# การประยุกต์ใช้การแปลงของซันด์แมนแบบวางนัยทั่วไปในการทำให้เป็นเชิงเส้นของระบบสมการเชิงอนุพันธ์สามัญอันดับสาม

# An Application of Generalized Sundman Transformations to Linearization of Systems of Third-order Ordinary Differential Equations

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## บทคัดย่อ

บทความวิจัยนี้ศึกษาปัญหาการทำให้เป็นเชิงเส้น สำหรับระบบสมการเชิงอนุพันธ์อันดับสาม

$$\begin{aligned} x''' &= F(t,x,y,x',y',x'',y''), \\ y''' &= G(t,x,y,x',y',x'',y'') \end{aligned}$$

โดยใช้การแปลงของซันด์แมนแบบวางนัยทั่วไป

$$u = f^x(t,x,y), \qquad v = f^y(t,x,y), \qquad d\tau = g(t,x,y)dt$$

เมื่อ $g \neq 0$ และ $f_x^x f_y^y - f_y^x f_x^y \neq 0$ จากผลการศึกษา ทำให้ได้เงื่อนไขจำเป็นและเพียงพอสำหรับระบบสมการข้างต้นที่สมมูลกับระบบสมการเชิงอนุพันธ์เชิงเส้นอันดับสาม

$$u''' = 0, \qquad v''' = 0$$

นอกจากนี้ ยังได้ผลเฉลยที่สมบูรณ์ สำหรับกรณี $f_y^x = 0,\ f_t^y = 0$ และ $f_x^y = 0$ ด้วย

## ABSTRACT

This paper is devoted to the study of the linearization problem of a system of two third-order ordinary differential equations

$$\begin{aligned} x''' &= F(t,x,y,x',y',x'',y''), \\ y''' &= G(t,x,y,x',y',x'',y'') \end{aligned}$$

by the generalized Sundman transformation

$$u = f^x(t,x,y), \qquad v = f^y(t,x,y), \qquad d\tau = g(t,x,y)dt,$$

where $g \neq 0$ and $f_x^x f_y^y - f_y^x f_x^y \neq 0$. In this paper, we obtain the necessary and sufficient conditions for the above system to be linearizable into the system of linear third-order ordinary differential equations

$$u''' = 0, \qquad v''' = 0.$$

Moreover, a complete solution is given for the case $f_y^x = 0,\ f_t^y = 0$, and $f_x^y = 0$.

**คำสำคัญ:** ปัญหาการทำให้เป็นเชิงเส้น การแปลงของซันด์แมนแบบวางนัยทั่วไป ระบบสมการเชิงอนุพันธ์สามัญอันดับสามไม่เชิงเส้น
**Keywords:** Linearization problem, Generalized Sundman transformations, System of nonlinear third-order ordinary differential equations

## INTRODUCTION

Differential equations have been used as an important tool to solve problems in science and physical phenomena. In general, these equations are very difficult to find solutions. One of the fundamental methods for solving them makes use of a change of variables that transform a given differential equation into another differential equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there arises the problem of transforming given differential equations into linear equations. This problem is called the **linearization problem**, which is a particular case of an equivalence problem. Transformations used for solving a linearization problem are point transformations, contact transformations, generalized Sundman transformations, and tangent transformations.

The generalized Sundman transformation was considered earlier for second-order ordinary differential equations by Duarte et al. (1994) using the Laguerre form. Nakpim and Meleshko (2010a) gave examples which show that the Laguerre form is not sufficient for the linearization problem via generalized Sundman transformations. Criteria for a third-order ordinary differential equation to be equivalent to the linear equation

$$u''' = 0$$

with respect to the generalized Sundman transformation were presented in Euler et al. (2003). Nakpim and Meleshko (2010b) obtained necessary and sufficient conditions for a third-order ordinary differential equation to be linearized into $u''' + \alpha u = 0,$ where $\alpha$ is a constant. Some applications of generalized Sundman transformations to ordinary differential equations were considered in Berkovich (2001) and earlier papers, which were summarized in Berkovich (2002). Srisuntorn and Nakpim (2016) studied the linearization problem of a third-order ordinary differential equation to be equivalent to the general form of a linear third-order ordinary differential equation $u''' + \beta u'' + \alpha u' + \gamma u = \eta$, where $\beta, \alpha, \gamma$, and $\eta$ are constants. A complete solution was only given for the case where $F_x = 0$.

The linearization problem for a system of two second-order ordinary differential equations via generalized Sundman transformations was studied by Moyo and Meleshko (2011). They obtained necessary and sufficient conditions for a system of two second-order ordinary differential equations

$$\begin{aligned} x'' &= F(t, x, y, x', y'), \\ y'' &= G(t, x, y, x', y') \end{aligned}$$

to be linearizable into a system of two linear second-order ordinary differential equations

$$u'' = k_{11}u' + k_{12}v', \qquad v'' = k_{21}u' + k_{22}v',$$

where $k_{ij}$ are constants for all $i, j \in \{1,2\}$.

In this paper, we find necessary and sufficient conditions which allow a system of two third-order ordinary differential equations

$$\begin{aligned} x''' &= F(t, x, y, x', y', x'', y''), \\ y''' &= G(t, x, y, x', y', x'', y'') \end{aligned}$$

to be transformed into the system of two linear third-order ordinary differential equations

$$u''' = 0, \qquad v''' = 0$$

via the generalized Sundman transformation

$$u = f^x(t,x,y),\ v = f^y(t,x,y),\ d\tau = g(t,x,y)dt$$

where $f^x, f^y,\ g$ are sufficiently many-time continuously differentiable functions. Furthermore, we present a complete solution for the case $f_y^x = 0,\ f_t^y = 0,$ and $f_x^y = 0$.

## GENERALIZED SUNDMAN TRANSFORMATIONS

A generalized Sundman transformation is a non-point transformation defined by the formulae

$$u = f^x(t,x,y),\ v = f^y(t,x,y),\ d\tau = g(t,x,y)dt \tag{1}$$

where $f^x, f^y,\ g$ are sufficiently many-time continuously differentiable functions. Here, it is assumed that $g \neq 0$ and $f_x^x f_y^y - f_y^x f_x^y \neq 0.$

Let us explain how a generalized Sundman transformation maps one function into another. Assume that $x_0(t), y_0(t)$ are given functions. Integrating the last equation of (1), that is,

$$\frac{d\tau}{dt} = g(t, x_0(t), y_0(t)),$$

we obtain $\tau = Q(t)$ for some function $Q(t)$. Using the inverse function theorem, we find that $t = Q^{-1}(\tau)$. Substituting $t$ into the functions $f^x(t, x_0(t), y_0(t))$ and $f^y(t, x_0(t), y_0(t))$, we obtain the transformed functions

$$u_0(\tau) = f^x\left(Q^{-1}(\tau), x_0\left(Q^{-1}(\tau)\right), y_0\left(Q^{-1}(\tau)\right)\right),$$
$$v_0(\tau) = f^y(Q^{-1}(\tau), x_0(Q^{-1}(\tau)), y_0(Q^{-1}(\tau))).$$

Conversely, let $u_0(\tau), v_0(\tau)$ be given functions of $\tau$. Using the inverse function theorem, we solve the equations

$$u_0(\tau) = f^x(t,x,y), \qquad v_0(\tau) = f^y(t,x,y)$$

with respect to $x$ and $y$, where $x = \phi(\tau,t)$ and $y = \psi(\tau,t)$ for some functions $\phi(\tau,t)$ and $\psi(\tau,t)$. Integrating the ordinary differential equation

$$\frac{d\tau}{dt} = g(t, \phi(\tau,t), \psi(\tau,t)),$$

we find that $\tau = H(t)$ for some function $H(t)$. Substituting $\tau = H(t)$ into the functions $\phi(\tau,t)$ and $\psi(\tau,t)$, the transformed functions $x_0(t) = \phi(H(t),t)$ and $y_0(t) = \psi(H(t),t)$ are obtained.

