ผลเฉลยสมดุลบนการส่งสองมิติเชิงเส้นเป็นช่วง

Equilibrium Solution on a Two-dimensional Piecewise Linear Map

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บทคัดย่อ

ในบทความนี้เราศึกษาพฤติกรรมของการส่งเชิงเส้นเป็นช่วงที่มีเงื่อนเริ่มต้นอยู่ในบริเวณเฉพาะในจตุภาคที่สอง การส่งดังกล่าวมีจุดสมดุลเพียงหนึ่งเดียว เราพบบริเวณของเงื่อนไขเริ่มต้นของผลเฉลยที่เป็นจุดสมดุล เราทำการแบ่งบริเวณเฉพาะในจตุภาคที่สองดังกล่าวเป็นบริเวณย่อยและทำการค้นหาพฤติกรรมของผลเฉลยในแต่ละบริเวณย่อยด้วยการคำนวณทางตรงและการใช้ประพจน์ที่พิสูจน์ด้วยหลักอุปนัยเชิงคณิตศาสตร์ ซึ่งได้แสดงว่าแต่ละบริเวณย่อยของเงื่อนไขเริ่มต้นจะมีผลเฉลยเป็นจุดสมดุลในที่สุดโดยไม่ใช้ทฤษฎีบทเสถียรภาพ

ABSTRACT

In this paper we study a behavior of a piecewise linear map with specific initial condition in the second quadrant. We show that there is a unique equilibrium point of the map. We found regions of initial conditions that solutions become an equilibrium point. We separate the region into sub-regions and investigate behaviors of solutions in each sub-region by direct calculations. We use inductive statements to assert that solutions of the map in each region of initial conditions are an eventually equilibrium point without using stability theorem.

คำสำคัญ: การส่งเชิงเส้นเป็นช่วง จุดสมดุล จุดสมดุลในที่สุด

Keywords: Piecewise Linear Map, Equilibrium Point, Eventually Equilibrium Point

INTRODUCTION

Lozi map (Lozi, 1978) is a well-known piecewise linear map (PWL) which is a simplest case of piecewise smooth map (PWS). It is a simplified version of Hénon map and it has a strange attractor. A map with absolute value is one kind of the piecewise linear map. We know that multistability (Simpson, 2010; Zhusubaliyev et al., 2008), and infinitely many coexisting attractors (Simpson, 2014a; Simpson, 2014b) can

occur in PWS maps. An open problem about a piecewise linear system were mentioned in Grove et al. (2012) as follows:

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, n = 0,1,2,\ldots \quad (1)$$

where parameters a, b, c and d are in $\{-1, 0, 1\}$ and the initial condition $(x_0, y_0) \in \mathbb{R}^2$. Gove et al. (2012) found that every solution of a special case of system (1) is eventually prime period-3 solution except for the unique equilibrium solution. In article Tikjha et al. (2010; 2015; 2017) and Tikjha and Lapiere (2020), they studied some special cases of System (1), by showing that there are periodic attractors. They showed that every solution is eventually either that attractors or the equilibrium point by using direct calculation and inductive statement. Our goal is to make generalization of parameter b of special case of System (1) given as follow,

$$\begin{cases} x_{n+1} = |x_n| - y_n + b \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, n = 0,1,2,\ldots \quad (2)$$

where b is any real number. Krinket and Tikjha (2015) studied a special case of System (2) by taking $b = -1$, and found that solutions are eventually a prime period 4 with a specific initial condition on $y-$axis. Tikjha and Piasu (2020) studied the solution of a special case of System (2) by taking $b = -3$:

$$\begin{cases} x_{n+1} = |x_n| - y_n - 3 \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, n = 0,1,2,\ldots \quad (3)$$

with the initial condition being a specific region in first quadrant is an eventually equilibrium point or prime period 4. Therefore we continue to investigate System (3) with the initial condition in a specific region in second quadrant.

