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สูตรสำเร็จของความยาววิ่งเฉลี่ยสำหรับกระบวนการ ARFIMAX (1,d,1)

The Explicit Expression of Average Run Length for ARFIMAX (1, d, 1) Procedure

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บทคัดย่อ

วัตถุประสงค์ของงานวิจัยนี้คือเพื่อหาสูตรสำเร็จของค่าความยาววิ่งเฉลี่ยสำหรับแผนภูมิควบคุมค่าเฉลี่ยเคลื่อนที่ถ่วงน้ำหนักแบบเลขชี้กำลังเมื่อค่าสังเกตถูกกำหนดโดยกระบวนการ ARFIMAX (1,d,1) พร้อมกับความคลาดเคลื่อนมีการแจกแจงแบบเลขชี้กำลังและเป็นอิสระกัน ผลลัพธ์ของค่าความยาววิ่งเฉลี่ยที่ได้จากสูตรสำเร็จและสมการปริพันธ์ให้ผลที่สอดคล้องกันดี และสูตรสำเร็จที่ได้ใช้เวลาในการคำนวณค่าความยาววิ่งเฉลี่ยน้อยกว่า

ABSTRACT

The purpose of this research was to derive the Explicit Expression of ARL for Exponential Weighted Moving Average (EWMA) control chart when observations are explained by ARFIMAX (1,d,1) process with exponential white noise. The results of the ARL obtained from the Explicit Expression and Integral Equation (IE) were in good accordance, with the Explicit Expression taking time to calculate ARL.

คำสำคัญ: กระบวนการ ARFIMAX แผนภูมิควบคุมค่าเฉลี่ยเคลื่อนที่ถ่วงน้ำหนักแบบเลขชี้กำลัง ความยาววิ่งเฉลี่ย

Keywords: ARFIMAX procedure, EWMA control chart, Average Run Length (ARL).

1. INTRODUCTION

Control charts are graphical procedures used to detect assignable causes. Observations are obtained from the process and the values of some statistics that are computed and plotted over time. The first control charts were proposed by Shewhart (1931), resulting in them often being called Shewhart control charts. These charts are designed for the detection of large deviations in the parameter process monitoring. However, these control charts are not good for detecting relatively small changes in parameter processing. Other types of control charts include the cumulative sum (CUSUM) chart and the exponentially-weighted moving average (EWMA) chart, proposed by Page (1954) and Roberts (1959), respectively. In contrast to Shewhart charts, which have the advantage of accumulating information from past observations, these types of charts usually have some tuning parameters that must be specified by the user that are optimal detecting specific changed size. Therefore, performance for detecting other changes in size could be compromised. This implies that, for these charts, the user needs to have relatively good information about the changed parameters that would likely occur when the process is out of control. The underlying assumption in a control chart is that observations from the process are independent and identically distributed (i.i.d.).

The most commonly used autoregressive moving-average (ARMA) models (Box and Jenkins, 1976) are short-memory processes. On the other hand, a long-memory process shows very strong persistence and observations more like a non-stationary process. Based on this observation, a class of continuous-time long-memory processes (fractional Brownian motions) was proposed by Mandelbrot and van Ness (1968). However, such models tend to be quite restricted because they depend on only one parameter. The most commonly used long-memory processes are fractionally integrated ARMA (ARFIMA) processes, which were first introduced by Granger and Joyeux (1980) and Hosking (1981). ARFIMA models can represent a variety of dependent structures, including both short memory and long memory. There are more interesting applications. Besides, seasonal effects can be incorporated into an ARFIMA process in various ways to construct flexible models (Porter-Hudak, 1990; Ray 1993).

Criticisms of Shewhart charts are slow to show the small shifts in the process average. Two methods often used because of their efficiency in more quickly finding such small shifts are CUSUM and EWMA charts. These charts handle the data as a time series rather than iid. The CUSUM is an infinite length time series in that the difference from the long term average value of each point is added algebraically to the cumulative algebraic sum of all previous differences, after which it is plotted (Reproduced with permission of the copyright owner; Further reproduction prohibited without permission). The technique is somewhat cumbersome as it requires special techniques to determine when a point is out of control. The EWMA uses a weighted series of previous data points, along with the current data point, to determine the expected value of that point. The calculated value is then plotted against standard Shewhart control limits looking for a trend or out-of-control point. Thus, the two methods commonly used to overcome a weakness in the Shewhart method are in fact time series methods and have been clearly shown and accepted by practitioners in the field to be better than conventional charts for detecting small shifts in the process average. Therefore, the main goal of this paper is to study the Fredholm type integral equations method to derive a closed-form solution for Average Run Length (ARL) of the EWMA control chart for Autoregressive Fractionally-Integrated Moving Average with exponential white noise.