## NECESSARY CONDITIONS FOR LINEARIZATION

We start by obtaining necessary conditions for the linearization problem. First, we find the general form of a system of two third-order ordinary differential equations

$$\begin{aligned} x''' &= F(t,x,y,x',y',x'',y''),\\ y''' &= G(t,x,y,x',y',x'',y'') \end{aligned} \tag{2}$$

which can be mapped by the generalized Sundman transformation

$$u = f^x(t,x,y),\ v = f^y(t,x,y),\ d\,\tau = g(t,x,y)dt \tag{3}$$

into the system of two linear third-order ordinary differential equations

$$u''' = 0,\ v''' = 0. \tag{4}$$

The function $u, v$ and their derivatives $u', u'', u''', v', v''$, and $v'''$ are defined by equation (3) and their derivatives with respect to $t$. Substituting them into (4), we have

$$x''' + a_1 x'y'' + a_2 x''y' + a_3 x''x' + a_4 y''y' + a_5 x'' + a_6 y'' + a_7 x'^3 + a_8 y'^3 + a_9 x'^2 y'$$
$$+ a_{10} y'^2 x' + a_{11} x'^2 + a_{12} y'^2 + a_{13} x'y' + a_{14} x' + a_{15} y' + a_{16} = 0,$$
$$y''' + b_1 x'y'' + b_2 x''y' + b_3 x''x' + b_4 y''y' + b_5 x'' + b_6 y'' + b_7 x'^3 + b_8 y'^3 + b_9 x'^2 y'$$
$$+ b_{10} y'^2 x' + b_{11} x'^2 + b_{12} y'^2 + b_{13} x'y' + b_{14} x' + b_{15} y' + b_{16} = 0, \quad (5)$$

where the coefficients $a_i(x,y)$ and $b_i(x,y)$ $(\text{for } i = 1,2,3,\ldots,16)$ are related to the functions $f^x, f^y$, and $g$ in the following way:

$$a_1 = \frac{1}{gL}\left(3f^x_{xy}f^y_y g - f^x_x f^y_y g_y - 3f^x_y f^y_{xy} g + f^x_y f^y_x g_y\right), \quad (6)$$
$$a_2 = \frac{3}{gL}\left(f^x_{xy}f^y_y g - f^x_x f^y_y g_y - f^x_y f^y_{xy} g + f^x_y f^y_x g_y\right), \quad (7)$$
$$a_3 = \frac{1}{gL}\left(3f^x_{xx}f^y_y g - 4f^x_x f^y_y g_x - 3f^x_y f^y_{xx} g + 4f^x_y f^y_x g_x\right), \quad (8)$$
$$a_4 = \frac{3}{L}\left(f^x_{yy}f^y_y - f^x_y f^y_{yy}\right), \quad (9)$$
$$a_5 = \frac{1}{gL}\left(3f^x_{tx}f^y_y g - f^x_t f^y_y g_x - 3f^x_x f^y_y g_t - 3f^x_y f^y_{tx} g + f^x_y f^y_t g_x + 3f^x_y f^y_x g_t\right), \quad (10)$$
$$a_6 = \frac{1}{gL}\left(3f^x_{ty}f^y_y g - f^x_t f^y_y g_y - 3f^x_y f^y_{ty} g + f^x_y f^y_t g_y\right), \quad (11)$$
$$a_7 = \frac{1}{g^2 L}(f^x_{xxx}f^y_y g^2 - 3f^x_{xx}f^y_y g_x g - f^x_x f^y_y g_{xx} g + 3f^x_x f^y_y g_x^2 - f^x_y f^y_{xxx} g^2 + 3f^x_y f^y_{xx} g_x g$$
$$+ f^x_y f^y_x g_{xx} g - 3f^x_y f^y_x g_x^2), \quad (12)$$
$$a_8 = \frac{1}{gL}\left(f^x_{yyy}f^y_y g - 3f^x_{yy}f^y_y g_y - f^x_y f^y_{yyy} g + 3f^x_y f^y_{yy} g_y\right), \quad (13)$$
$$a_9 = \frac{1}{g^2 L}(-6f^x_{xy}f^y_y g_x g + 3f^x_{xxy}f^y_y g^2 - 3f^x_{xx}f^y_y g_y g - 2f^x_x f^y_y g_{xy} g + 6f^x_x f^y_y g_x g_y)$$
$$+ \frac{f^x_y}{g^2 L}(6f^y_{xy} g_x g - 3f^y_{xxy} g^2 + 3f^y_{xx} g_y g + 2f^y_x g_{xy} g - 6f^y_x g_x g_y), \quad (14)$$
$$a_{10} = \frac{1}{g^2 L}(3f^x_{xyy}f^y_y g^2 - 6f^x_{xy}f^y_y g_y g - f^x_x f^y_y g_{yy} g + 3f^x_x f^y_y g_y^2 - 3f^x_{yy}f^y_y g_x g)$$
$$+ \frac{f^x_y}{g^2 L}(-3f^y_{xyy} g^2 + 6f^y_{xy} g_y g + f^y_x g_{yy} g - 3f^y_x g_y^2 + 3f^y_{yy} g_x g), \quad (15)$$
$$a_{11} = \frac{1}{g^2 L}(3f^x_{txx}f^y_y g^2 - 6f^x_{tx}f^y_y g_x g - f^x_t f^y_y g_{xx} g + 3f^x_t f^y_y g_x g - f^x_t f^y_y g_{xx} g$$
$$+ 3f^x_t f^y_y g_x^2 - 3f^x_{xx}f^y_y g_t g - 2f^x_x f^y_y g_{tx} g + 6f^x_x f^y_{xy} g_t g_x) + \frac{f^x_y}{g^2 L}(-3f^y_{txx} g^2$$
$$+ 6f^y_{tx} g_x g + f^y_t g_{xx} g - 3f^y_t g_x^2 + 3f^y_{xx} g_t g + 2f^y_x g_{tx} g - 6f^y_x g_t g_x), \quad (16)$$
$$a_{12} = \frac{1}{g^2 L}(3f^x_{tyy}f^y_y g^2 - 6f^x_{ty}f^y_y g_y g - f^x_t f^y_y g_{yy} g + 3f^x_t f^y_y g_y^2 - 3f^x_{yy}f^y_y g_t g)$$
$$+ \frac{f^x_y}{g^2 L}(-3f^y_{tyy} g^2 + 6f^y_{ty} g_y g + f^y_t g_{yy} g - 3f^y_t g_y^2 + 3f^y_{yy} g_t g), \quad (17)$$
$$a_{13} = \frac{2}{g^2 L}(3f^x_{txy}f^y_y g^2 - 3f^x_{tx}f^y_y g_y g - 3f^x_{ty}f^y_y g_x g - f^x_t f^y_y g_{xy} g + 3f^x_t f^y_y g_x g_y$$
$$- 3f^x_{xy}f^y_y g_t g - f^x_x f^y_y g_{ty} g + 3f^x_x f^y_y g_t g_y) + \frac{f^x_y}{g^2 L}(-3f^y_{txy} g^2 + 3f^y_{tx} g_y g$$
$$+ 3f^y_{ty} g_x g + f^y_t g_{xy} g - 3f^y_t g_x g_y + 3f^y_{xy} g_t g + f^y_x g_{ty} g - 3f^y_x g_t g_y), \quad (18)$$
$$a_{14} = \frac{1}{g^2 L}(-6f^x_{tx}f^y_y g_t g + 3f^x_{ttx}f^y_y g^2 - 3f^x_{tt}f^y_y g_x g - 2f^x_t f^y_y g_{tx} g + 6f^x_t f^y_y g_t g_x$$
$$- f^x_x f^y_y g_{tt} g + 3f^x_x f^y_y g_t^2) + \frac{f^x_y}{g^2 L}(6f^y_{tx} g_t g - 3f^y_{ttx} g^2 + 3f^y_{tt} g_x g + 2f^y_t g_{tx} g$$
$$- 6f^y_t g_t g_x + f^y_x g_{tt} g - 3f^y_x g_t^2), \quad (19)$$
$$a_{15} = \frac{1}{g^2 L}(-6f^x_{ty}f^y_y g_t g + 3f^x_{tty}f^y_y g^2 - 3f^x_{tt}f^y_y g_y g - 2f^x_t f^y_y g_{ty} g + 6f^x_t f^y_y g_t g_y)$$
$$+ \frac{f^x_y}{g^2 L}(6f^y_{ty} g_t g - 3f^y_{tty} g^2 + 3f^y_{tt} g_y g + 2f^y_t g_{ty} g - 6f^y_t g_t g_y), \quad (20)$$

