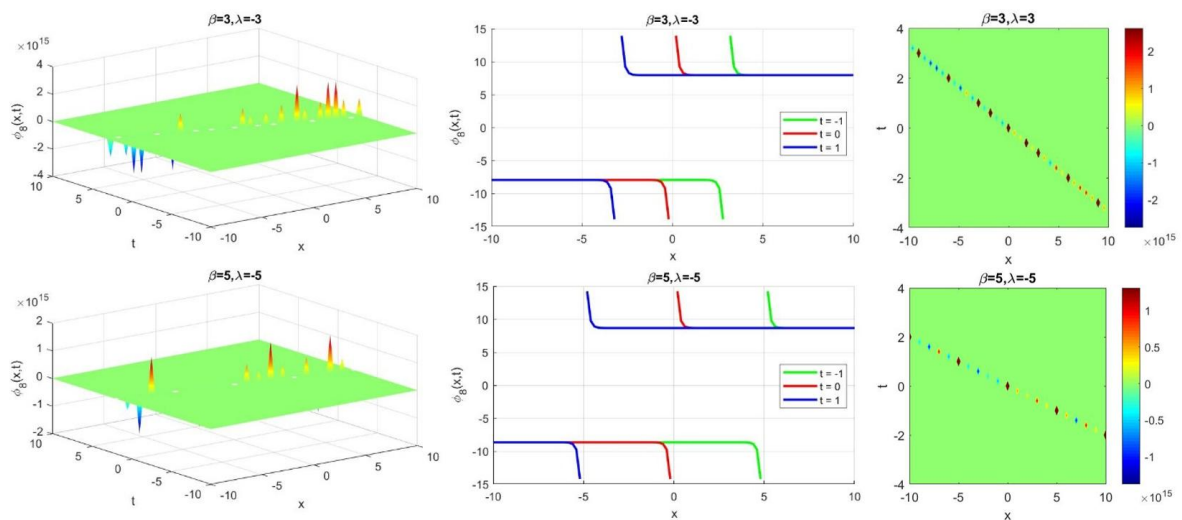


Figure 3 3D, 2D, and contour representations of the solution (69) show a kink wave.

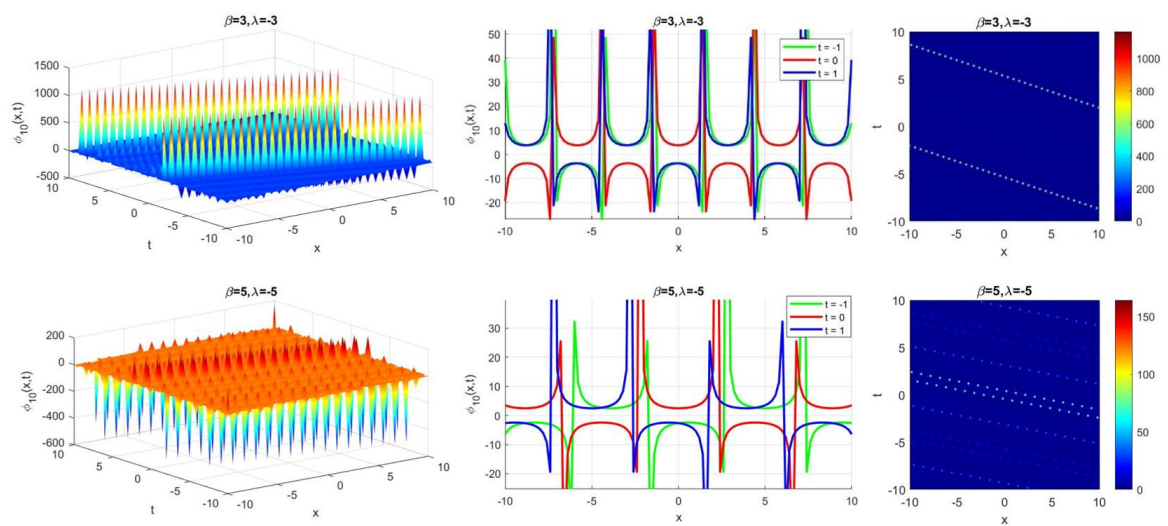


Figure 4 3D, 2D, and contour representations of the solution (71) show a periodic wave.

