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สมการไดโอแฟนไทน์ $a^x + (a+2)^y = z^2$

On the Diophantine Equation $a^x + (a+2)^y = z^2$

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บทคัดย่อ

ในงานวิจัยนี้ได้ศึกษาผลเฉลยของสมการไดโอแฟนไทน์ $a^x + (a+2)^y = z^2$ เมื่อ a เป็นจำนวนเต็มบวก และ x, y, z เป็นจำนวนเต็มที่ไม่เป็นลบ ให้ S เป็นเซตผลเฉลยจำนวนเต็มที่ไม่เป็นลบ (x, y, z) ของสมการ ผลการวิจัยพบว่า 1) ถ้า a เป็นจำนวนเฉพาะ และ $a \equiv 5 (\mathrm{mod}\ 8)$ แล้ว $S = \{(0, 1, \sqrt{a+3})\}$ เมื่อ $\sqrt{a+3}$ เป็นจำนวนเต็ม มิเช่นนั้นแล้ว $S = \emptyset$ 2) ถ้า $a+2$ เป็นจำนวนเฉพาะ และ x เป็นจำนวนคู่ และสมการมีผลเฉลย แล้ว $y = 1$ และ $z = 2$ 3) ให้ p เป็นจำนวนเฉพาะโดยที่ $p \equiv 5, 7\ (\mathrm{mod}\ 8)$ และ $a \equiv -2\ (\mathrm{mod}\ p)$ จะได้ว่า $S = \{(1, 0, \sqrt{a+1})\}$ เมื่อ $\sqrt{a+1}$ เป็นจำนวนเต็ม มิเช่นนั้นแล้ว $S = \emptyset$ ถ้าสอดคล้องกับกรณีใดกรณีหนึ่งต่อไปนี้ กรณีที่ 1 $a \equiv 3\ (\mathrm{mod}\ 4)$ หรือ กรณีที่ 2 มีจำนวนเฉพาะ q ซึ่งทำให้ $q \equiv 3, 5\ (\mathrm{mod}\ 8)$ และ $a \equiv -1\ (\mathrm{mod}\ q)$

ABSTRACT

In this paper, we investigated the solutions of the Diophantine equation $a^x + (a+2)^y = z^2$, where a is a positive integer and x, y, z are non-negative integers. Let S be the set of non-negative integer solutions (x, y, z) of the equation. The results showed that 1) if a is a prime number with $a \equiv 5 (\mathrm{mod}\ 8)$, then $S = \{(0, 1, \sqrt{a+3})\}$, where $\sqrt{a+3}$ is an integer, otherwise $S = \emptyset$. 2) If $a+2$ is a prime number and x is even and the equation has a solution, then $y = 1$ and $z = 2$. 3) Let p be a prime number such that $p \equiv 5,7 (\mathrm{mod}\ 8)$ and $a \equiv -2 (\mathrm{mod}\ p)$. Then $S = \{(1, 0, \sqrt{a+1})\}$, where $\sqrt{a+1}$ is an integer, otherwise $S = \emptyset$, when it satisfies one of the following cases: case 1 $a \equiv 3\ (\mathrm{mod}\ 4)$ or case 2 there exists a prime number q such that $q \equiv 3, 5\ (\mathrm{mod}\ 8)$ and $a \equiv -1\ (\mathrm{mod}\ q)$.

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INTRODUCTION

In the past ten years, many researchers studied the Diophantine equation in the form