$$a_{16} = \frac{1}{g^2L}(f_{ttt}^x f_y^y g^2 - 3f_{tt}^x f_y^y g_t g - f_t^x f_y^y g_{tt} g + 3f_t^x f_y^y g_t^2 - f_y^x f_{ttt}^y g^2 + 3f_y^x f_{tt}^y g_t g$$
$$+f_y^x f_t^y g_{tt} g - 3f_y^x f_t^y g_t^2), \quad (21)$$
$$b_1 = \frac{3}{gL}\left(-f_{xy}^x f_x^y g - f_x^x f_{xy}^y g - f_x^x f_y^y g_x + f_y^x f_x^y g_x\right), \quad (22)$$
$$b_2 = \frac{1}{gL}\left(-3f_{xy}^x f_x^y g + 3f_x^x f_{xy}^y g - f_x^x f_y^y g_x + f_y^x f_x^y g_x\right), \quad (23)$$
$$b_3 = \frac{3}{L}\left(-f_{xx}^x f_x^y + f_x^x f_{xx}^y\right), \quad (24)$$
$$b_4 = \frac{1}{gL}\left(3f_x^x f_{yy}^y g - 4f_x^x f_y^y g_y - 3f_{yy}^x f_x^y g + 4f_y^x f_x^y g_y\right), \quad (25)$$
$$b_5 = \frac{1}{gL}\left(-3f_{tx}^x f_x^y g + f_t^x f_x^y g_x + 3f_x^x f_{tx}^y g - f_x^x f_t^y g_x\right), \quad (26)$$
$$b_6 = \frac{1}{gL}\left(-3f_{ty}^x f_x^y g + f_t^x f_x^y g_y + 3f_x^x f_{ty}^y g - f_x^x f_t^y g_y - 3f_x^x f_y^y g_t + 3f_y^x f_x^y g_t\right), \quad (27)$$
$$b_7 = \frac{1}{gL}(-f_{xxx}^x f_x^y g + 3f_{xx}^x f_y^y g_x - f_x^x f_{xxx}^y g - 3f_x^x f_{xx}^y g_x), \quad (28)$$
$$b_8 = \frac{1}{g^2L}(f_x^x f_{yyy}^y g^2 - 3f_x^x f_{yy}^y g_y g - f_x^x f_y^y g_{yy} g + 3f_x^x f_y^y g_y^2 - f_{yyy}^x f_x^y g^2 + 3f_{yy}^x f_x^y g_y g$$
$$+f_y^x f_x^y g_{yy} g - 3f_y^x f_x^y g_y^2), \quad (29)$$
$$b_9 = \frac{1}{g^2L}(6f_{xy}^x f_x^y g_x g - 3f_{xxy}^x f_x^y g^2 + 3f_{xx}^x f_y^y g_y g - 6f_x^x f_{xy}^y g_x g + 3f_x^x f_{xxy}^y g^2$$
$$-3f_x^x f_{xx}^y g_y g - f_x^x f_y^y g_{xx} g + 3f_x^x f_y^y g_x^2 + f_y^x f_x^y g_{xx} g - 3f_y^x f_x^y g_x^2), \quad (30)$$
$$b_{10} = \frac{1}{g^2L}(-3f_{xyy}^x f_x^y g^2 + 6f_{xy}^x f_x^y g_y g + 3f_x^x f_{xyy}^y - 6f_x^x f_{xy}^y g_y g - 3f_x^x f_{yy}^y g_x g$$
$$-2f_x^x f_y^y g_{xy} g + 6f_x^x f_y^y g_x g_y + 3f_{yy}^x f_x^y g_x g + 2f_y^x f_x^y g_{xy} g - 6f_y^x f_x^y g_x g_y), \quad (31)$$
$$b_{11} = \frac{1}{g^2L}(-3f_{txx}^x f_y^y g^2 + 6f_{tx}^x f_y^y g_x g + f_t^x f_x^y g_{xx} g + 3f_t^x f_y^y g_x^2 - 3f_t^x f_x^y g_x^2$$
$$+3f_{xx}^x f_x^y g_t g + 3f_x^x f_{txx}^y g^2 - 6f_x^x f_{tx}^y g_x g - f_x^x f_t^y g_{xx} g + 3f_x^x f_t^y g_x^2 - 3f_x^x f_{xx}^y g_t g), \quad (32)$$
$$b_{12} = \frac{1}{g^2L}(-3f_{tyy}^x f_x^y g^2 + 6f_{ty}^x f_x^y g_y g + f_t^x f_x^y g_{yy} g - 3f_t^x f_x^y g_y^2 + 3f_{yy}^x f_x^y g_t g$$
$$+2f_y^x f_x^y g_{ty} g - 6f_y^x f_x^y g_t g_y) + \frac{f_x^x}{g^2L}(3f_{tyy}^y g^2 - 6f_{ty}^y g_y g - f_t^y g_{yy} g + 3f_t^y g_y^2$$
$$-3f_{yy}^y g_t g - 2f_y^y g_{ty} g + 6f_y^y g_t g_y), \quad (33)$$
$$b_{13} = \frac{2}{g^2L}(-3f_{txy}^x f_x^y g^2 + 3f_{tx}^x f_x^y g_y g + 3f_{ty}^x f_x^y g_x g + f_t^x f_x^y g_{xy} g - 3f_t^x f_x^y g_x g_y$$
$$+3f_{xy}^x f_x^y g_t g + 3f_y^x f_{txy}^y g^2 + f_y^x f_x^y g_{tx} g - 3f_y^x f_x^y g_t g_x) + \frac{2f_x^x}{g^2L}(-3f_{tx}^y g_y g$$
$$-3f_{ty}^y g_x g - f_t^y g_{xy} g + 3f_t^y g_x g_y - 3f_{xy}^y g_t g - f_y^y g_{tx} g - 3f_y^y g_t g_x), \quad (34)$$
$$b_{14} = \frac{1}{g^2L}(6f_{tx}^x f_x^y g_t g - 3f_{ttx}^x f_x^y g^2 + 3f_{tt}^x f_x^y g_x g + 2f_t^x f_x^y g_{tx} g - 6f_t^x f_x^y g_t g_x$$
$$-6f_x^x f_{tx}^y g_t g + 3f_x^x f_{ttx}^y g^2 - 3f_x^x f_{tt}^y g_x g - 2f_x^x f_t^y g_{tx} g + 6f_x^x f_t^y g_t g_x), \quad (35)$$
$$b_{15} = \frac{1}{g^2L}(6f_{ty}^x f_{yx}^y g_t g - 3f_{tty}^x f_x^y g^2 + 3f_{tt}^x f_x^y g_y g + 2f_t^x f_x^y g_{ty} g - 6f_t^x f_x^y g_t g_y$$
$$-6f_x^x f_{ty}^y g_t g - f_y^x f_y^y g_{tt} g + 3f_x^x f_y^y g_t^2 + f_y^x f_x^y g_{tt} g - 3f_y^x f_x^y g_t^2), \quad (36)$$
$$b_{16} = \frac{1}{g^2L}(-f_{ttt}^x f_y^y g^2 + 3f_{tt}^x f_x^y g_t g + f_t^x f_x^y g_{tt} g - 3f_t^x f_x^y g_t^2 + f_x^x f_{ttt}^y g^2 - 3f_x^x f_{tt}^y g_t g$$
$$-f_x^x f_t^y g_{tt} g + 3f_x^x f_t^y g_t^2), \quad (37)$$

where $g \neq 0$ and $L = f_x^x f_y^y - f_y^x f_x^y \neq 0$.

Equations (5) present the necessary form of a system of two third-order ordinary differential equations which can be mapped into a system of two linear third-order ordinary differential equations (4) via a generalized Sundman transformation.

## SUFFICIENT CONDITIONS FOR LINEARIZATION

To obtain sufficient conditions, we have to solve the compatibility problem of the system of equations (6)-(37) by considering these equations as an overdetermined system of partial differential equations, where the coefficients $a_i(x,y)$ and $b_i(x,y)$ (for $i = 1,2,3,\dots,16$) are known functions, and $f^x, f^y, g$ are unknowns with the independent variables $t, x, y.$ Here, a complete solution for the case $f_y^x = 0,\ f_x^y = 0,$ and $f_t^y = 0$ will be determined.