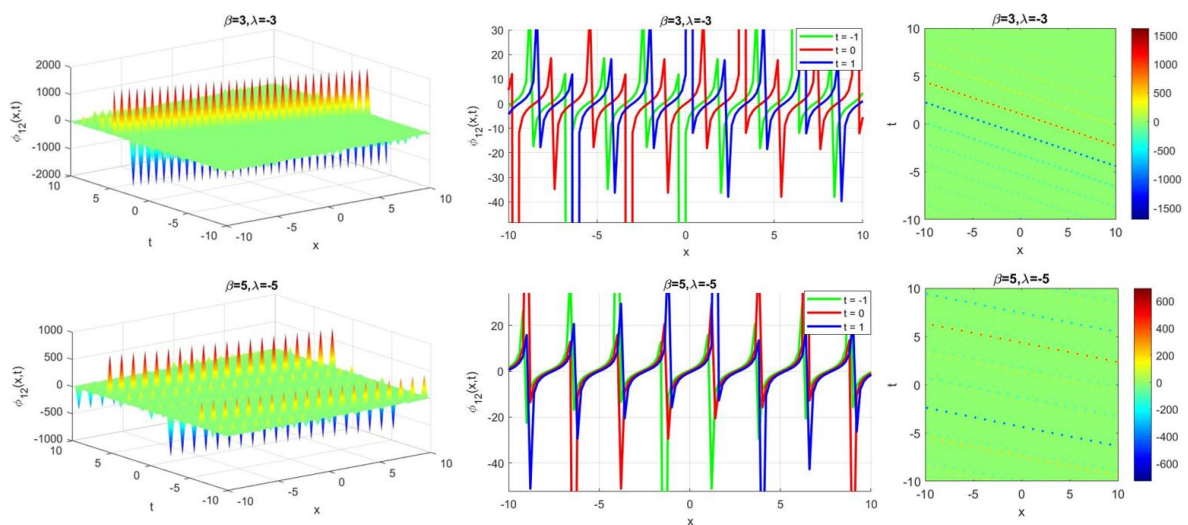


Figure 5 3D, 2D, and contour representations of the solution (73) show a periodic wave.

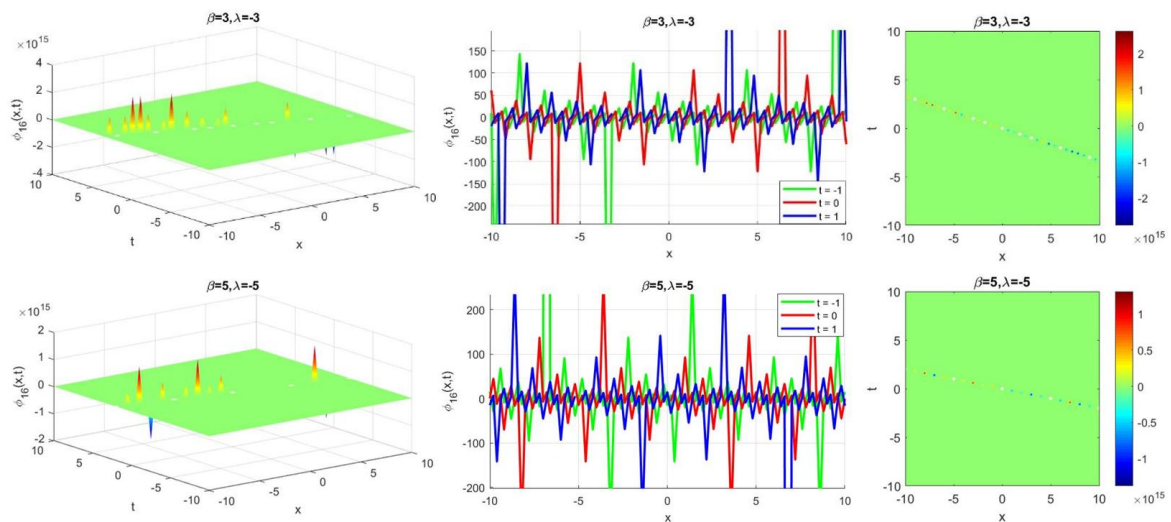


Figure 6 3D, 2D, and contour representations of the solution (77) show a periodic wave.