$$a^x + (a+2)^y = z^2, \quad (1)$$

where a is a positive integer and x, y, z are non-negative integers. For example, Sroysang (2012) proved that if $a = 3$, then the equation has the unique non-negative integer solution. That is $(x, y, z) = (1, 0, 2)$. Later, Sroysang (2013a; 2013b; 2013c) showed that if $a \in \{5,47,89\}$, then the equation has no non-negative integer solution. In the same year, Rabago (2013) proved that if $a \in \{17, 71\}$, then the equation has the unique non-negative integer solution, i.e. $(x, y, z) = (1, 1, 6)$ and $(x, y, z) = (1, 1, 12)$, respectively. In 2014, Sroysang (2014) proved that if $a = 143$, then the equation has the unique non-negative integer solution. That is $(x, y, z) = (1, 0, 12)$. Sugandha *et al.* (2018) showed that if $a = 11$, then the equation has no non-negative integer solution. Gupta *et al.* (2020) proved that if a and $a + 2$ are prime numbers, then the equation has infinitely many solutions of the form $(a, x, y, z) = (6n - 1, 1, 1, 2\sqrt{3n})$ for some positive integer n. Pandichelvi and Vanaja (2022) studied all non-negative integer solutions of the equation, where a is a prime number with $a \equiv 1 (\text{mod } 4)$ and $1 \leq x + y \leq 3$. Dokchann and Pakapongpun (2020) proved that if $a \equiv 5(\text{mod } 42)$, then the equation has no non-negative integer solution. In 2022, Pakapongpun and Chattae (2022) proved that if $a \equiv 3(\text{mod } 20)$, then the non-negative integer solution of the equation is $(x, y, z) = (1, 0, \sqrt{a+1})$, where $a = (10k - 2)^2 - 1$ and k is an integer. Recently, Viriyapong *et al.* (2023, 2024) proved that if $a \equiv 5(\text{mod } 21)$ or $a \equiv 19\ (\text{mod } 28)$, then the equation has no non-negative integer solution. In this paper, we will generalize the results of the above research, by using elementary methods.

PRELIMINARIES

In the beginning of this section, we recall the definition of the Legendre symbol and its properties. (Karaivanov and Vassilev, 2016)

Definition 1. Let a be an integer and let p be an odd prime number with $\gcd(a, p) = 1$. The Legendre symbol, $\left(\frac{a}{p}\right)$, is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a\ (\text{mod } p) \text{ is solvable,} \\ -1 & \text{if } x^2 \equiv a\ (\text{mod } p) \text{ is not solvable.} \end{cases}$$

Theorem 2. Let a, b be integers and let p be an odd prime number with $\gcd(a, p) = \gcd(b, p) = 1$. Then $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

Theorem 3. Let p be an odd prime number. Then

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1(\text{mod } 4) \\ -1 & \text{if } p \equiv 3(\text{mod } 4), \end{cases}$$

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 7\ (\text{mod } 8) \\ -1 & \text{if } p \equiv 3, 5\ (\text{mod } 8). \end{cases}$$

By Theorem 2 and 3, we have the following theorem.

Theorem 4. Let p be an odd prime number. Then

$$\left(\frac{-2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 3 \pmod 8, \\ -1 & \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

Now, we present an important theorem, which was proved by Mihăilescu (2004):

Theorem 5. *(Mihăilescu's Theorem)* The Diophantine equation $a^x - b^y = 1$, where a, b, x and y are positive integers with $\min\{a, b, x, y\} > 1$, has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$.

Corollary 6. Let a be a positive integer. Then the set S of non-negative integer solutions (x, z) of the Diophantine equation $a^x + 1 = z^2$ is

$$S = \begin{cases} \{(3,3)\} & \text{if } a = 2, \\ \{(1, \sqrt{a+1})\} & \text{if } \sqrt{a+1} \text{ is an integer}, \\ \emptyset & \text{otherwise}. \end{cases}$$

Proof. Let x and z be non-negative integers such that $a^x + 1 = z^2$ or $z^2 - a^x = 1$. It is easy to see that $a > 1$, $z > 1$ and $x \geq 1$. Suppose that $x = 1$. Therefore, $z = \sqrt{a+1}$. If $\sqrt{a+1}$ is an integer, then $S = \{(1, \sqrt{a+1})\}$. If $\sqrt{a+1}$ isn't an integer, then $S = \emptyset$. Now, we consider $x > 1$. By Theorem 5, we get $a = 2$ and $S = \{(3,3)\}$. ■

Corollary 7. Let a be a positive integer. Then the set S of non-negative integer solutions (y, z) of the Diophantine equation $1 + (a+2)^y = z^2$ is

$$S = \begin{cases} \{(1, \sqrt{a+3})\} & \text{if } \sqrt{a+3} \text{ is an integer}, \\ \emptyset & \text{otherwise}. \end{cases}$$

Proof. Let y and z be non-negative integers such that $1 + (a+2)^y = z^2$ or $z^2 - (a+2)^y = 1$. It is easy to see that $a + 2 > 1$, $z > 1$ and $y \geq 1$. Assume that $y > 1$. By Theorem 5, we obtain $a = 0$, a contradiction. Thus $y = 1$ and so $z = \sqrt{a+3}$. If $\sqrt{a+3}$ is an integer, then $S = \{(1, \sqrt{a+3})\}$, otherwise $S = \emptyset$. ■

MAIN RESULTS

In this section, we present our results.