From equations (6)-(10), (12)-(16), (19), (21), (22), (23), (24)-(36), and (37), we find that

$$g_y = -a_1\, g, \tag{38}$$

$$-3a_1 + a_2 = 0, \tag{39}$$

$$f_{xx}^x = \frac{f_x^x}{9}(3a_3 - 4b_1), \tag{40}$$

$$a_4 = 0, \tag{41}$$

$$f_{tx}^x = \frac{1}{9}(-f_t^x\, b_1 + 3\, f_x^x\, a_5 - f_x^x\, b_6), \tag{42}$$

$$g_{xx} = \frac{g}{27}(-3a_3b_1 + 5\, b_1^2\, g + 9\,\lambda_1), \tag{43}$$

$$a_8 = 0, \tag{44}$$

$$-3\, a_{3y} + 2\, b_{1y} - 3\, a_1a_3 + 3\, a_9 = 0, \tag{45}$$

$$-a_{1y} - 2a_1^2 + a_{10} = 0, \tag{46}$$

$$g_{tx} = \frac{g}{27\, f_x^x}(-f_t^x b_1^2 - 3f_x^x a_5 b_1 + 6\, f_x^x b_1 b_6 + 9\, f_x^x\,\lambda_2), \tag{47}$$

$$\begin{aligned} g_{tt} = \frac{g}{54\,(f_x^x)^2}(&-18\, f_{tt}^x f_x^x b_1 - f_t^{x2} b_1^2 - 6 f_t^x f_x^x b_1 b_6 + 9\, f_t^x f_x^x \lambda_2 + 9 f_x^{x2} b_6^2 \\ &+ 9\, f_x^{x2} g^2\,\lambda_3\,), \end{aligned} \tag{48}$$

$$\begin{aligned} f_{ttt}^x = \frac{1}{54\, f_x^{x2}}(&-18\, f_{tt}^x f_t^x f_x^x b_1 - 54\, f_{tt}^x f_x^{x2} b_6 - f_t^{x3} b_1^2 - 6 f_t^{x2} f_x^x b_1 b_6 \\ &+9\, f_t^{x2} f_x^x \lambda_2 - 9\, f_t^x f_x^{x2} b_6^2 + 9\, f_t^x f_t^{x2}\lambda_3 + 54\, f_x^{x3} a_{16}), \end{aligned} \tag{49}$$

$$g_x = \frac{-b_1\, g}{3}, \tag{50}$$

$$-b_1 + 3b_2 = 0, \tag{51}$$

$$b_3 = 0, \tag{52}$$

$$f_{yy}^y = \frac{f_y^y}{3}(-4a_1 + b_4), \tag{53}$$

$$b_5 = 0, \tag{54}$$

$$g_t = \frac{-b_6\, g}{3}, \tag{55}$$

$$b_7 = 0, \tag{56}$$

$$3a_{1y} - 3b_{4y} + 2a_1^2 - a_1b_4 - b_4^2 + 9\, b_8 = 0, \tag{57}$$

$$-3\, b_{1x} - 2\, b_1^2 + 9b_9 = 0, \tag{58}$$

$$-2\, b_{1y} - b_1b_4 + 3b_{10} = 0, \tag{59}$$

$$b_{11} = 0, \tag{60}$$

$$-2\, b_{6y} - 4b_1 b_6 + 9\, b_{13} = 0, \tag{61}$$

$$-6\, b_{6x} - 4b_1 b_6 + 9b_{13} = 0, \tag{62}$$

$$b_{14} = 0, \tag{63}$$

$$-3b_{6t} + 9b_{15} - 2b_6^2 = 0, \tag{64}$$

$$b_{16} = 0, \tag{65}$$

where

$$\begin{aligned} \lambda_1 &= -3a_{3x} - a_3^2 + 9a_7, \\ \lambda_2 &= -3a_{5x} + 3a_{11} - a_3 a_5, \\ \lambda_3 &= -3a_{5t} + 3\, a_{14} - a_5^2. \end{aligned}$$

Comparing the mixed derivatives

$$\begin{gathered} (g_y)_{xx} = (g_{xx})_y, (g_y)_{tx} = (g_{tx})_y, (g_y)_{tt} = (g_{tt})_y, (g_y)_x = (g_x)_y, (g_y)_t = (g_t)_y, \\ (g_{xx})_t = (g_{tx})_x, (g_{xx})_{tt} = (g_{tt})_{xx}, g_{xx} = (g_x)_x,\ (g_{xx})_t = (g_t)_{xx}, (g_{tx})_t = (g_{tt})_x, \\ g_{tx} = (g_x)_t, g_{tx} = (g_t)_x, (g_{tt})_x = (g_x)_{tt}, g_{tt} = (g_t)_t, (g_x)_t = (g_t)_x, \\ (f_{xx}^x)_t = (f_{tx}^x)_x, (f_{xx}^x)_{ttt} = (f_{ttt}^x)_{xx} \text{ and } (f_{tx}^x)_{tt} = (f_{ttt}^x)_x, \end{gathered}$$

we obtain the equations

$$\begin{aligned} &-27a_{1xx} + 18a_{1x}b_1 + 3a_{3y}b_1 + 9b_{1x}a_1 + 3b_{1y}a_3 - 10b_{1y}b_1 - 9\lambda_{1y} - 3\, a_1 a_3 b_1 \\ &+2a_1 b_1^2 + 9a_1\lambda_1 = 0, \end{aligned} \tag{66}$$

$$\begin{aligned} &3\, f_x^x\, g(-9a_{1tx} + 3a_{1t}b_1 + 3a_{1x}b_6 + a_{5y}b_1 + b_{1y}a_5 - 2\, b_{1y}b_6 + 3b_{6x}a_1 - 2b_{6y}b_1 - 3\lambda_{2y} \\ &-a_1 a_5 b_1 + a_1 b_1 b_6 + 3a_1\lambda_2) + f_t^x b_1\, g(2\, b_{1y} - a_1 b_1) = 0, \end{aligned} \tag{67}$$

$$\begin{aligned} &f_x^{x2}\left(-54\, a_{1tt} + 36\, a_{1t}b_6 + 2b_{1y}b_1 + 18b_{6t}a_1 - 18\, b_{6y}b_6 - 9\lambda_{3y} + 3\, a_1 b_6^2 + 9a_1\lambda_3\right) \\ &+f_x^x\left(18b_{1y}\, f_{tt}^x + 6b_{1y}f_t^x b_6 + 6b_{6y}f_t^x b_1 - 18f_{tt}^x a_1 b_1 - 9f_t^x\lambda_{2y} - 6f_t^x a_1 b_1 b_6 + 9f_t^x a_1\lambda_2\right) \\ &+2b_{1y}f_t^{x2}b_1 - f_t^{x2}a_1 b_1^2 = 0, \end{aligned} \tag{68}$$

$$-3a_{1x} + b_{1y} = 0, \tag{69}$$

$$-3a_{1t} + b_{6y} = 0, \tag{70}$$

$$\begin{aligned} &f_x^x(-9a_{3t}b_1 - 9b_{1t}a_3 + 30\, b_{1t}b_1 + 9\, b_{1x}a_5 - 18\, b_{1x}b_6 - 18\, b_{6x}b_1 + 27\,\lambda_{1t} - 27\lambda_{2x} \\ &+9a_{11}b_1 - 3a_3 a_5 b_1 + 3a_3 b_1 b_6 - 2a_5 b_1^2 + 6b_1\lambda_2 - 9b_6\lambda_1) + f_t^x b_1\,(6b_{1x} - a_3 b_1) = 0, \end{aligned} \tag{71}$$