Table 2. Parameter settings for Eqs. (89), (93), (95), (96) and (99).

Eqs.	Parameters	Figures	Wave effects
(89)	$\alpha = -5, \beta = 3/5, \lambda = -3/-5, \eta = 0, -10 \leq x, t \leq 10$	7	kink
(93)	$\alpha = -5, \beta = 3/5, \lambda = -3/-5, G = 5, H = 2, \eta_0 = 0, -10 \leq x, t \leq 10$	8	kink
(95)	$\alpha = 5, \beta = 3/5, \lambda = -3/-5, \eta_0 = 0, -10 \leq x, t \leq 10$	9	periodic
(96)	$\alpha = 5, \beta = 3/5, \lambda = -3/-5, \eta_0 = 0, -10 \leq x, t \leq 10$	10	periodic
(99)	$\alpha = 5, \beta = 3/3, \lambda = -3/-5, G = 5, H = 2, \eta_0 = 0, -10 \leq x, t \leq 10$	11	periodic

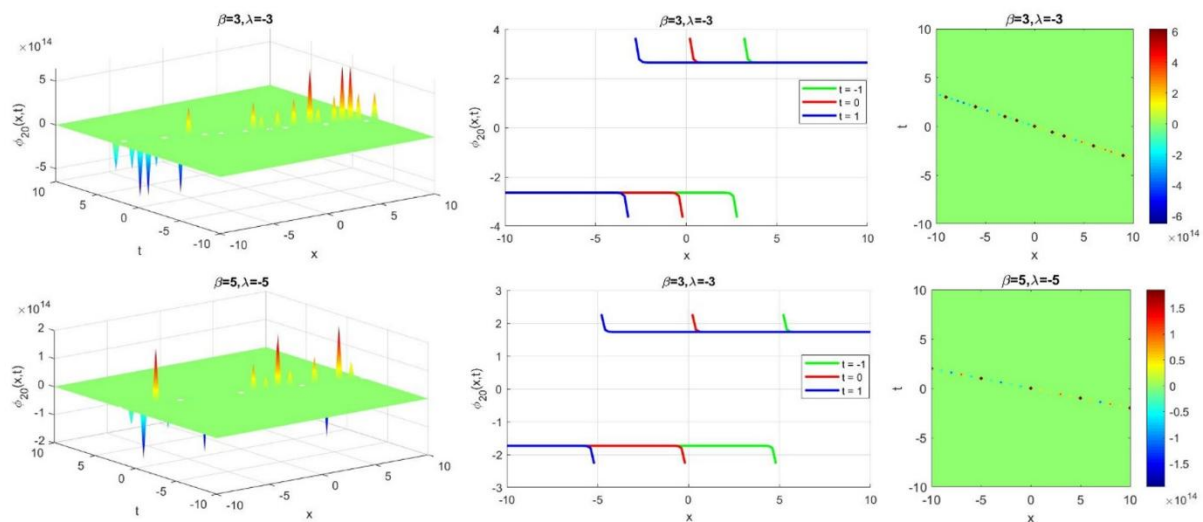


Figure 7 3D, 2D, and contour representations of the solution (89) show a kink wave.