Lemma 8. Let a be a positive integer with $a \equiv 1 \pmod 4$. If the equation (1) has a non-negative integer solution, then y is odd.

Proof. Let x, y and z be non-negative integers such that $a^x + (a+2)^y = z^2$. Since $a \equiv 1 \pmod 4$, we have $a^x + (a+2)^y \equiv 1 + (-1)^y \pmod 4$. Then $z^2 \equiv 1 + (-1)^y \pmod 4$. Assume that y is even. Therefore, $z^2 \equiv 2 \pmod 4$, which contradicts the fact that $z^2 \equiv 0, 1 \pmod 4$. Thus y is odd. ■

Lemma 9. Let a be a positive integer with $a \equiv 3 \pmod 4$. If the equation (1) has a non-negative integer solution, then x is odd.

Proof. Let x, y and z be non-negative integers such that $a^x + (a+2)^y = z^2$. Since $a \equiv 3(\text{mod } 4)$, we have $a^x + (a+2)^y \equiv (-1)^x + 1(\text{mod } 4)$. Then $z^2 \equiv (-1)^x + 1(\text{mod } 4)$. Assume that x is even. Therefore, $z^2 \equiv 2(\text{mod } 4)$, which contradicts the fact that $z^2 \equiv 0, 1(\text{mod } 4)$. Thus x is odd. ■

Theorem 10. Let a be a prime number with $a \equiv 5(\text{mod } 8)$. Then the set S of non-negative integer solutions (x, y, z) of the equation (1) is

$$S = \begin{cases} \{(0, 1, \sqrt{a+3})\} & \text{if } \sqrt{a+3} \text{ is an integer,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Let x, y and z be non-negative integers such that $a^x + (a+2)^y = z^2$. Assume that $x > 0$. Then $a^x + (a+2)^y \equiv 2^y(\text{mod } a)$ and so $z^2 \equiv 2^y(\text{mod } a)$. Therefore, $\left(\frac{2}{a}\right)^y = \left(\frac{2^y}{a}\right) = 1$. Since $a \equiv 1(\text{mod } 4)$, by Lemma 8, it implies that y is odd. This implies that $\left(\frac{2}{a}\right) = 1$. By Theorem 3, we get $a \equiv 1, 7 \ (\text{mod } 8)$, which is impossible since $a \equiv 5(\text{mod } 8)$. Thus $x = 0$. By Corollary 7, it implies that if $\sqrt{a+3}$ is an integer, then $S = \{(0, 1, \sqrt{a+3})\}$, otherwise $S = \emptyset$. ■

By Theorem 10, we prove the result of Sroysang (2013a).

Corollary 11. (Sroysang, 2013a) If $a = 5$, then the equation (1) has no non-negative integer solution.

Proof. Since $a = 5$, we get $a \equiv 5(\text{mod } 8)$ and $\sqrt{a+3} = \sqrt{8}$ isn't an integer, by Theorem 10, the equation (1) has no non-negative integer solution. ■

Theorem 12. Let a be a positive integer such that $a + 2$ is a prime number. If the equation (1) has a solution (x, y, z) with even x, then $y = 1$ and $z = 2$.

Proof. Let x be even. Then $x = 2k$ for some non-negative integer k. From the equation (1), it follows that

$$(z - a^k)(z + a^k) = (a+2)^y. \tag{2}$$

Since $a + 2$ is a prime number, there exists a non-negative integer v such that $z - a^k = (a+2)^v$ and $z + a^k = (a+2)^{y-v}$. Then

$$2 \cdot a^k = (a+2)^v[(a+2)^{y-2v} - 1]. \tag{3}$$

If $v > 0$, then $(a+2)|2$ or $(a+2)|a$, a contradiction. Thus $v = 0$ and so

$$2 \cdot a^k = (a+2)^y - 1 = (a+1)[(a+2)^{y-1} + (a+2)^{y-2} + \cdots + 1]. \tag{4}$$