METHODOLOGY

We will use a direct calculation by substituting an initial conditions in each region into the system (3) and verify that it is negative or non-negative in each iteration and then use that property to compute the next iteration. The following definitions (Grove and Ladas, 2005) are used in this article. A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a system of difference equations is called *eventually an equilibrium point* if there exists an integer $N > 0$ such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is the equilibrium point $(\bar{x}, \bar{y})$; that is, $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq N$. A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a System (3) is called *eventually periodic with prime period p* or an *eventually prime period p solution* (or *p-cycle*) if there exists an integer $N > 0$ and *p* is the smallest positive integer such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is periodic with period *p*; that is, $(x_{n+p}, y_{n+p}) = (x_n, y_n)$ for all $n \geq N$. It is worth noting that a solution is eventually periodic with period *p* (a solution is an eventually equilibrium point) when *orbit* (forward iterations) contains a point of the cycle (equilibrium point). We will show that every solution of the system (3) with specific initial condition is an equilibrium point by finding a pattern of

solutions. Then, we verify that the closed form is true by using mathematical induction. We also use direct calculations to show that some solutions will become the equilibrium point in some regions.

RESULTS

In this section, we will investigate the behavior of System (3) with the initial condition belonging to the set $A:=\{(x,y)\in\mathbb{R}^{-}\times\mathbb{R}^{+}|-x-y-3>0\ and\ -2x-2y-7<0\ and\ -2x+2y-9<0\}$ as Figure 1.

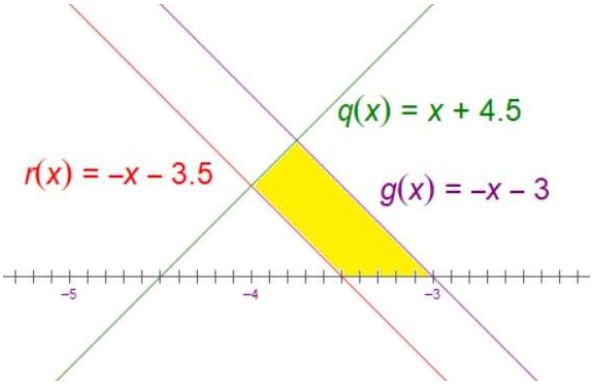


Figure 1 The yellow region is the initial condition of the system.

We claim that the system (3) has a unique equilibrium point which is $(-1,-1)$ obtained by solving a system of equations $\bar{x}=|\bar{x}|-\bar{y}-3$ and $\bar{y}=\bar{x}-|\bar{y}|+1$. Let $(x_0,y_0)\in A$. We have the first two iterations as

$$x_1=|x_0|-y_0-3=-x_0-y_0-3\geq 0 \quad \text{and} \quad y_1=x_0-|y_0|+1=x_0-y_0+1<0, \tag{4}$$

$$x_2=|x_1|-y_1-3=-2x_0-7 \quad \text{and} \quad y_2=x_1-|y_1|+1=-2y_0-1<0. \tag{5}$$

If (x_0,y_0) is in the region for which $-\frac{7}{2}\leq x_0<-3$ then $x_2\leq 0$ and so

$$x_3=2x_0+2y_0+5<0 \quad \text{and} \quad y_3=-2x_0-2y_0-7<0,$$
$$x_4=-1 \quad \text{and} \quad y_4=-1.$$

So, we have the following lemma.

Lemma 1 *Let* $(x_0,y_0)\in A$ *be an initial condition where* $x_0\in\left[-\frac{7}{2},-3\right)$. *Then the fourth iteration of the solution of the system* (3) *is the equilibrium point* $(-1,-1)$.

If (x_0,y_0) is in the region for which $-4<x_0<-\frac{7}{2}$ then $x_2>0$ and so the next two iterations will be in the following form:

$$x_3=-2x_0+2y_0-9\leq 0 \quad \text{and} \quad y_3=-2x_0-2y_0-7<0, \tag{6}$$

$$x_4=4x_0+13<0 \quad \text{and} \quad y_4=-4x_0-15. \tag{7}$$

If $x_0 \in \left[-\frac{15}{4},-\frac{7}{2}\right)$, then $y_4 \le 0$ and so $(x_5,y_5)=(-1,-1)$. Thus, we have the following lemma.