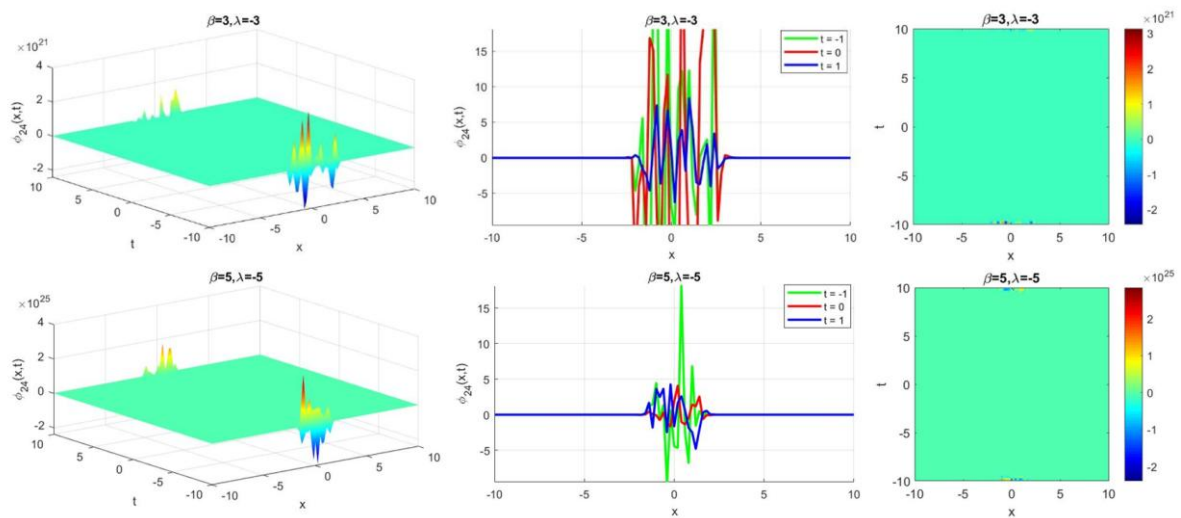


Figure 8 3D, 2D, and contour representations of the solution (93) show a kink wave.

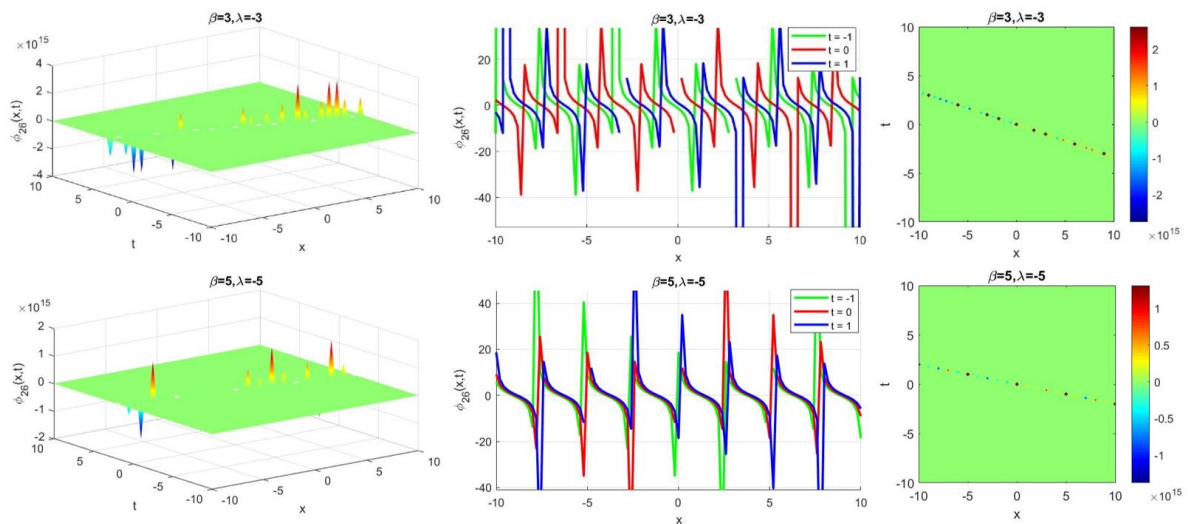


Figure 9 3D, 2D, and contour representations of the solution (95) show a periodic wave.