$$\begin{aligned} &f_x^{x2}\,(54a_{14x}b_1 - 54\, a_{3tt}b_1 - 108a_{3t}b_{1t} + 36\, a_{3t}\, b_1 b_6 - 54\, b_{1tt}a_3 + 180\, b_{1tt}b_1 + 180\, b_{1t}^2 \\ &+36b_{1t}a_3 b_6 - 6b_{1t}a_5 b_1 - 114\, b_{1t}b_1 b_6 - 108b_{1x}b_{6t} + 108\, b_{1x}a_{14} - 36\, b_{1x}a_5 b_6 + 27b_{1x}b_6^2 \\ &-9\, b_{1x}\lambda_3 - 54\, b_{6tx}b_1 + 36\, b_{6t}a_3 b_1 - 12\, b_{6t}b_1^2 - 54\, b_{6t}\lambda_1 - 162\, b_{6xx}b_6 - 162b_{6x}^2 \\ &+90\, b_{6x}b_1 b_6 + 27\, b_{6x}\lambda_2 + 162\,\lambda_{1tt} - 108\,\lambda_{1t}b_6 - 54\,\lambda_{2x}a_5 + 54\lambda_{2x}b_6 - 81\,\lambda_{3xx} \\ &+36\lambda_{3x}b_1 - 18a_{11}b_1 b_6 - 27\, a_{11}\lambda_2 - 18\, a_{14}a_3 b_1 - 18\, a_{14}b_1^2 + 12\, a_3 a_5 b_1 b_6 + 18\, a_3 a_5\lambda_2 \\ &-6a_3 b_1 b_6^2 + 6\, a_3 b_1\lambda_3 - 9a_3 b_6\lambda_2 + 6a_5 b_1^2 b_6 + 9a_5 b_1\lambda_2 + b_1^2 b_6^2 - 3b_1^2\lambda_3 - 3b_1 b_6\lambda_2 \\ &+18\, b_6^2\lambda_1 + 9\,\lambda_2^2\,) + f_t^x f_x^x\,(-18\, b_{1tx}b_1 - 36\, b_{1t}b_{1x} + 12\, b_{1t}a_3 b_1 + 54b_{1xx}b_6 + 108\, b_{1x}b_{6x} \\ &-36\, b_{1x}a_3 b_6 + 6b_{1x}a_5 b_1 - 6b_{1x}b_1 b_6 + 54\, b_{6xx}b_1 - 36b_{6x}a_3 b_1 - 81\,\lambda_{2xx} + 54\,\lambda_{2x}a_3 \\ &+12\, a_3^2 b_1 b_6 - 18\, a_3^2\lambda_2 - 2\, a_3 a_5 b_1^2 + 2\, a_3 b_1^2 b_6 + 2\, a_5 b_1^3 - 54\, a_7 b_1 b_6 + 81\, a_7\lambda_2 - 2b_1^3 b_6 \\ &+6\, b_1 b_6\lambda_1 - 9\,\lambda_1\lambda_2) + f_{tt}^x\,(162b_{1xx}f_x^x - 108\, b_{1x}f_{tt}^x f_x^x a_3 + 36\, f_{tt}^x f_x^x a_3^2 b_1 - 162\, f_{tt}^x f_x^x a_7 b_1 \\ &+18\, f_{tt}^x f_x^x b_1\lambda_1\,) + f_t^{x2}\,(\,18\, b_{1xx}b_1 + 18\, b_{1x}^2 - 24\, b_{1x}a_3 b_1 + 15\, b_{1x}b_1^2 + 6a_3^2 b_1^2 - 4\, a_3 b_1^3 \\ &-18\, a_7 b_1^2 + b_1^4 + 2\, b_1^2\lambda_1) = 0, \end{aligned} \tag{72}$$

$$9b_{1x} - 3\, a_3 b_1 + 2b_1^2 + 9\,\lambda_1 = 0, \tag{73}$$

$$\begin{aligned} &-9a_{3t}b_1 - 9\, b_{1t}a_3 + 30\, b_{1t}b_1 - 9\, b_{1x}b_6 + 27\, b_{6xx} - 18b_{6x}b_1 + 27\,\lambda_{1t} \\ &+3\, a_3 b_1 b_6 - 2b_1^2 b_6 - 9\, b_6\lambda_1 = 0, \end{aligned} \tag{74}$$

$$\begin{aligned} &3f_t^x f_x^x(-18b_{1t}b_1 + 18\, b_{1x}b_6 + 18b_{6x}b_1 - 27\,\lambda_{2x} - 6a_3 b_1 b_6 + 9a_3\lambda_2 + 2\, a_5 b_1^2) \\ &+f_x^{x2}\,(-54b_{1t}a_5 + 108\, b_{1t}b_6 + 54\, b_{6t}b_1 - 162\, b_{6x}b_6 + 162\,\lambda_{2t} - 81\,\lambda_{3x} + 18a_5^2 b_1 \\ &-27\, a_5\lambda_2 - 9\, b_1 b_6^2 + 27\, b_1\lambda_3 - 27\, b_6\lambda_2) + (162\, b_{1x}f_{tt}^x f_x^x + 18b_{1x}f_t^{x2}b_1 - 54f_{tt}^x f_x^x a_3 b_1 \end{aligned}$$

$$-18\, f_{tt}^{x} f_{x}^{x} b_1^2 \;-6 f_t^{x2} a_3 b_1^2 \;+ f_t^{x2} b_1^3) \;=\; 0, \tag{75}$$

$$9 b_{1t} f_x^x \;- f_t^x b_1^2 \;-3\, f_x^x a_5 b_1 \;+3\, f_x^x b_1 b_6 \;+9\, f_x^x \lambda_2 \;=\; 0, \tag{76}$$

$$9 b_{6x} f_x^x \;- f_t^x b_1^2 \;-3\, f_x^x a_5 b_1 \;+3\, f_x^x b_1 b_6 \;+9\, f_x^x \lambda_2 \;=\; 0, \tag{77}$$

$$\begin{aligned}
&f_x^{x2}(54 b_{1tt} - 36 b_{1t} b_6 + 54\, b_{6x} b_6 \;+27\, \lambda_{3x} - 18 a_{14} b_1 + 6 a_5 b_1 b_6 + 9 a_5 \lambda_2 - 3 b_1 b_6^2 \;-3\, b_1 \lambda_3 \\
&-9\, b_6 \lambda_2) + (6\, b_{1t} f_t^x f_x^x b_1 \;-54\, b_{1x} f_{tt}^x f_x^x \;-6\, b_{1x} f_t^{x2} b_1 \;-18\, b_{1x} f_t^x f_x^x b_6 \;-18\, b_{6x} f_t^x\, f_x^x b_1 \\
&+18\, f_{tt}^x f_x^x a_3 b_1 \;+2 f_t^{x2} a_3 b_1^2 \;- f_t^{x2} b_1^3 \;+27\, f_t^x f_x^x \lambda_{2x} \;+6\, f_t^x f_x^x a_3 b_1 b_6 \;-9 f_t^x f_x^x a_3 \lambda_2) \;=\; 0,
\end{aligned} \tag{78}$$

$$18 b_{6t} f_x^{x2} \;-18 f_{tt}^x f_x^x b_1 \;- f_t^{x2} b_1^2 \;-6 f_t^x f_x^x b_1 b_6 \;+9 f_t^x f_x^x \lambda_2 \;+3 f_x^{x2} b_6^2 \;+9 f_x^{x2} \lambda_3 \;=\; 0, \tag{79}$$

$$-b_{1t} \;+\; b_{6x} \;=\; 0, \tag{80}$$

$$\begin{aligned}
&f_x^x\,(9\, a_{3t} \;-12 b_{1t} \;+9 b_{6x} -\; 9 a_{11} \;+3\, a_3 a_5 \;+ a_5 b_1 \;- b_1 b_6 \;+3 \lambda_2) \;+ f_t^x\,(3 b_{1x} \\
&- a_3 b_1 \;+ b_1^2) \;=\; 0,
\end{aligned} \tag{81}$$