The control charts are typically compared by their average run length (ARL).The ARL is the average number of runs that occur before an out-of-control point presents itself. If a process is in-control (ARL_0), it is ideal for the ARL to be large. However, if a process is out-of-control (ARL_1), it is optimal for the ARL to be small. Methods for evaluating ARL_0 and ARL_1 of EWMA control chart include the Markov Chain Approach (MCA), Integral Equation approach (IE) and Monte Carlo Simulation (MC). In 1987, Crowder (1987) first used IE for approximating ARL_0 and ARL_1 for Gaussian distribution. Later, Lucus and Saccucci (1990) used MCA to evaluate the run length properties of EWMA control charts. Harris and Ross (1991) studied CUSUM and EWMA control charts with serially correlated observations via Monte Carlo simulation. Later, a simple and very accurate ARL calculation procedure based on an approximating equation was provided by Hawkins and Olwell (1998). Recently, Areepong and Novikov (2009) used the Nystrom method to present the error term of numerical integral equations for approximating ARL_0 and ARL_1 for EWMA control chart in the case of exponential distribution. Later, Mititelu et al. (2010) used the Fredholm integral equation of the second method for explicit formulas of ARL for a one-sided EWMA procedure. Vermatt and Does (2008) derived explicit easy-to-use expressions for EWMA statistics when the process observations were autoregressive of order 1. Variance was used to modify the control limits of the corresponding EWMA control chart. Suriyakat et al. (2012) presented the explicit formulas of ARL for EWMA control chart to monitor AR(1) process. Consequently, exact expression of ARL for EWMA control chart based on ARX (p) process were proposed by Paichit (2017). The IE for the ARL can be easy to compute and program. The focus of this paper is to find the analytical formulas and numerical methods of ARL for EWMA control chart for ARFIMAX (1,d,1) process with exponential white noise to detect change in the process mean. The Integral Equation is used to derive this explicit expression for ARL. The procedures of the paper are as follows: the characteristics of EWMA control chart for ARFIMAX (1, d, 1) process is introduced in Section 2; the derivation of closed-form expression for ARL is expressed in Section 3; the numerical method for solving the integral equation to obtain approximation of ARL is presented in Section 4; the conclusions are presented in Section 5.

2. THE ARFIMAX (1,D,1) PROCESS FOR EWMA CONTROL CHARTS

In this section, the characteristics of EWMA control chart for ARFIMAX (1,d,1) process effected the tool for detecting small changes in the process parameters. Given $\mathrm{Y_t}$ to be a sequence for an Autoregressive Fractionally Integrated Moving Average with explanatory variable (ARFIMAX (1, d, 1)) random process. The EWMA process regress the current value $\mathrm{Y_t}$ on the past values of itself $\mathrm{Y_{t-1}, Y_{t-2}, \dots, Y_{t-d}}$ and past random errors that occurred in past time periods $\xi_{\mathrm{t-1}}$. Thus, the current value is a linear combination of the most recent past values and most recent past (unobserved) white noise error terms.

The definition of EWMA statistics based on the ARFIMAX (1, d, 1) process is the following recursion:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t \; ; \; t = 1,2,\dots \tag{1}$$

where $\mathrm{Z_t}$ is the EWMA statistics, $\mathrm{Y_t}$ is the sequence Autoregressive Fractionally Integrated Moving Average (ARFIMAX (1,d,1)) process and $\lambda \in (0,1)$ is the smoothing parameter. The initial value is a constant $(\mathrm{Z_0} = \mathrm{u} = 0)$

The ARFIMA (Auto Regressive Fractionally Integrated Moving Average) model combining the fractional order of time series integration with the conventional autoregressive/moving average method has been widely applied to prove its empirical applicability, such as stock price series or international exchange rates (Henry and Olekalns, 2002). The ARFIMA model can include explanatory variables for deterministic trends, called the ARFIMAX model (Davidson and Sibbertsen, 2005).

The general Autoregressive Fractionally Integrated Moving Average with explanatory variable, denoted by the (ARFIMAX (1, d, 1)) process, can be written as:

$$\begin{aligned} Y_t &= dY_{t-1} - d\tfrac{(d-1)}{2!}Y_{t-2} + d(d-1)\tfrac{(d-2)}{3!}Y_{t-3} + \cdots + Y_{t-d} - X_t + \phi_1 X_{t-1} + \xi_t - \theta_1\xi_{t-1} \\ &+ \left(\phi_1 Y_{t-1} - \phi_1 dY_{t-2} + d\tfrac{(d-1)}{2!}\phi_1 Y_{t-3} - d(d-1)\tfrac{(d-2)}{3!}\phi_1 Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})\right) \end{aligned} \quad (2)$$

where ξ_t is the white noise process assumed with Exponential distribution. The initial value is normally the process mean, with an autoregressive coefficient $-1 < \phi_1 < 1$ and moving average coefficients $-1 < \theta_1 < 1$. It is assumed that the initial value of ARFIMAX (1, d, 1) process $Y_{t-1}, Y_{t-2}, \ldots, Y_{t-d} = 1$, difference operator $d = 0.3$ and $X_t, X_{t-1} = 1.$

In this paper, the case of a positive change in distribution crossing the upper control limit that raises alarm is discussed in detail. Given ξ_t, $t = 1,2, \ldots$ to be a sequence of independent, identical distribution random variables with exponential parameter (β). It is normally assumed that under in-control state, the parameter has a known in-control value of $\beta = \beta_0$. Parameter β could be changed to out-of-control value $\beta = \beta_1$, when $(\delta < \infty)$, δ is the change-point time.

The first passage time for the EWMA can be written as:

$$\tau_b = inf(t > 0: Z_t > b), \quad (3)$$

where b is a control limit.

The ARL is the expectation value of τ_b. Mostly, two characteristics are used for the performance of control chart, which are ARL_0 and ARL_1, as follows:

$$ARL_0 = E_\infty(\tau_b) \quad (4)$$

$$ARL_1 = E_\delta(\tau_b - \delta + 1 | \tau_b \geq \delta) \quad (5)$$

where $E_\infty(.)$ is the expectation corresponding to the target value and is assumed to be large enough.