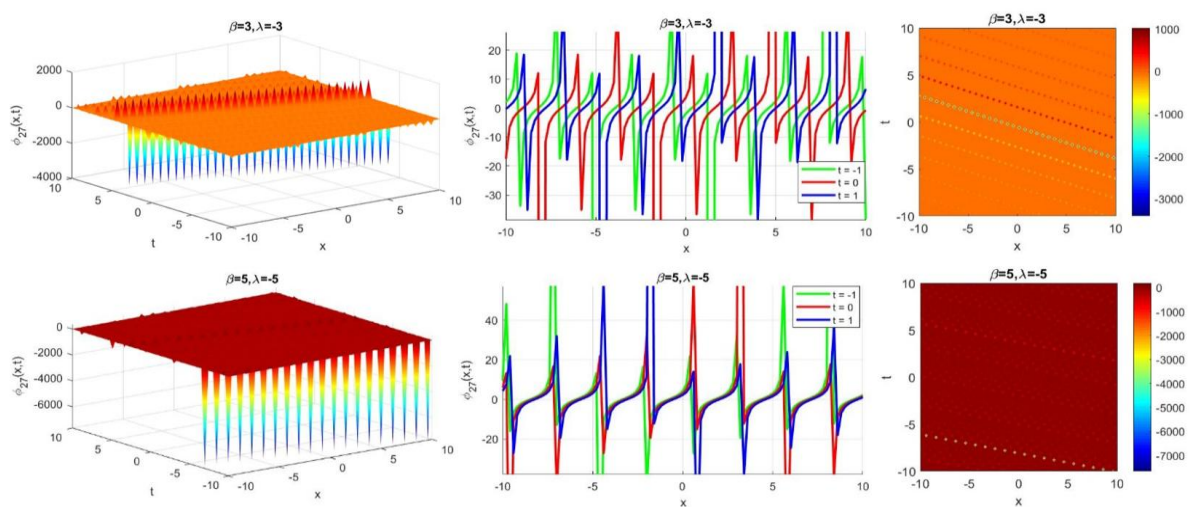


Figure 10 3D, 2D, and contour representations of the solution (96) show a periodic wave.

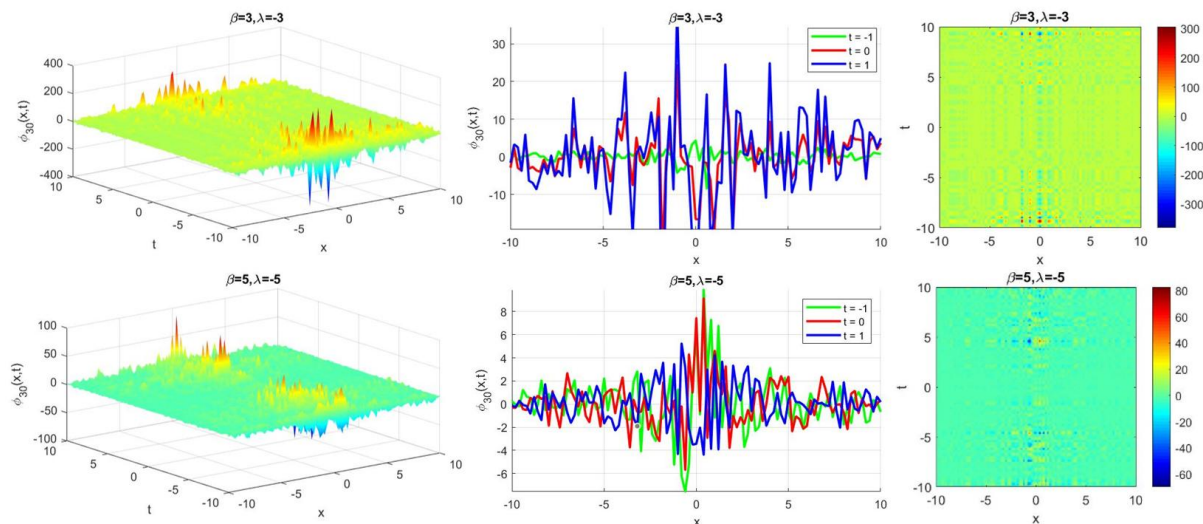


Figure 11 3D, 2D, and contour representations of the solution (99) show a periodic wave.