Lemma 2 *Let* $(x_0,y_0)\in A$ *be an initial condition where* $x_0 \in \left[-\frac{15}{4},-\frac{7}{2}\right)$. *Then the fifth iteration of the solution of the system* (3) *is the equilibrium point* $(-1,-1)$.

If (x_0,y_0) is in the region for which $-4 < x_0 < -\frac{15}{4}$, then $y_4 > 0$ and so the next two iterations will be in the following form:

$$x_5 = -1 \quad \text{and} \quad y_5 = 8x_0 + 29 < 0, \tag{8}$$

$$x_6 = -8x_0 - 31 \quad \text{and} \quad y_6 = 8x_0 + 29 < 0. \tag{9}$$

If $x_0 \in \left[-\frac{31}{8},-\frac{15}{4}\right)$ then $x_6 \le 0$ and so $(x_7,y_7)=(-1,-1)$. Thus, we have the following lemma.

Lemma 3 *Let* $(x_0,y_0)\in A$ *be an initial condition where* $x_0 \in \left[-\frac{31}{8},-\frac{15}{4}\right)$. *Then the seventh iteration of the solution of the system* (3) *is the equilibrium point* $(-1,-1)$.

We will investigate the solution with the initial condition (x_0,y_0) belonging to the remain region as the following lemma.

Lemma 4 *Let* $(x_0,y_0)\in A$ *be an initial condition where* $x_0 \in \left(-4,-\frac{31}{8}\right)$. *Then the solution of the system is eventually the equilibrium point* $(-1,-1)$.

Proof. We have a closed form of the first six iterations in the form of equations (4) – (9) and $x_6 = -8x_0 - 31 > 0$. The following sequences are used in the inductive statement of this lemma. $a_n = \frac{-2^{2n+3}+1}{2^{2n+1}}, b_n = \frac{-2^{2n+4}+1}{2^{2n+2}},$ and $\delta_n = 2^{2n+4}-1$. Let $P(n)$ be the following statement:

"for $x_0 \in (-4,a_n)$,

$x_{4n+3} = -2^{2n+2}x_0 - \delta_n$ and $y_{4n+3} = -1$.

If $x_0 \in [b_n,a_n)$ then $x_{4n+3} \le 0$ and so

$x_{4n+4} = 2^{2n+2}x_0 + \delta_n - 2 < 0$ and $y_{4n+4} = -2^{2n+2}x_0 - \delta_n \le 0$,

$x_{4n+5} = -1$ and $y_{4n+5} = -1$.

If $x_0 \in (-4,b_n)$ then $x_{4n+3} > 0$ and so

$x_{4n+4} = -2^{2n+2}x_0 - \delta_n - 2 < 0$ and $y_{4n+4} = -2^{2n+2}x_0 - \delta_n > 0$,

$x_{4n+5} = 2^{2n+3}x_0 + 2\delta_n - 1 < 0$ and $y_{4n+5} = -1$,

$x_{4n+6} = -2^{2n+3}x_0 - 2\delta_n - 1$ and $y_{4n+6} = 2^{2n+3}x_0 + 2\delta_n - 1 < 0$.

If $x_0 \in [a_{n+1},b_n)$ then $x_{4n+6} \le 0$ and so

$x_{4n+7} = -1$ and $y_{4n+7} = -1$.

If $x_0 \in (-4, a_{n+1})$ then $x_{4n+6} > 0$ ".

Firstly, we shall show that $P(1)$ is true. By letting $x_0 \in (-4, a_1) = \left(-4, \frac{-31}{8}\right)$ we have,

$x_{4(1)+3} = x_7 = |x_6| - y_6 - 3 = -16x - 63 = -2^{2(1)+2} x_0 - \delta_1$ and

$y_{4(1)+3} = y_7 = x_6 - |y_6| + 1 = -1$.