$E_\delta(.)$ is the expectation under the assumption that change-point occurs at time $\delta = 1.$

3. EXPLICIT EXPRESSION FOR ARL IN EWMA CONTROL CHART

In this section, explicit expression to evaluate the solution of EWMA control chart for ARFIMAX (1, d, 1) observations with exponential white noise is discussed. We follow the technique for approximating the ARL for Gaussian distribution developed in Crowder (1997). Given $L(u)$ is ARL for ARFIMAX (1, d, 1) observations with initial value $Z_0 = u,$ if is assumed that the lower control limit and the upper control limit are $LCL = 0$ and $UCL = b$. When EWMA statistics Z_1 in an in-control process, the inequality is:

$$\begin{aligned} 0 &< (1-\lambda)Z_0 + d\lambda Y_{t-1} - d\frac{(d-1)}{2!}\lambda Y_{t-2} + d(d-1)\frac{(d-2)}{3!}\lambda Y_{t-3} + \cdots + \lambda Y_{t-d} - \lambda X_t + \lambda\phi_t X_{t-1} + \lambda\xi_t \\ &-\lambda\theta_1\xi_{t-1} + \lambda\left(\phi_1 Y_{t-1} - \phi_1 dY_{t-2} + d\tfrac{(d-1)}{2!}\phi_1 Y_{t-3} - d(d-1)\tfrac{(d-2)}{3!}\phi_1 Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})\right) < b. \end{aligned}$$

Consider function $L(u) = 1 + \int L(Z_1) f(\xi_t) d\xi_t,$

Therefore,

$$L(u) = 1 + \int L\big((1-\lambda)u\big) + d\lambda Y_{t-1} - d\frac{(d-1)}{2!}\lambda Y_{t-2} + d(d-1)\frac{(d-2)}{3!}\lambda Y_{t-3} + \cdots + \lambda Y_{t-d} - \lambda X_t + \lambda\phi_t X_{t-1} + \lambda\xi_t$$
$$-\lambda\theta_1\xi_{t-1} + \lambda\phi_1 Y_{t-1} - \lambda\phi_1 dY_{t-2} + d\frac{(d-1)}{2!}\lambda\phi_1 Y_{t-3} - d(d-1)\frac{(d-2)}{3!}\lambda\phi_1 Y_{t-3} + \cdots + \lambda\phi_1(-Y_{t-d+1}))f(\omega)d\omega.$$

Change variables in the integration will be:

$$L(u) = 1 + \frac{1}{\lambda}\int_0^b L(\omega) f\Big(\frac{\omega-(1-\lambda)u}{\lambda} - dY_{t-1} + d\frac{(d-1)}{2!}Y_{t-2} - d(d-1)\frac{(d-2)}{3!}Y_{t-3} - \cdots - Y_{t-d} + X_t - \phi_1 X_{t-1} + \theta_1\xi_{t-1}$$
$$+\lambda\theta_1\xi_{t-1} - \phi_1 Y_{t-1} + \phi_1 dY_{t-2} - d\frac{(d-1)}{2!}\phi_1 Y_{t-3} + d(d-1)\frac{(d-2)}{3!}\phi_1 Y_{t-3} - \cdots - \phi_1(-Y_{t-d+1})\Big)d\omega$$

$$L(u) = 1 + \frac{1}{\lambda\beta}\int_0^b L(\omega)e^{-\frac{\omega}{\lambda\beta}} exp\Big(\frac{(1-\lambda)u}{\lambda\beta} + \frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + Y_{t-d} - X_t + \phi_1 X_{t-1} - \theta_1\xi_{t-1}}{\beta}$$
$$\frac{-\lambda\theta_1\xi_{t-1} + \phi_1 Y_{t-1} - \phi_1 dY_{t-2} + d\frac{(d-1)}{2!}\phi_1 Y_{t-3} - d(d-1)\frac{(d-2)}{3!}\phi_1 Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})}{\beta}\Big)d\omega. \qquad (6)$$

Let P_z and E_z be the probability measure and induced expectation corresponding to the initial value u, respectively. Then the ARL is a unique solution to the integral equation, defined as ARL $= L(u) = E_\infty(\tau) < \infty$. The solution for the integral equation of EWMA control chart is defined in Equation (6). The right hand side of Equation (6) is continuous. Therefore, the solution of the integral for Equation (6) is a continuous function.

Considering the complete matric space $(C(I), \|\ \|_\infty)$ where C(I) denotes the space of all continuous functions on I, where I is a compact interval, with the norm $\|L\|_\infty = \sup_{u\in I}|L(u)|$. The operator T is named as a contraction if there exists a number $0 \le q < 1$ such $\|T(L_1)-T(L_2)\| \le q\|L_1-L_2\|$ for all $L_1, L_2 \in C(I)$. In this case, let T be an operation in this class of all continuous functions C(I), where I = [0, b] is defined by:

$$T(L(u)) = 1 + \frac{1}{\lambda\beta}\exp\Big(\frac{(1-\lambda)u}{\lambda\beta} + \frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + Y_{t-d} - X_t + \phi_1 X_{t-1} - \theta_1\xi_{t-1}}{\beta}$$
$$\frac{-\lambda\theta_1\xi_{t-1} + \phi_1 Y_{t-1} - \phi_1 dY_{t-2} + d\frac{(d-1)}{2!}\phi_1 Y_{t-3} - d(d-1)\frac{(d-2)}{3!}\phi_1 Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})}{\beta}\Big)\int_0^b L(\omega)e^{-\frac{\omega}{\lambda\beta}}d\omega. \qquad (7)$$

Thus, the integral equation in (7) can be written as T(L(u))=L(u). Now, according to Banach's Fixed Point Theorem, if the operator T is a contraction, then the fixed point equation T(L(u))=L(u) has a unique solution. We will show that operator T is a contraction in Proposition1.