Figures 1 - 3 and 7 - 8 display kink-type wave structures, where the solutions transition smoothly from one asymptotic state to another, approaching constant values as $x \rightarrow \pm\infty$. The wave solutions maintain their monotonic character, and the position of the wave front varies at different time levels while preserving the overall amplitude. This asymptotic behavior satisfies the boundary conditions in shallow water problems, where the wave connects two constant levels. Moreover, since the solutions remain bounded and approach finite values, the corresponding wave energy remains finite as $x \rightarrow \pm\infty$. In contrast, Figures 4 - 6 and 9 - 11 exhibit periodic wave behavior characterized by oscillatory patterns with successive crests and troughs. The amplitude remains bounded, and the oscillatory structure is preserved at different time levels, confirming the periodic nature of the obtained solutions.

From Tables 1 and 2, it is observed that variations in the selected parameter sets significantly affect the amplitude and overall shape of the traveling wave solutions. In particular, the hyperbolic-form solutions generate kink-type waves, which increase or decrease from one asymptotic state to another and approach constant values at infinity. On the other hand, the trigonometric-form solutions produce periodic traveling waves characterized by oscillatory behavior with repeating patterns. These observations indicate that the model parameters play a crucial role in controlling both the qualitative structure (kink or periodic) and the quantitative features such as amplitude of the wave solutions.

Comparison

The smCH equation has been previously investigated using several analytical methods. For instance, the exp-function method (Irshad *et al.*, 2012) produced nine solutions expressed in exponential forms. The improved (G'/G) -expansion method (Naher and Begum, 2015) generated nine solutions in rational, trigonometric, and hyperbolic forms. The $\exp(\phi(\eta))$ -expansion method (Ali *et al.*, 2016) yielded five solutions in trigonometric and hyperbolic forms. More recently, the modified extended tanh-function method (Sanjun *et al.*, 2025) provided eight solutions expressed in trigonometric and hyperbolic forms. In contrast, the present study with the IMSSEM gives sixteen distinct solution forms, while the IGREMM gives

fourteen distinct solution forms for the smCH equation. This comparison shows that the proposed methods can generate a wider range of exact traveling wave solutions with more diverse functional structures.

CONCLUSIONS

In this study, the improved modified Sardar sub-equation method (IMSSEM) and the improved generalized Riccati equation mapping method (IGREMM) were successfully applied to obtain new exact traveling wave solutions of the simplified modified Camassa–Holm (smCH) equation. These solutions exhibit diverse functional structures involving exponential, trigonometric, and hyperbolic functions, which describe various nonlinear behaviors such as periodic and kink-type waves.

Such a variety of exact solutions provides valuable insights into the nonlinear dynamics of shallow water waves and related physical systems where dispersive and nonlinear effects coexist. The obtained results can be utilized to model real physical phenomena such as soliton propagation, fluid motion, and wave interaction in hydrodynamics and optical fiber communications. Furthermore, the analytical frameworks of IMSSEM and IGREMM demonstrate strong potential for extension to other nonlinear partial differential equations encountered in applied mathematics and physics, making them powerful tools for exploring complex wave behaviors in diverse scientific and engineering contexts.

In addition, the advantages of IMSSEM and IGREMM lie in their systematic construction and efficiency in deriving exact closed-form solutions. Once the balancing procedure is completed, the solution process reduces to solving an algebraic system in a structured and straightforward manner. Moreover, these methods are capable of generating diverse functional forms (exponential, trigonometric, and hyperbolic), facilitating clear physical interpretations such as kink-type and periodic wave structures.

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