Assume that $a > 1$. Since $a + 2$ is a prime number, we have $a + 1 \geq 4$. Then there exists a prime number p such that $p|\frac{a+1}{2}$. From the equation (4), we get $a^k = \frac{a+1}{2}[(a+2)^{y-1} + (a+2)^{y-2} + \cdots + 1]$. Then $p|a$. However, since $p|\frac{a+1}{2}$, we also have $p|(a+1)$. Hence, $p|[(a+1) - a]$, i.e., $p|1$, which is a contradiction. Thus $a = 1$. From the equation (4), it implies that $2 = 3^y - 1$. Then $y = 1$ and so $z = 2$. ■

Theorem 13. Let a be a positive integer and let p be a prime number with $p \equiv 5,7 (\mathrm{mod}\ 8)$ and $a \equiv -2 (\mathrm{mod}\ p)$. Then the set S_{odd} of non-negative integer solutions (x, y, z) of the equation (1) with odd x is

$$S_{odd} = \begin{cases} \{(1,0,\sqrt{a+1})\} & \text{if } \sqrt{a+1} \text{ is an integer,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Let x be odd. Since $a \equiv -2 (\mathrm{mod}\ p)$, we obtain $a^x + (a+2)^y \equiv (-2)^x + p^y\ (\mathrm{mod}\ p)$. From the equation (1), we have $z^2 \equiv (-2)^x + p^y\ (\mathrm{mod}\ p)$. Assume that $y > 0$. Therefore, $z^2 \equiv (-2)^x\ (\mathrm{mod}\ p)$. This implies that $\left(\frac{(-2)^x}{p}\right) = 1$. Since x is odd and $\left(\frac{(-2)^x}{p}\right) = \left(\frac{-2}{p}\right)^x$, we obtain $\left(\frac{-2}{p}\right) = 1$. By Theorem 4, we have $p \equiv 1, 3\ (\mathrm{mod}\ 8)$. This is impossible since $p \equiv 5,7 (\mathrm{mod}\ 8)$. Thus $y = 0$. By Corollary 6, if $\sqrt{a+1}$ is an integer, then $S_{odd} = \{(1,0,\sqrt{a+1})\}$, otherwise $S_{odd} = \emptyset$. ■

Theorem 14. Let a be a positive integer with $a \equiv 3 (\mathrm{mod}\ 4)$ and let p be a prime number with $p \equiv 5,7 (\mathrm{mod}\ 8)$ and $a \equiv -2 (\mathrm{mod}\ p)$. Then the set S of non-negative integer solutions (x, y, z) of the equation (1) is

$$S = \begin{cases} \{(1,0,\sqrt{a+1})\} & \text{if } \sqrt{a+1} \text{ is an integer,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Since $a \equiv 3\ (\mathrm{mod}\ 4)$, by Lemma 9, it implies that x is odd. By Theorem 13, if $\sqrt{a+1}$ is an integer, then $S = \{(1,0,\sqrt{a+1})\}$, otherwise $S = \emptyset$. ■

Next, we use Theorem 14 to prove some previous research.

Corollary 15. (Sroysang, 2012). If $a = 3$, then the equation (1) has the unique non-negative integer solution, i.e. $(x, y, z) = (1, 0, 2)$.

Proof. Since $a = 3$, we get $a \equiv 3\ (\mathrm{mod}\ 4)$ and $a \equiv -2\ (\mathrm{mod}\ 5)$. By Theorem 14, it implies that $(x, y, z) = (1, 0, 2)$. ■

Corollary 16. (Sroysang, 2013b). If $a = 47$, then the equation (1) has no non-negative integer solution.

Proof. Since $a = 47$, we obtain $a \equiv 3 (\mathrm{mod}\ 4)$, $a \equiv -2 (\mathrm{mod}\ 7)$, and $\sqrt{a+1} = \sqrt{48}$ isn't an integer. By Theorem 14, it follows that the equation (1) has no non-negative integer solution. ■

Corollary 17. (Sugandha *et al.*, 2018). If $a = 11$, then the equation (1) has no non-negative integer solution.