Proposition 1 On the matric space $(C(I), \|\|_\infty)$ with the norm $\|L\|_\infty = \sup_{u\in I}|L(u)|$ the operator T is a contraction.

Proof.

To show that $\mathbf{T}$ is a contraction for any $u \in I$ and $L_1, L_2 \in C(I)$. The inequality $\|T(L_1)-T(L_2)\| \le q\|L_1-L_2\|$ for all $L_1, L_2 \in C(I)$ with $0 \le q < 1$. According to (7), then

$$\|T(L_1) - T(L_2)\|_\infty \le \sup_{u\in[0,b]}\Big|L_1(0) - L_2(0)\ \frac{1}{\lambda\beta}\exp\Big(\frac{(1-\lambda)u}{\lambda\beta} + \frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + Y_{t-d} - X_t + \phi_1 X_{t-1} - \theta_1\xi_{t-1}}{\beta}$$
$$\frac{-\lambda\theta_1\xi_{t-1} + \phi_1 Y_{t-1} - \phi_1 dY_{t-2} + d\frac{(d-1)}{2!}\phi_1 Y_{t-3} - d(d-1)\frac{(d-2)}{3!}\phi_1 Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})}{\beta}\Big)\int_0^b L(\omega)e^{-\frac{\omega}{\lambda\beta}}d\omega\Big|$$

$$\le \sup_{u\in[0,b]}\Big|\|L_1 - L_2\|\frac{1}{\lambda\beta}\ \exp\Big(\frac{(1-\lambda)u}{\lambda\beta} + \frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + Y_{t-d} - X_t + \phi_1 X_{t-1} - \theta_1\xi_{t-1}}{\beta}$$
$$\frac{-\lambda\theta_1\xi_{t-1} + \phi_1 Y_{t-1} - \phi_1 dY_{t-2} + d\frac{(d-1)}{2!}\phi_1 Y_{t-3} - d(d-1)\frac{(d-2)}{3!}\phi_1 Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})}{\beta}\Big)(-\lambda\beta)\ \Big(e^{-\frac{b}{\lambda\beta}} - 1\Big)\Big|$$

$$= \|L_1 - L_2\|_\infty \sup_{u\in[0,b]}\Big[\Big|\exp\Big(\frac{(1-\lambda)u}{\lambda\beta} + \frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + Y_{t-d} - X_t + \phi_1 X_{t-1} - \theta_1\xi_{t-1}}{\beta}$$

$$\left.\left.\frac{-\lambda\theta_1\xi_{t-1}+\phi_1Y_{t-1}-\phi_1 dY_{t-2}+d\frac{(d-1)}{2!}\phi_1Y_{t-3}-d(d-1)\frac{(d-2)}{3!}\phi_1Y_{t-3}+\cdots+\phi_1(-Y_{t-d+1})}{\beta}\right)\left(1-e^{-\frac{b}{\lambda\beta}}\right)\right|\Bigg]$$

$$=\|L_1-L_2\|_\infty\left|1-e^{-\frac{b}{\lambda\beta}}\right|\sup_{u\in[0,b]}\Bigg[\Bigg|\frac{1}{\beta\lambda}\exp\Bigg(\frac{(1-\lambda)u}{\lambda\beta}+\frac{dY_{t-1}-d\frac{(d-1)}{2!}Y_{t-2}+d(d-1)\frac{(d-2)}{3!}Y_{t-3}+\cdots+Y_{t-d}-X_t+\phi_1X_{t-1}-\theta_1\xi_{t-1}}{\beta}$$

$$\left.\left.\frac{-\lambda\theta_1\xi_{t-1}+\phi_1Y_{t-1}-\phi_1 dY_{t-2}+d\frac{(d-1)}{2!}\phi_1Y_{t-3}-d(d-1)\frac{(d-2)}{3!}\phi_1Y_{t-3}+\cdots+\phi_1(-Y_{t-d+1})}{\beta}\right)\right|\Bigg]$$

$$=q\|L_1-L_2\|_\infty,$$

where $0\le q=\left|1-e^{-\frac{b}{\lambda\beta}}\right|\sup_{u\in[0,b]}\Bigg[\Bigg|\frac{1}{\beta\lambda}\exp\Bigg(\frac{(1-\lambda)u}{\lambda\beta}+\frac{dY_{t-1}-d\frac{(d-1)}{2!}Y_{t-2}+d(d-1)\frac{(d-2)}{3!}Y_{t-3}+\cdots+Y_{t-d}-X_t+\phi_1X_{t-1}-\theta_1\xi_{t-1}}{\beta}$

$$\left.\left.\frac{-\lambda\theta_1\xi_{t-1}+\phi_1Y_{t-1}-\phi_1 dY_{t-2}+d\frac{(d-1)}{2!}\phi_1Y_{t-3}-d(d-1)\frac{(d-2)}{3!}\phi_1Y_{t-3}+\cdots+\phi_1(-Y_{t-d+1})}{\beta}\right)\right|\Bigg]<1,$$

$0<\lambda<1$, $\beta>0$ and $y_i,\xi_i=1.$

Triangular inequality has been used and the fact is:

$$|L_1(0)-L_2(0)|\le\sup_{u\in[0,b]}|L_1(u)-L_2(u)|=\|L_1-L_2\|_\infty.$$

Therefore, the uniqueness of the solution is guaranteed via Proposition 1. Next, we derive explicit expression of the Fredholm integral equation, which is called the explicit expression of equation (7), as shown by Theorem 1.