$$\begin{aligned}
&f_x^{x3}\,(162 a_{14t} a_3 - 216\, a_{14t}\, b_1 + 486\, a_{14x}\, b_6 \;-1458\, a_{16xx} \;-972\, a_{16x}\, a_3 + 1296\, a_{16x} b_1 \\
&+486\, a_{3ttt} \;+486\, a_{3tt} a_5 \;-486\, a_{3tt} b_6 \;-486 a_{3t} b_{6t} \;+486 a_{3t} a_{14} - 324\, a_{3t} a_5 b_6 \\
&+162\, a_{3t} b_6^2 \;-162\, a_{3t} \lambda_3 \;-648\, b_{1ttt} \;-648 b_{1tt} a_5 + 648 b_{1tt} b_6 \;+648 b_{1t} b_{6t} \;-648\, b_{1t} a_{14} \\
&+378\, b_{1t} a_5 b_6 \;-162 b_{1t} b_6^2 \;+216\, b_{1t} \lambda_3 + 648\, b_{1x} a_{16} \;-486\, b_{6tx} b_6 \;-162\, b_{6tt} a_3 \\
&+216 b_{6tt} b_1 \;-972\, b_{6t} b_{6x} \;-162 b_{6t} a_3 a_5 \;+108\, b_{6t} a_5 b_1 \;+162\, b_{6t} b_1 b_6 \;+972 b_{6x} a_{14} \\
&-648\, b_{6x} a_5 b_6 \;+243\, b_{6x} b_6^2 \;-243\, b_{6x} \lambda_3 \;\;-54 \lambda_{3t} a_3 \;+72\, \lambda_{3t} b_1 \;-162\, \lambda_{3x} a_5 \;-243\, a_{11} b_6^2 \\
&-81 a_{11} \lambda_3 \;+54\, a_{14} a_3 a_5 \;+36\, a_{14} a_5 b_1 \;-162\, a_{14} b_1 b_6 \;+378\, a_{16} a_3 b_1 - 1458\, a_{16} a_7 \\
&-216\, a_{16} b_1^2 \;+162\, a_{16} \lambda_1 \;+54\, a_3 a_5 b_6^2 \;-18 a_3 a_5 \lambda_3 \;+9 a_3 b_6^3 \;+27\, a_3 b_6 \lambda_3 \;\;-54\, a_5^2 b_1 b_6 \\
&-54\, a_5^2 \lambda_2 \;+135\, a_5 b_1 b_6^2 + 33\, a_5 b_1 \lambda_3 \;+108\, a_5 b_6 \lambda_2 \;-39\, b_1 b_6^3 \;+9\, b_1 b_6 \lambda_3 \;+27\, b_6^2 \lambda_2 \\
&+27\, \lambda_2 \lambda_3\,) + f_x^x\,(f_t^x\,(162 a_{14x} b_1 \;-162\, a_{3tt} b_1 \;-162\, a_{3t} b_{1t} \;\;-54 a_{3t} a_5 b_1 \;\;+54\, a_{3t} b_1 b_6 \\
&-162\, b_{1tx} b_6 - 54 b_{1tt} a_3 \;+288 b_{1tt} b_1 \;+216\, b_{1t}^2 \;-324 b_{1t} b_{6x} \;-18 b_{1t} a_3 a_5 \;+18 b_{1t} a_3 b_6 \\
&+42 b_{1t} a_5 b_1 \;-6 b_{1t} b_1 b_6 \;-324 b_{1x} b_{6t} \;\;+324\, b_{1t} a_{14} \;-54 b_{1x} a_5 b_6 \;-81\, b_{1x} b_6^2 \;-81\, b_{1x} \lambda_3 \\
&-162\, b_{6tx} b_1 + 90 b_{6t} a_3 b_1 \;-66 b_{6t} b_1^2 \;+486\, b_{6xx} b_6 \;+486\, b_{6x}^2 \;-162\, b_{6x} b_1 b_6 \;+162\, b_{6x} \lambda_2 \\
&-324 \lambda_{2x} a_5 \;+324 \lambda_{2x} b_6 \;-243\, \lambda_{3xx} \;-54 a_{11} b_1 b_6 \;-162\, a_{11} \lambda_2 \;-90 a_{14} a_3 b_1 \;+66 a_{14} b_1^2 \\
&+6 a_3 a_5^2 b_1 \;+30 a_3 a_5 b_1 b_6 \;+108\, a_3 a_5 \lambda_2 \;+21\, a_3 b_1 b_6^2 \;+21\, a_3 b_1 \lambda_3 \;-54 a_3 b_6 \lambda_2 \;-8 a_5^2 b_1^2 \\
&-4 a_5 b_1^2 b_6 \;-\; 18\, a_5 b_1 \lambda_2 \;+19 b_1^2 b_6^2 \;-13 b_1^2 \lambda_3 \;+36\, b_1 b_6 \lambda_2 \;+54\, \lambda_2^2) + (-162\, a_{3t} f_{tt}^x\;\, b_1 \\
&-108 b_{1t} f_{tt}^x a_3 \;+360\, b_{1t} f_{tt}^x b_1 \;+324\, b_{1x} f_{tt}^x a_5 \;-486\, b_{1x} f_{tt}^x b_6 \;+1458 b_{6xx} f_{tt}^x \\
&-486 b_{6x} f_{tt}^x b_1 \;+162\, f_{tt}^x a_{11} b_1 \;-126\, f_{tt}^x a_3 a_5 b_1 \;+126\, f_{tt}^x a_3 b_1 b_6 \;+42\, f_{tt}^x a_5 b_1^2 \\
&-96 f_{tt}^x b_1^2 b_6 - 54 f_{tt}^x b_1 \lambda_2\;\,)) +\; f_t^{x3}\,(54 b_1 + 54 b_{1x}^2 \;-72 b_{1x} a_3 b_1 \;+75 b_{1x} b_1^2 \;+18 a_3^2 b_1^2 \\
&-19\, a_3 b_1^3 \;-54 a_7 b_1^2 \;\;+7 b_1^4 \;+6 b_1^2 \lambda_1) \;+ f_t^{x2}\, f_x^x\,(-54 b_{1tx}\;\, b_1 \;-108 b_{1t} b_{1x} \;-36\, b_{1t} b_3 b_1 \\
&-24 b_{1t} b_1^2 \;+162\, b_{1xx} b_6 \;-108\, b_{1x} a_3 b_6 \;+54 b_{1x} a_5 b_1 \;+54\, b_{1x} b_1 b_6 \;-54\, b_{1x\lambda_2} \\
&+162\, b_{6xx} b_1 \;-108\, b_{6x} a_3 b_1 +\; 63 b_{6x} b_1^2 \;-243\, \lambda_{2xx} \;+162\, \lambda_{2x} a_3 \;-108\, \lambda_{2x} b_1 \;+9\, a_{11} b_1^2 \\
&+36\, a_3^2 b_1 b_6 \;-54\, a_3^2 \lambda_2 \;-18\, a_3 a_5 b_1^2 \;-3\, a_3 b_1^2 b_6 \;+27\, a_3 b_1 \lambda_2 \;+13\, a_5 b_1^3 \;-162\, a_7 b_1 b_6 \\
&+243\, a_7 \lambda_2 \;\;-13\, b_1^3 b_6 \;\;-3\, b_1^2 \lambda_2 \;\;+18\, b_1 b_6 \lambda_1 \;-27\, \lambda_1 \lambda_2) + f_{tt}^x f_t^x f_x^x\,(486 b_{1xx} \\
&-324\, b_{1x} a_3 \;+324\, b_{1x} b_1 \;+108\, a_3^2 b_1 \;-54\, a_3 b_1^2 \;-468\, a_7 b_1 \;+54\, b_1 \lambda_1\;\,) \;=\; 0,
\end{aligned} \tag{82}$$

$$\begin{aligned}
&f_x^{x3}\,(54 a_{14t} \;-162 a_{16x} \;-54 b_{tt} \;-54\, b_{6t} a_5 \;-18 \lambda_{3t} \;+18 a_{14} a_5 \;-54\, a_{16} a_3 \;+54\, a_{16} b_1 \\
&-9 a_5 b_6^2 \;-15 a_5 \lambda_3 \;+3 b_6^3 \;+9 b_6 \lambda_3) + f_x^{x2}\,(-18 b_{1tt} f_t^x \;-36\, b_{1t} f_{tt}^x \;- b_{1t} f_t^x a_5 \;-12\, b_{1t} f_t^x b_6 \\
&-6 b_{6t} f_t^x b_1 \;+162\, b_{6x} f_{tt}^x \;+54\, b_{6x} f_t^x b_6 \;+12\, f_{tt}^x a_5 b_1 \;-12\, f_{tt}^x b_1 b_6 \;-27 f_t^x \lambda_{3x} \;+6\, f_t^x a_{14} b_1 \\
&+2 f_t^x a_5^2 b_1 - 2\, f_t^x a_5 b_1 b_6 - 18\, f_t^x a_5 \lambda_2 - 2 f_t^x b_1 b_6^2 - 2 f_t^x b_1 \lambda_3 + 18 f_t^x b_6 \lambda_2) + (-6 b_{1t} f_t^{x2} f_x^x b_1 \\
&+54 b_{1x} f_{tt}^x f_t^x f_x^x \;+6\, b_{1x} f_t^{x3} b_1 \;+18\, b_{1x} f_t^{x2} f_x^x b_6 \;+18\, b_{6x} f_t^{x2} f_x^x b_1 \;-18\, f_{tt}^x f_t^x f_x^x a_3 b_1 \\
&+18\, f_{tt}^x f_t^x f_x^x b_1^2 \;-2 f_t^{x3} a_3 b_1^2 \;+2 f_t^{x3} b_1^3 \;-27 f_t^{x2} f_x^x \lambda_{2x} \;-6 f_t^{x2} f_x^x a_3 b_1 b_6 \\
&+9 f_t^{x2} f_x^x a_3 \lambda_{2t} f_t^{x2} f_x^x a_3 b_1^2 \;+5 f_t^{x2} f_x^x b_1^2 b_6 \;-9 f_t^{x2} f_x^x b_1 \lambda_2) \;=\; 0.
\end{aligned} \tag{83}$$

Recall (11) and consider the equation

$$-f_t^x\, a_1 \;+ f_x^x\, a_6 \;=\; 0. \tag{84}$$

Further analysis of the compatibility depends on $a_1$.