Proof. Since $a = 11$, we obtain $a \equiv 3\ (\mathrm{mod}\ 4)$, $a \equiv -2 (\mathrm{mod}\ 13)$, and $\sqrt{a+1} = \sqrt{12}$ isn't an integer. By Theorem 14, it follows that the equation (1) has no non-negative integer solution. ■

Corollary 18. (Pakapongpun and Chattae, 2022). Let a be a positive integer with $a \equiv 3 (\mathrm{mod}\ 20)$. Then the set S of non-negative integer solutions (x, y, z) of the equation (1) is

$$S = \begin{cases} \{(1,0,\sqrt{a+1})\} & \text{if } \sqrt{a+1} \text{ is an integer,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Since $a \equiv 3 (\mathrm{mod}\ 20)$, we have $a \equiv 3 \ (\mathrm{mod}\ 4)$ and $a \equiv -2 \ (\mathrm{mod}\ 5)$. By Theorem 14, it implies that if $\sqrt{a+1}$ is an integer, then $S = \{(1, 0, \sqrt{a+1})\}$, otherwise $S = \emptyset$. ∎

Corollary 19. (Viriyapong *et al.*, 2024). Let a be a positive integer. If $a \equiv 19 \ (\mathrm{mod}\ 28)$, then the equation (1) has no non-negative integer solution.

Proof. Assume that there exist non-negative integers x, y and z such that $a^x + (a+2)^y = z^2$. Since $a \equiv 19 \ (\mathrm{mod}\ 28)$, we have $a \equiv 3 \ (\mathrm{mod}\ 4)$ and $a \equiv -2 \ (\mathrm{mod}\ 7)$. By Theorem 14, it implies that $z^2 = a + 1 \equiv 6 \ (\mathrm{mod}\ 7)$, which contradicts the fact that $z^2 \equiv 0, 1, 2, 4 \ (\mathrm{mod}\ 7)$. ∎

Theorem 20. Let a be a positive integer and let p, q be prime numbers such that $p \equiv 5,7 (\mathrm{mod}\ 8)$ and $q \equiv 3,5 (\mathrm{mod}\ 8)$. If $a \equiv -2 (\mathrm{mod}\ p)$ and $a \equiv -1 (\mathrm{mod}\ q)$, then the set S of non-negative integer solutions (x, y, z) of the equation (1) is

$$S = \begin{cases} \{(1, 0, \sqrt{a+1})\} & \text{if } \sqrt{a+1} \text{ is an integer,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Since $a \equiv -1 (\mathrm{mod}\ q)$, we get $a^x + (a+2)^y \equiv (-1)^x + 1 (\mathrm{mod}\ q)$. From the equation (1), we have $z^2 \equiv (-1)^x + 1 (\mathrm{mod}\ q)$. Assume that x is even. Therefore, $z^2 \equiv 2 (\mathrm{mod}\ q)$ and so $\left(\frac{2}{q}\right) = 1$. By Theorem 3, we obtain $q \equiv 1, 7 \ (\mathrm{mod}\ 8)$. This is impossible since $q \equiv 3, 5 \ (\mathrm{mod}\ 8)$. Thus x is odd. By Theorem 13, it implies that if $\sqrt{a+1}$ is an integer, then $S = \{(1, 0, \sqrt{a+1})\}$, otherwise $S = \emptyset$. ∎

Next, we use Theorem 20 to prove some previous research.

Corollary 21. (Sroysang, 2013c). If $a = 89$, then the equation (1) has no non-negative integer solution.

Proof. Since $a = 89$, we obtain $a \equiv -2 (\mathrm{mod}\ 7)$, $a \equiv -1 \ (\mathrm{mod}\ 5)$, and $\sqrt{a+1} = \sqrt{90}$ isn't an integer. By Theorem 20, it follows that the equation (1) has no non-negative integer solution. ∎

Corollary 22. (Sroysang, 2014). If $a = 143$, then the equation (1) has the unique non-negative integer solution, i.e. $(x, y, z) = (1, 0, 12)$.

Proof. Since $a = 143$, we have $a \equiv -2 (\mathrm{mod}\ 5)$ and $a \equiv -1 \ (\mathrm{mod}\ 3)$. By Theorem 20, it implies that $(x, y, z) = (1, 0, 12)$. ∎

Corollary 23. (Viriyapong *et al.*, 2023). Let a be a positive integer. If $a \equiv 5 \ (\mathrm{mod}\ 21)$, then the equation (1) has no non-negative integer solution.