Theorem 1 The solutions for the integral equation $T(L(u))=L(u)$ given by:

$$L(u)=1-\frac{e^{\left(\frac{(1-\lambda)u}{\lambda\beta}+\frac{dY_{t-1}-d\frac{(d-1)}{2!}Y_{t-2}+d(d-1)\frac{(d-2)}{3!}Y_{t-3}+\cdots+\phi_1(-Y_{t-d+1})}{\beta}\right)}\left(e^{-\frac{b}{\lambda\beta}}-1\right)}{1+\frac{e^{\left(\frac{dY_{t-1}-d\frac{(d-1)}{2!}Y_{t-2}+d(d-1)\frac{(d-2)}{3!}Y_{t-3}+\cdots+\phi_1(-Y_{t-d+1})}{\beta}\right)}\left(e^{-\frac{b}{\beta}}-1\right)}{\lambda}}$$

Proof.

To consider the integral equation in (6):

$$L(u)=1+\frac{1}{\lambda\beta}\int_0^b L(\omega)e^{-\frac{\omega}{\lambda\beta}}exp\Bigg(\frac{(1-\lambda)u}{\lambda\beta}+\frac{dY_{t-1}-d\frac{(d-1)}{2!}Y_{t-2}+d(d-1)\frac{(d-2)}{3!}Y_{t-3}+\cdots+Y_{t-d}-X_t+\phi_1X_{t-1}-\theta_1\xi_{t-1}}{\beta}$$
$$\left.\frac{-\lambda\theta_1\xi_{t-1}+\phi_1Y_{t-1}-\phi_1 dY_{t-2}+d\frac{(d-1)}{2!}\phi_1Y_{t-3}-d(d-1)\frac{(d-2)}{3!}\phi_1Y_{t-3}+\cdots+\phi_1(-Y_{t-d+1})}{\beta}\right)d\omega.$$

where $\xi_t\sim Exp(\beta)$.

Let $C(u)=exp\Bigg(\frac{(1-\lambda)u}{\lambda\beta}+\frac{dY_{t-1}-d\frac{(d-1)}{2!}Y_{t-2}+d(d-1)\frac{(d-2)}{3!}Y_{t-3}+\cdots+Y_{t-d}-X_t+\phi_1X_{t-1}-\theta_1\xi_{t-1}}{\beta}$

$$\left.\frac{-\lambda\theta_1\xi_{t-1}+\phi_1Y_{t-1}-\phi_1 dY_{t-2}+d\frac{(d-1)}{2!}\phi_1Y_{t-3}-d(d-1)\frac{(d-2)}{3!}\phi_1Y_{t-3}+\cdots+\phi_1(-Y_{t-d+1})}{\beta}\right)$$

Then $L(u)=1+\frac{C(u)}{\lambda\beta}\int_0^b L(\omega)e^{-\frac{\omega}{\lambda\beta}}d\omega\quad;0\le u\le b.$

Let $d=\int_0^b L(\omega)\,e^{-\frac{\omega}{\lambda\beta}}d\omega$, we have $L(u)=1+\frac{C(u)}{\lambda\beta}d$,

then

$$L(u)=1+\frac{d}{\lambda\beta}\,exp\Bigg(\frac{(1-\lambda)u}{\lambda\beta}+\frac{dY_{t-1}-d\frac{(d-1)}{2!}Y_{t-2}+d(d-1)\frac{(d-2)}{3!}Y_{t-3}+\cdots+Y_{t-d}-X_t+\phi_tX_{t-1}-\theta_1\xi_{t-1}}{\beta}$$
$$\left.\frac{-\lambda\theta_1\xi_{t-1}+\phi_1Y_{t-1}-\phi_1 dY_{t-2}+d\frac{(d-1)}{2!}\phi_1Y_{t-3}-d(d-1)\frac{(d-2)}{3!}\phi_1Y_{t-3}+\cdots+\phi_1(-Y_{t-d+1})}{\beta}\right).$$

Solving a constant $d = \int_0^b L(\omega)\, e^{-\frac{\omega}{\lambda\beta}} d\omega$.

$$d = \int_0^b \left(1 + \frac{C(\omega)}{\lambda\beta} d\right) e^{-\frac{\omega}{\lambda\beta}} d\omega$$

$$= \int_0^b e^{-\frac{\omega}{\lambda\beta}}\, d\omega + \int_0^b C(\omega)\, e^{-\frac{\omega}{\lambda\beta}} d\omega$$

$$= \lambda\beta\left(1 - e^{-\frac{b}{\lambda\beta}}\right) \frac{d}{\lambda\beta}\, exp\left(\frac{(1-\lambda)u}{\lambda\beta} + \frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + Y_{t-d} - X_t + \phi_t X_{t-1} - \theta_1 \xi_{t-1}}{\beta}\right.$$
$$\left.\frac{-\lambda\theta_1\xi_{t-1} + \phi_1 Y_{t-1} - \phi_1 dY_{t-2} + d\frac{(d-1)}{2!}\phi_1 Y_{t-3} - d(d-1)\frac{(d-2)}{3!}\phi_1 Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})}{\beta}\right) \int_0^b e^{-\frac{\omega}{\beta}} d\omega$$

$$d = \frac{(-\lambda\beta)\left(e^{-\frac{b}{\lambda\beta}} - 1\right)}{1 + \frac{e^{\left(\frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})}{\beta}\right)\left(e^{-\frac{b}{\beta}} - 1\right)}}{\lambda}}$$