If $x_0 \in [b_1, a_1) = \left[\frac{-63}{16}, \frac{-31}{8}\right)$ then $x_7 = -16x_0 - 63 \le 0$ and so

$x_{4(1)+4} = x_8 = 16x_0 + 61 = 2^{2(1)+2} x_0 + \delta_1 - 2 < 0$ and

$y_{4(1)+4} = y_8 = -16x_0 - 63 = -2^{2(1)+2} x_0 - \delta_1 \le 0$,

$x_{4(1)+5} = x_9 = -1$ and $y_{4(1)+5} = y_9 = -1$.

If $x_0 \in (-4, b_1) = \left(-4, \frac{-63}{16}\right)$ then $x_7 = -16x_0 - 63 > 0$ and so

$x_{4(1)+4} = x_8 = -16x_0 - 65 = -2^{-2(1)+2} x_0 - \delta_1 - 2 < 0$ and

$y_{4(1)+4} = y_8 = -16x_0 - 63 = -2^{2(1)+2} x_0 - \delta_1 > 0$,

$x_{4(1)+5} = x_9 = 32x_0 + 125 = 2^{2(1)+3} x_0 + 2\delta_1 - 1 < 0$ and $y_{4(1)+5} = y_9 = -1$,

$x_{4(1)+6} = x_{10} = -32x_0 - 127 = -2^{2(1)+3} x_0 - 2\delta_1 - 1$ and

$y_{4(1)+6} = y_{10} = 32x_0 + 125 = 2^{2(1)+3} x_0 + 2\delta_1 - 1 < 0$.

If $x_0 \in [a_2, b_1) = \left[\frac{-127}{32}, \frac{-63}{16}\right)$ then $x_{10} = -32x_0 - 127 \le 0$ and so

$x_{4(1)+7} = x_{11} = -1$ and $y_{4(1)+7} = y_{11} = -1$.

If $x_0 \in (-4, a_2) = \left(-4, \frac{-127}{32}\right)$ then $x_{10} = -32x_0 - 127 > 0$.

Hence $P(1)$ is true. Now we suppose further that $P(k)$ is true for a positive integer k. We have $x_{4k+6} = -2^{2k+3} x_0 - 2\delta_k - 1 > 0$ and $y_{4k+6} = 2^{2k+3} x_0 + 2\delta_k - 1 < 0$ for $x_0 \in (-4, a_{k+1}) = \left(-4, \frac{-2^{2k+5}+1}{2^{2k+3}}\right)$.

We determine the sign of x_{4k+6} by substituting -4 and a_{k+1} into x_0 of a linear function x_{4k+6} of variable x_0 as follow:

$$x_{4k+6}\left(\frac{-2^{2k+5}+1}{2^{2k+3}}\right) = -2^{2k+3}\left(\frac{-2^{2k+5}+1}{2^{2k+3}}\right) - 2\left(2^{2k+4} - 1\right) - 1 = 2^{2k+5} - 1 - 2^{2k+5} + 2 - 1 = 0,$$

$x_{4k+6}(-4) = -2^{2k+3}(-4) - 2\left(2^{2k+4} - 1\right) - 1 = 2^{2k+5} - 2^{2k+5} + 2 - 1 = 1 > 0$. From now on, we will determine the sign of solutions by using this method. Then

$x_{4(k+1)+3} = x_{4k+7} = |x_{4k+6}| - y_{4k+6} - 3 = -2^{2(k+1)+2} x_0 - 4\delta_k - 3 = -2^{2(k+1)+2} x_0 - \delta_{k+1}$. We note that

$$4\delta_k + 3 = 4\left(2^{2k+4} - 1\right) + 3 = 2^{2k+6} - 4 + 3 = 2^{2(k+1)+4} - 1 = \delta_{k+1}.$$

$y_{4(k+1)+3} = y_{4k+7} = x_{4k+6} - |y_{4k+6}| + 1 = -2^{2k+3}x_0 - 2\delta_k - 1 + 2^{2k+3}x_0 + 2\delta_k - 1 + 1 = -1$.