**Case $\boldsymbol{a_1 \neq 0}$.**

From (84), we can find the derivatives

$$f_t^x = f_x^x \lambda_4, \tag{85}$$

where $\lambda_4 = \frac{a_6}{a_1}$.

Comparing the mixed derivatives $(f_t^x)_{xx} = (f_{xx}^x)_t, (f_t^x)_x = f_{tx}^x$, and $(f_t^x)_{tt} = f_{ttt}^x$, we obtain the conditions

$$-9a_{3t} + 12b_{1t} - 12b_{1x}\lambda_4 + 27\lambda_{4xx} + 18\lambda_{4x}a_3 - 24\lambda_{4x}b_1 - 3a_3a_5 - 7a_3b_1\lambda_4 + 3a_3b_6$$
$$+4a_5b_1 + 27a_7\lambda_4 + 4b_1^2\lambda_4 - 4b_1b_6 - 3\lambda_1\lambda_4 = 0, \tag{86}$$

$$3\lambda_{4x} + a_3\lambda_4 - a_5b_1\lambda_4 + b_6 = 0, \tag{87}$$

$$-18b_{1t}\lambda_4^2 - 54b_{6t}\lambda_4 + 162\lambda_{4tt} + 108\lambda_{4t}a_5 + 54\lambda_{4t}b_6 + 54a_{14}\lambda_4 - 162a_{16} + 6a_5b_1\lambda_4^2$$
$$+18a_5b_6\lambda_4 - b_1^2\lambda_4^3 - 6b_1b_6\lambda_4^2 - 9b_6^2\lambda_4 - 27\lambda_2\lambda_4^2 - 45\lambda_3\lambda_4 = 0. \tag{88}$$

Equations (67), (68), (71), (72), (75)-(79), (81), (82), and (83) become

$$-27a_{1tx} + 9a_{1t}b_1 + 9a_{1x}b_6 + 3a_{5y}b_1 + 3b_{1y}a_5 + 2b_{1y}b_1\lambda_4 - 6b_{1y}b_6 + 9b_{6x}a_1 - 6b_{6y}b_1$$
$$-9\lambda_{2y} - 3a_1a_5b_1 - a_1b_1^2\lambda_4 + 3a_1b_1b_6 + 9a_1\lambda_2 = 0, \tag{89}$$

$$-54a_{1tt} + 36a_{1t}b_6 + 18b_{1y}\lambda_{4t} + 6b_{1y}a_5\lambda_4 + 18b_{6t}a_1 + 6b_{6y}b_1\lambda_4 - 18b_{6y}b_6 - 9\lambda_{2y}\lambda_4$$
$$-9\lambda_{3y} - 18\lambda_{4t}a_1b_1 - 6a_1a_5b_1\lambda_4 + a_1b_1^2\lambda_4^2 + 3a_1b_6^2 + 9a_1\lambda_2\lambda_4 + 9a_1\lambda_3 = 0, \tag{90}$$

$$-9a_{3t}b_1 - 9b_{1t}a_3 + 30b_{1t}b_1 + 9b_{1x}a_5 + 6b_{1x}b_1\lambda_4 - 18b_{1x}b_6 - 18b_{6x}b_1 + 27\lambda_{1t} - 27\lambda_{2t}$$
$$+9a_{11}b_1 - 3a_3a_5b_1 - a_3b_1^2\lambda_4 + 3a_3b_1b_6 - 2a_5b_1^2 + 6b_1\lambda_2 - 9b_6\lambda_1 = 0, \tag{91}$$

$$54a_{14x}b_1 - 54a_{3tt}b_1 - 108a_{3t}b_{1t} + 36a_{3t}b_1b_6 - 18b_{1tx}b_1\lambda_4 - 54b_{1tt}a_3 + 180b_{1tt}b_1$$
$$+180b_{1t}^2 - 36b_{1t}b_{1x}\lambda_4 + 12b_{1t}a_3b_1\lambda_4 + 36b_{1t}a_3b_6 - 6b_{1t}a_5b_1 - 114b_{1t}b_1b_6$$
$$+162b_{1xx}\lambda_{4t} + 54b_{1xx}a_5\lambda_4 + 18b_{1x}^2\lambda_4^2 - 108b_{1x}b_{6t} + 108b_{1x}b_{6x}\lambda_4 - 108b_{1x}\lambda_{4t}b_3$$
$$+108b_{1x}a_{14} - 36b_{1x}a_3a_5\lambda_4 - 12b_{1x}a_3b_1\lambda_4^2 + 6b_{1x}a_5b_1\lambda_4 - 36b_{1x}a_5b_6 + 15b_{1x}b_1^2\lambda_4^2$$
$$-6b_{1x}b_1b_6\lambda_4 + 27b_{1x}b_6^2 - 9b_{1x}\lambda_3 - 54b_{6tx}b_1 + 36b_{6t}a_3b_1 - 12b_{6t}b_1^2 - 54b_{6t}\lambda_1$$
$$+54b_{6xx}b_1\lambda_4 - 162b_{6xx}b_6 - 162b_{6x}^2 - 36b_{6x}a_3b_1\lambda_4 + 90b_{6x}b_1b_6 + 27b_{6x}\lambda_2 + 162\lambda_{1tt}$$
$$-108\lambda_{1t}b_6 - 81\lambda_{2xx}\lambda_4 + 54\lambda_{2x}a_3\lambda_4 - 54\lambda_{2x}a_5 + 54\lambda_{2x}b_6 - 81\lambda_{3xx} + 36\lambda_{3x}b_1$$
$$+36\lambda_{4t}a_3^2b_1 - 162\lambda_{4t}a_7b_1 + 18\lambda_{4t}b_1\lambda_1 - 18a_{11}b_1b_6 - 27a_{11}\lambda_2 - 18a_{14}a_3b_1 - 18a_{14}b_1^2$$
$$+12a_3^2a_5b_1\lambda_4 + 2a_3^2b_1^2\lambda_4^2 - 18a_3^2\lambda_2\lambda_4 - 2a_3a_5b_1^2\lambda_4 + 12a_3a_5b_1b_6 + 18a_3a_5\lambda_2 - 4a_3b_1^3\lambda_4^2$$
$$+2a_3b_1^2b_6\lambda_4 - 6a_3b_1b_6^2 + 6a_3b_1\lambda_3 - 9a_3b_6\lambda_2 - 54a_5a_7b_1\lambda_4 + 2a_5b_1^3\lambda_4 + 6a_5b_1^2b_6$$
$$+6a_5b_1\lambda_1\lambda_4 + 9a_5b_1\lambda_2 + 81a_7\lambda_2\lambda_4b_1^4\lambda_4^2 - 2b_1^3b_6\lambda_4 + b_1^2b_6^2 - 3b_1^2\lambda_3 - 3b_1b_6\lambda_2 + 18b_6^2\lambda_1$$
$$-9\lambda_1\lambda_2\lambda_4 + 9\lambda_2^2 = 0, \tag{92}$$