Substitute constant d into function$L(u)$, then

$$L(u) = 1 - \frac{e^{\left(\frac{(1-\lambda)u}{\lambda\beta} + \frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})}{\beta}\right)\left(e^{-\frac{b}{\lambda\beta}} - 1\right)}}{1 + \frac{e^{\left(\frac{dY_{t-1} - d\frac{(d-1)}{2!}Y_{t-2} + d(d-1)\frac{(d-2)}{3!}Y_{t-3} + \cdots + \phi_1(-Y_{t-d+1})}{\beta}\right)\left(e^{-\frac{b}{\beta}} - 1\right)}}{\lambda}}.$$

By Theorem 1, the explicit expression for ARL_0 is as follows:

$$ARL_0 = 1 - \frac{\lambda e^{\left(\frac{(1-\lambda)u}{\lambda\beta_0}\right)}\left(e^{-\frac{b}{\lambda\beta_0}} - 1\right)}{\lambda\, e^{\left(\frac{-dY_{t-1} + d\frac{(d-1)}{2!}Y_{t-2} - d(d-1)\frac{(d-2)}{3!}Y_{t-3} - \cdots - \phi_1(-Y_{t-d+1})}{\beta_0}\right)} e^{\left(-\frac{b}{\beta_0}\right)} - 1} \quad (8).$$

On the contrary, the explicit expression for ARL1 when the process is out-of-control involves parameter $\beta = \beta_1$, which can be written as follows:

$$ARL_1 = 1 - \frac{\lambda e^{\left(\frac{(1-\lambda)u}{\lambda\beta_1}\right)}\left(e^{-\frac{b}{\lambda\beta_1}} - 1\right)}{\lambda\, e^{\left(\frac{-dY_{t-1} + d\frac{(d-1)}{2!}Y_{t-2} - d(d-1)\frac{(d-2)}{3!}Y_{t-3} - \cdots - \phi_1(-Y_{t-d+1})}{\beta_1}\right)} e^{\left(-\frac{b}{\beta_1}\right)} - 1} \quad (9)$$

where $-1 < \phi_1 < 1$ and $-1 < \theta_1 < 1$ is Autoregressive Moving Average coefficient and $0 < \lambda < 1$, $Y_{t-1}, Y_{t-2}, \ldots, Y_{t-d}$ and ξ_t, ξ_{t-1} are the smoothing parameters and initial values, respectively.

4. NUMERICAL INTEGRAL EQUATION (IE)

Generally, the Integral Equation cannot be solved analytically for $L(u)$. Thus, it is necessary to use numerical methods to solve them. Kantorovich and Krylov (1958), and Atkinson and Han (2001) developed numerical schemes for solving integral equations. We use a quadrature rule to approximate the integral by a finite sum of an area of rectangles with base b/m with heights chosen as the values of f at the midpoint of intervals for length b/m beginning at zero. Particularly, once the choice of a quadrature rule is made, the interval $[0, b]$ is divided into a partition $0 \le a_1 \le a_2 \le \ldots \le a_m \le b$ and set of constant weight $k_j = (b\,/\,m) \ge 0$.

The approximation for an integral is of the form

$$\int_0^b L(\omega) f(\omega) d\omega \cong \sum_{j=1}^{m} k_j f(a_j)$$

where $a_j = \frac{b}{m}\left(\frac{2j-1}{2}\right)$ and $k_j = \frac{b}{m}$; $j = 1, 2, \ldots, m$.

Let $\tilde{L}(a_j)$ denote the numerical approximation to integral equation $L(u)$, which can be found as the solution for the linear equation as follows:

$$\tilde{L}(a_j) \cong 1 + \frac{1}{\lambda}\sum_{j=1}^{m} k_j L(a_j) f\left(\frac{a_j-(1-\lambda)a_i}{\lambda} - dY_{t-1} + d\frac{(d-1)}{2!}Y_{t-2} - d(d-1)\frac{(d-2)}{3!}Y_{t-3} - \cdots - \phi_1(-Y_{t-d+1})\right) \quad (10)$$

where ; $i = 1,2, \dots m,$ the set of m equations is m unknowns, which can be written in matrix form.

Let $\boldsymbol{L}_{m\times 1} = [\tilde{L}(a_1),\ \tilde{L}(a_2), \dots, \tilde{L}(a_m)]^{/}$ be the column vector of $\tilde{L}(a_j)$ and $\mathbf{1}_{m\times 1} = [1, 1, \dots, 1]^{/}$ be a column vector of one. Also, let $\mathrm{R}_{\mathrm{m\times m}}$ be a matrix and define the $(\mathrm{m},\mathrm{m})^{\mathrm{th}}$ as an element of matrix R, as follows:

$$[R_{ij}] \cong \frac{1}{\lambda}\sum_{j=1}^{m} k_j L(a_j) f\left(\frac{a_j-(1-\lambda)a_i}{\lambda} - dY_{t-1} + d\frac{(d-1)}{2!}Y_{t-2} - d(d-1)\frac{(d-2)}{3!}Y_{t-3} - \cdots - \phi_1(-Y_{t-d+1})\right).$$

and $I_{m\times 1} = diag[1,1, \dots ,1]$ is unit matrix order m. If $(\boldsymbol{I} - \boldsymbol{R})^{-1}$ exists, the numerical approximation for the integral equation in terms of matrix is:

$$\boldsymbol{L}_{m\times 1} = (\boldsymbol{I} - \boldsymbol{R})^{-1} 1_{m\times 1}.$$