If $x_0 \in [b_{k+1}, a_{k+1}) = \left[\frac{-2^{2k+6}+1}{2^{2k+4}}, \frac{-2^{2k+5}+1}{2^{2k+3}}\right)$ then $x_{4k+7} = -2^{2k+4}x_0 - \delta_{k+1} \le 0$ and so $x_{4(k+1)+4} = x_{4k+8} = 2^{2k+4}x_0 + \delta_{k+1} - 2 < 0$ and $y_{4(k+1)+4} = y_{4k+8} = -2^{2k+4}x_0 - \delta_{k+1} \le 0$. Thus $(x_{4(k+1)+5}, y_{4(k+1)+5}) = (-1,-1)$.

If $x_0 \in (-4, b_{k+1}) = \left(-4, \frac{-2^{2k+6}+1}{2^{2k+4}}\right)$ then $x_{4k+7} = -2^{2k+4}x_0 - \delta_{k+1} > 0$ and so $x_{4(k+1)+4} = x_{4k+8} = -2^{2k+4}x_0 - \delta_{k+1} - 2 < 0$ and $y_{4(k+1)+4} = y_{4k+8} = -2^{2k+4}x_0 - \delta_{k+1} > 0$,
$x_{4(k+1)+5} = x_{4k+9} = 2^{2k+5}x_0 + 2\delta_{k+1} - 1 < 0$ and $y_{4(k+1)+5} = y_{4k+9} = -1$,
$x_{4(k+1)+6} = x_{4k+10} = -2^{2k+5}x_0 - 2\delta_{k+1} - 1$ and $y_{4(k+1)+6} = y_{4k+10} = 2^{2k+5}x_0 + 2\delta_{k+1} - 1 < 0$,

If $x_0 \in [a_{k+2}, b_{k+1}) = \left[\frac{-2^{2k+7}+1}{2^{2k+5}}, \frac{-2^{2k+6}+1}{2^{2k+4}}\right)$ then $x_{4k+10} = -2^{2k+5}x_0 - 2\delta_{k+1} - 1 \le 0$ and so $x_{4(k+1)+7} = x_{4k+11} = -1$ and $y_{4(k+1)+7} = y_{4k+11} = -1$.

If $x_0 \in (-4, a_{k+2}) = \left(-4, \frac{-2^{2k+7}+1}{2^{2k+5}}\right)$ then $x_{4k+10} = -2^{2k+5}x_0 - 2\delta_{k+1} - 1 > 0$. Hence $P(k+1)$ is true. By the mathematical induction we conclude that $P(n)$ is true for every positive integer n. We note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = -4$. By the inductive statement $P(n)$, we can conclude that the solution is an eventually equilibrium point.

By the above lemmas, we immediately have the following theorem.

Theorem *Let* $(x_0, y_0) \in A$ *be an initial condition. Then the solution of the system* (3) *is an eventually equilibrium point* $(-1,-1)$..

DISCUSSION

The result of system (3) agrees with result of Tikjha and Piasu (2020). There are regions of initial condition in A that a solution is an eventually equilibrium point, however in the article Tikjha and Piasu (2020) there is 4-cycle of the map (3) given by $\{((-5,-1),(3,-5),(5,-1),(3,5))\}$. It is not easy to prove all behaviors of the map (3), because the map (3) is not differentiable. So, we cannot use the stability theorem apply to it. We believe that our results will be a tool for studying this system with another initial condition region of $\mathbb{R}^2$. We conjecture that the solution of system (3) will become only an equilibrium point or 4-cycles.

CONCLUSIONS

We separate region A into 4 sub-regions. The first region when initial condition $(x_0, y_0) \in A$ and $x_0 \in \left[-\frac{7}{2}, 3\right)$, the solution is an equilibrium point within 4 iterations. The second region when initial

condition $(x_0, y_0) \in A$ and $x_0 \in \left[-\frac{15}{4}, -\frac{7}{2}\right)$, the solution is an equilibrium point within 5 iterations. The third region when initial condition $(x_0, y_0) \in A$ and $x_0 \in \left[-\frac{31}{8}, -\frac{15}{4}\right)$, the solution is an equilibrium point within 7 iterations. We used inductive statement to prove Lemma 4. If we choose initial condition closer to point $\left(-4, -\frac{1}{2}\right)$, then it will take more iteration of the solution being equilibrium point.

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