Proof. Assume that the equation (1) has a non-negative integer solution. Since $a \equiv 5 \ (\mathrm{mod}\ 21)$, we have $a \equiv -2 (\mathrm{mod}\ 7)$ and $a \equiv -1 (\mathrm{mod}\ 3)$. By Theorem 20, we obtain $z^2 = a + 1$. Since $a \equiv -2 \equiv 5 (\mathrm{mod}\ 7)$, we have $z^2 \equiv 6 \ (\mathrm{mod}\ 7)$, which contradicts the fact that $z^2 \equiv 0, 1, 2, 4 \ (\mathrm{mod}\ 7)$. ∎

Corollary 24. (Dokchann and Pakapongpun, 2020). Let a be a positive integer. If $a \equiv 5 \ (\mathrm{mod}\ 42)$, then the equation (1) has no non-negative integer solution.

Proof. Since $a \equiv 5 \ (\mathrm{mod}\ 42)$, we have $a \equiv 5 \ (\mathrm{mod}\ 21)$. So, we can prove in the same way as Corollary 23. Hence, the equation (1) has no non-negative integer solution. ∎

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REFERENCES

Dokchann, R. and Pakapongpun, A. (2020). On the Diophantine equation $a^x + (a+2)^y = z^2$, where $a \equiv 5 (\mathrm{mod}\ 42)$. Tatra Mountains Mathematical Publications 77: 39 – 42. doi: 10.2478/tmmp-2020-0030.

Gupta, S., Kumar, R. and Kumar, S. (2020). On the non-linear Diophantine equation $p^x + (p+2)^y = z^2$. Ilkogretim Online – Elementary Education Online 19(1): 472 – 475. doi: 10.17051/ilkonline.2020.661876.

Karaivanov, B. and Vassilev, T.S. (2016). On certain sums involving the Legendre symbol. Integers. 16:1 - 10.

Mihăilescu, P. (2004). Primary cyclotomic units and a proof of Catalan's conjecture. Journal für die Reine und Angewandte Mathematik 572: 167 - 195.

Pakapongpun, A. and Chattae, B. (2022). On the Diophantine equation $a^x + (a+2)^y = z^2$, where $a \equiv 3 (\mathrm{mod}\ 20)$. International Journal of Mathematics and Computer Science 17(2): 711 - 716.

Pandichelvi, V. and Vanaja, R. (2022). Inspecting integer solutions for an exponential Diophantine equation $p^x + (p+2)^y = z^2$. Advances and Applications in Mathematical Sciences 21(8): 4693 - 4701.

Rabago, J.F.T. (2013). A note on two Diophantine equations $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$. Mathematical Journal of Interdisciplinary Sciences 2(1): 19 – 24. doi: 10.15415/mjis.2013.21002.

Sroysang, B. (2012). On the Diophantine equation $3^x + 5^y = z^2$. International Journal of Pure and Applied Mathematics 81(4): 605 - 608.

Sroysang, B. (2013a). On the Diophantine equation $5^x + 7^y = z^2$. International Journal of Pure and Applied Mathematics 89(1): 115 - 118. doi: 10.12732/ijpam.v89i1.14.

Sroysang, B. (2013b). On the Diophantine equation $47^x + 49^y = z^2$. International Journal of Pure and Applied Mathematics 89(2): 279 – 282. doi: 10.12732/ijpam.v89i2.11.

Sroysang, B. (2013c). On the Diophantine equation $89^x + 91^y = z^2$. International Journal of Pure and Applied Mathematics 89(2): 283 - 286. doi: 10.12732/ijpam.v89i2.12.

Sroysang, B. (2014). On the Diophantine equation $143^x + 145^y = z^2$. International Journal of Pure and Applied Mathematics 91(2): 265 - 268. doi: 10.12732/ijpam.v91i2.13.

Sugandha, A., Tripena, A., Prabowo, A. and Sukono, F. (2018). Nonlinear Diophantine equation $11^x + 13^y = z^2$. IOP Conf. Series: Materials Science and Engineering 332: 1 - 4. doi: 10.1088/1757-899X/332/1/012004.

Viriyapong, C. and Viriyapong, N. (2023). On the Diophantine equation $a^x + (a+2)^y = z^2$, where $a \equiv 5 (\mathrm{mod}\ 21)$. International Journal of Mathematics and Computer Science 18(3): 525 - 527.

Viriyapong, C. and Viriyapong, N. (2024). On the Diophantine equation $a^x + (a+2)^y = z^2$, where a is congruent to $\mathbf{19}$ modulo $\mathbf{28}$. International Journal of Mathematics and Computer Science 19(2): 449 - 451.

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