When a_i is replaced by u in $\tilde{L}(a_j)$, the numerical approximation for function $\mathrm{L(u)}$ is as follows:

$$\tilde{L}(u) \cong 1 + \frac{1}{\lambda}\sum_{j=1}^{m} k_j L(a_j) f\left(\frac{a_j-(1-\lambda)a_i}{\lambda} - dY_{t-1} + d\frac{(d-1)}{2!}Y_{t-2} - d(d-1)\frac{(d-2)}{3!}Y_{t-3} - \cdots - \phi_1(-Y_{t-d+1})\right) \quad (11).$$

NUMERICAL RESULTS

In this section, the results of ARL0 and ARL1 for ARFIMAX(1,d,1) process, which are obtained from the explicit expression with numerical solution for the integral equation (IE) method with m=1,000, are compared. The results of ARL are expressed in Tables 1 to 3. The parameter value for in-control parameter is equal to 1 and the parameter value for out-of-control β_1=1.01, 1.02, 1.03, 1.04, 1.05, 1.06, 1.07, 1.08, 1.09, 1.10, 1.20, 1.30, 1.40, 1.50, 2, 3, 4, 5. Performance of the proposed explicit expression in terms of computational times and absolute percentage difference is considered.

$$Diff(\%) = \frac{\left|\tilde{L}(u) - L(u)\right|}{L(u)} \times 100\%.$$

Table 1. Comparison of ARL_0 and ARL_1 for EWMA control chart by explicit expression with numerical integral equation for ARFIMAX(1,0.3,1) process with $\phi_1 = 0.1$, $\theta_1 = 0.1$, and $\lambda = 0.01$

Parameter value of EWMA chart for ARL_0=370 b = 0.0162051 and for ARL_1=500 b = 0.01621058

β	ARL_0=370		Diff(%)	ARL_0=500		Diff(%)
	Explicit expression	IE (Time used: second)		Explicit expression	IE (Time used: second)	
1.00	370.229	370.229 (33.3)	0.00003	500.299	500.299 (32.6)	0.00004
1.01	77.8823	77.8823 (32.7)	0.00001	82.2747	82.2747 (32.2)	0.00002
1.02	44.0443	44.0443 (32.4)	0.00002	45.3825	45.3825 (32.7)	0.00007
1.03	30.9537	30.9537 (32.7)	0.00006	31.5934	31.5934 (32.8)	0.00006
1.04	24.0057	24.0057 (32.8)	0.00008	24.3799	24.3799 (33.1)	0.00008
1.05	19.6981	19.6981 (32.1)	0.00005	19.9438	19.9438 (32.9)	0.00005
1.06	16.7659	16.7659 (31.7)	0.00006	16.9397	16.9397 (32.4)	0.00006
1.07	14.6409	14.6409 (32.6)	0.00007	14.7704	14.7704 (32.3)	0.00007

Table 1. Comparison of ARL_0 and ARL_1 for EWMA control chart by explicit expression with numerical integral equation for ARFIMAX(1,0.3,1) process with $\phi_1 = 0.1$, $\theta_1 = 0.1$, and $\lambda = 0.01$ (continue)

Parameter value of EWMA chart for ARL_0=370 b = 0.0162051 and for ARL_1=500 b = 0.01621058						
β	ARL_0=370		Diff(%)	ARL_0=500		Diff(%)
	Explicit expression	IE (Time used: second)		Explicit expression	IE (Time used: second)	
1.08	13.0300	13.0300 (32,2)	0.00008	13.1303	13.1303 (32.1)	0.00008
1.09	11.7666	11.7666 (32.5)	0.00007	11.8467	11.8467 (31.8)	0.00004
1.10	10.7492	10.7492 (31.8)	0.00008	10.8147	10.8147 (32.5)	0.00006
1.20	6.09145	6.09145 (32.4)	0.00002	6.10890	6.1089 (32.6)	0.00003
1.30	4.50352	4.50352 (32.5)	0.00004	4.51169	4.51169 (32.3)	0.00007
1.40	3.69825	3.69825 (32.9)	0.00005	3.70335	3.70335 (32.7)	0.00005
1.50	3.20958	3.20958 (33.1)	0.00009	3.21282	3.21282 (32.8)	0.00006
2.00	2.20256	2.20256 (32.6)	0.00004	2.20358	2.20358 (31.9)	0.00005
3.00	1.65908	1.65908 (31.9)	0.00006	1.65945	1.65945 (32.1)	0.00006
4.00	1.46133	1.46133 (32.2)	0.00004	1.46155	1.46155 (32.2)	0.00003
5.00	1.35659	1.35659 (32.7)	0.00007	1.35675	1.35675 (31.7)	0.00005

Table 2. Comparison of ARL_0 and ARL_1 for EWMA control chart by explicit expression with numerical integral equation for ARFIMAX(1,0.3,1) process with $\phi_1 = 0.1$, $\theta_1 = -0.1$, and $\lambda = 0.01$

Parameter value of EWMA chart for ARL_0 =370 b = 0.01324593 and for ARL_0=500 b = 0.0132512						
β	ARL_0=370		Diff(%)	ARL_0=500		Diff(%)
	Explicit expression	IE (Time used: second)		Explicit expression	IE (Time used: second)	
1.00	370.076	370.076 (31.6)	0.00006	500.840	500.840 (32.3)	0.00004
1.01	65.4492	65.4492 (31.3)	0.00003	68.5295	68.5295 (32.7)	0.00006
1.02	36.4500	36.4500 (32.1)	0.00005	37.3610	37.3610 (32.4)	0.00005
1.03	25.5152	25.5152 (32.4)	0.00008	25.9469	25.9469 (31.8)	0.00008
1.04	19.7736	19.7736 (32.5)	0.00006	20.0255	20.0255 (31.9)	0.00005
1.05	16.2351	16.2351 (31.7)	0.00006	16.4004	16.4004 (32.1)	0.00007
1.06	13.8356	13.8356 (31.5)	0.00007	13.9525	13.9525 (33.6)	0.00008
1.07	12.1010	12.1010 (32.8)	0.00005	12.1883	12.1883 (32.8)	0.00003
1.08	10.7885	10.7885 (32.5)	0.00004	10.8563	10.8563 (32.5)	0.00007
1.09	9.76062	9.76062 (32.9)	0.00003	9.81485	9.81485 (32.4)	0.00002
1.10	8.93369	8.93369 (32.3)	0.00004	8.97812	8.97812 (33.1)	0.00003
1.20	5.15563	5.15563 (32.1)	0.00006	5.16779	5.16779 (33.2)	0.00006
1.30	3.86831	3.86831 (32.5)	0.00003	3.87415	3.87415 (32.9)	0.00005
1.40	3.21459	3.21459 (31.8)	0.00009	3.21811	3.21811 (32.5)	0.00003
1.50	2.81664	2.81664 (31.4)	0.00005	2.81905	2.81905 (32.1)	0.00004
2.00	1.99276	1.99276 (32.1)	0.00004	1.99357	1.99357 (31.8)	0.00005
3.00	1.54429	1.54429 (31.9)	0.00006	1.54461	1.54461 (32.2)	0.00008
4.00	1.38052	1.38052 (31.8)	0.00007	1.38071	1.38071 (32.3)	0.00007
5.00	1.29378	1.29378 (32.2)	0.00008	1.29393	1.29393 (32.6)	0.00006

Table 3. Comparison of ARL_0 and ARL_1 for EWMA control chart by explicit expression with numerical integral equation for ARFIMAX(1,0.3,1) process with $\phi_1 = -0.1$, $\theta_1 = 0.1$, and $\lambda = 0.01$

Parameter value of EWMA chart for ARL_0=370 b = 0.02236853 and for ARL_0=500 b = 0.02237498						
β	ARL_0=370		Diff(%)	ARL_0=500		Diff(%)
	Explicit expression	IE (Time used: second)		Explicit expression	IE (Time used: second)	
1.00	370.085	370.085 (31.4)	0.00005	500.224	500.224 (32.4)	0.00004
1.01	133.610	133.61 (30.7)	0.00008	147.216	147.216 (32.8)	0.00007
1.02	81.2722	81.2722 (31.2)	0.00006	86.0315	86.0315 (31.7)	0.00003
1.03	58.2966	58.2966 (31.7)	0.00003	60.6655	60.6655 (32.3)	0.00005
1.04	45.4027	45.4027 (30.5)	0.00004	46.8035	46.8035 (33.1)	0.00004
1.05	37.1582	37.1582 (32.1)	0.00005	38.0760	38.0760 (32.3)	0.00005
1.06	31.4385	31.4385 (30.4)	0.00003	32.0824	32.0824 (32.5)	0.00003
1.07	27.2419	27.2419 (31.6)	0.00007	27.7164	27.7164 (33.2)	0.00009
1.08	24.0344	24.0344 (31.9)	0.00008	24.3971	24.3971 (31.8)	0.00004
1.09	21.5054	21.5054 (31.3)	0.00005	21.7907	21.7907 (32.9)	0.00005
1.10	19.4616	19.4616 (32.2)	0.00007	19.6914	19.6914 (32.6)	0.00008
1.20	10.0995	10.0995 (31.8)	0.00006	10.1524	10.1524 (32.5)	0.00006
1.30	6.97878	6.97878 (31.2)	0.00003	7.00074	7.00074 (33.1)	0.00003
1.40	5.44556	5.44556 (32.5)	0.00004	5.45734	5.45734 (33.4)	0.00004
1.50	4.54225	4.54225 (32.1)	0.00007	4.54957	4.54957 (32.6)	0.00009
2.00	2.7909	2.79090 (31.8)	0.00004	2.7927	2.79270 (31.6)	0.00007
3.00	1.93752	1.93752 (31.4)	0.00008	1.93806	1.93806 (32.2)	0.00005
4.00	1.64682	1.64682 (30.7)	0.00002	1.64712	1.64712 (32.5)	0.00008
5.00	1.49685	1.49685 (30.5)	0.00003	1.49705	1.49705 (32.7)	0.00007

The results from Tables 1 to 3 show that these methods are in good agreement. Our analytical results agree with the numerical approximation per an absolute percentage difference of less than 0.01% for $m = 1000$ iterations and computational durations of approximately 30 seconds. The computational durations for the proposed analytical explicit expression are less than 1 second.

5. CONCLUSION

We derived explicit expression formulas for the ARL of EWMA control chart in the case of ARFIMAX (1,d,1) process with exponential white noise. We have shown that our formulas are very accurate as well as being easy to calculate and programmable. In addition, explicit expression consumes much less computational time compared to numerical integral equation.

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