$$-18b_{1t}a_5 - 18b_{1t}b_1\lambda_4 + 36b_{1t}b_6 + 54b_{1x}\lambda_{4t} + 18b_{1x}a_5\lambda_4 + 18b_{6t}b_1 + 18b_{6x}b_1\lambda_4$$
$$-54b_{6x}b_6 + 54\lambda_{2t} - 27\lambda_{2x}\lambda_4 - 27\lambda_{3x} - 18\lambda_{4t}a_3b_1 - 6\lambda_{4t}b_1^2 - 6a_3a_5b_1\lambda_4 + 9a_3\lambda_2\lambda_4$$
$$+6a_5^2b_1 - 9a_5\lambda_2 + b_1^3\lambda_4^2 + 2b_1^2b_6\lambda_4 - 3b_1b_6^2 + 9b_1\lambda_3 - 9b_6\lambda_2 = 0, \tag{93}$$

$$9b_{1t} - 3a_5b_1 - b_1^2\lambda_4 + 3b_1b_6 + 9\lambda_2 = 0, \tag{94}$$

$$9b_{6x} - 3a_5b_1 - b_1^2\lambda_4 + 3b_1b_6 + 9\lambda_2 = 0, \tag{95}$$

$$54b_{1tt} + 6b_{1t}b_1\lambda_4 - 36b_{1t}b_6 - 54b_{1x}\lambda_{4t} - 18b_{1x}a_5\lambda_4 - 18b_{6x}b_1\lambda_4 + 54b_{6x}b_6 + 27\lambda_{2x}\lambda_4$$
$$+27\lambda_{3x} + 18\lambda_{4t}a_3b_1 - 18a_{14}b_1 + 6a_3a_5b_1\lambda_4 - 9a_3\lambda_2\lambda_4 + 6a_5b_1b_6 + 9a_5\lambda_2 - b_1^3\lambda_4^2 - 3b_1b_6^2$$
$$-3b_1\lambda_3 - 9b_6\lambda_2 = 0, \tag{96}$$

$$18b_{6t} - 18\lambda_{4t}b_1 - 6a_5b_1\lambda_4 + b_1^2\lambda_4^2 + 3b_6^2 + 9\lambda_2\lambda_4 + 9\lambda_3 = 0, \tag{97}$$

$$9a_{3t} - 12b_{1t} + 3b_{1x}\lambda_4 + 9b_{6x} - 9a_{11} + 3a_3a_5 - a_3b_1\lambda_4 + a_5b_1 + b_1^2\lambda_4 - b_1b_6 + 3\lambda_2 = 0, \tag{98}$$

$$
\begin{aligned}
&486a_{14t}a_3-648a_{14t}b_1+486\,a_{14x}b_1\lambda_4+1458a_{14xb_6}-4374a_{16xx}-2916a_{16x}a_3+3888a_{16x}b_1\\
&+1458\,a_{3ttt}+1458a_{3tt}a_5-486a_{3tt}b_1\lambda_4-1458a_{3tt}b_6-486\,a_{3t}b_{1t}\lambda_4-1458\,a_{3t}b_{6t}\\
&-486\,a_{3t}\lambda_{4t}b_1+1458a_{3t}a_{14}-324\,a_{3t}a_5b_1\lambda_4-972\,a_{3t}a_5b_6+54\,a_{3t}b_1^2\lambda_4^2+324\,a_{3t}b_1b_6\lambda_4\\
&+486\,a_{3t}b_6^3-486\,a_{3t}\lambda_3-162\,b_{1tx}b_1\lambda_4^2-486b_{1tx}b_6\lambda_4-1944b_{1ttt}-162\,b_{1tt}a_3\lambda_4\\
&-1944\,b_{1tt}a_5+864\,b_{1tt}b_1\lambda_4+1944\,b_{1tt}b_6+648\,b_{1t}^2\lambda_4-324\,b_{1t}b_{1x}\lambda_4^2+1944b_{1t}b_{6t}\\
&-972\,b_{1t}b_{6x}\lambda_4-324b_{1t}\lambda_{4t}a_3+1080\,b_{1t}\lambda_{4t}b_1-1944\,b_{1t}a_{14}-162\,b_{1t}a_3a_5\lambda_4+144\,b_{1t}a_3b_1\lambda_4^2\\
&+162\,b_{1t}a_3b_6\lambda_4+486b_{1t}a_5b_1\lambda_4+1134\,b_{1t}a_5b_6-192\,b_{1t}b_1^2\lambda_4^2-378\,b_{1t}b_1b_6\lambda_4-486\,b_{1t}b_6^2\\
&+648b_{1t}\lambda_3+1458\,b_{1xx}\lambda_{4t}\lambda_4+486b_{1xx}a_5\lambda_4^2+162\,b_{1x}^2\lambda_4^3-972\,b_{1x}b_{6t}\lambda_4+972\,b_{1x}b_{6x}\lambda_4^2\\
&-972\,b_{1x}\lambda_{4t}a_3\lambda_4+972\,b_{1x}\lambda_{4t}a_5+972\,b_{1x}\lambda_{4t}b_1\lambda_4-1458\,b_{1x}\lambda_{4t}b_6+972\,b_{1x}a_{14}\lambda_4\\
&+1944\,b_{1x}a_{16}-324\,b_{1x}a_3a_5\lambda_4^2-108\,b_{1x}a_3b_1\lambda_4^3+324b_{1x}a_5^2\lambda_4+378\,b_{1x}a_5b_1\lambda_4^2\\
&-972b_{1x}a_5b_6\lambda_4+117b_{1x}b_1^2\lambda_4^3+243\,b_{1x}b_6^2\lambda_4-162\,b_{1x}\lambda_2\lambda_4^2-243\,b_{1x}\lambda_3\lambda_4-486b_{6tx}b_1\lambda_4\\
&-1458\,b_{6tx}b_6-486b_{6tt}a_3+648\,b_{6tt}b_1-2916b_{6t}b_{6x}-486\,b_{6t}a_3a_5+270\,b_{6t}a_3b_1\lambda_4\\
&+324\,b_{6t}a_5b_1-198\,b_{6t}b_1^2\lambda_4+486b_{6t}b_1b_6+4374\,b_{6xx}\lambda_{4t}+1458\,b_{6xx}a_5\lambda_4+1458\,b_{6x}^2\lambda_4\\
&-1458b_{6x}\lambda_{4t}b_1+2916\,b_{6x}a_{14}-324\,b_{6x}a_3b_1\lambda_4^2-486b_{6x}a_5b_1\lambda_4-1944\,b_{6x}a_5b_6\\
&+351\,b_{6x}b_1^2\lambda_4^2+729\,b_{6x}b_6^2+486\,b_{6x}\lambda_2\lambda_4-729\,b_{6x}\lambda_3-729\,\lambda_{2xx}\lambda_4^2+486\lambda_{2x}a_3\lambda_4^2\\
&-972\,\lambda_{2x}a_5\lambda_4-324\,\lambda_{2x}b_1\lambda_4^2+972\,\lambda_{2x}b_6\lambda_4-162\,\lambda_{3t}a_3+216\,\lambda_{3t}b_1-729\,\lambda_{3xx}\lambda_4\\
&-486\,\lambda_{3x}a_5+486\,\lambda_{4t}a_{11}b_1+324\,\lambda_{4t}a_3^2b_1\lambda_4-378\,\lambda_{4t}a_3a_5b_1-162\,\lambda_{4t}a_3b_1^2\lambda_4\\
&+378\lambda_{4t}a_3b_1b_6+126\,\lambda_{4t}a_5b_1^2-1458\,\lambda_{4t}a_7b_1\lambda_4-288\,\lambda_{4t}b_1^2b_6+162\,\lambda_{4t}b_1\lambda_1\lambda_4\\
&-162\,\lambda_{4t}b_1\lambda_2+162\,a_{11}a_5b_1\lambda_4-27a_{11b_1^2\lambda_4^2}-324\,a_{11}b_1b_6\lambda_4-729\,a_{11}b_6^2-486a_{11}\lambda_2\lambda_4\\
&-243\,a_{11}\lambda_3+162\,a_{14}a_3a_5-270\,a_{14}a_3b_1\lambda_4+108\,a_{14}a_5b_1+198\,a_{14}b_1^2\lambda_4-486\,a_{14}b_1b_6\\
&+1134\,a_{16}a_3b_1-4374\,a_{16}a_7-648\,a_{16}b_1^2+486\,a_{16}\lambda_1+108a_3^2a_5b_1\lambda_4^2+18\,a_3^2b_1^2\lambda_4^3\\
&-162\,a_3^2\lambda_2\lambda_4^2-108a_3a_5^2b_1\lambda_4-66\,a_3a_5b_1^2\lambda_4^2+342\,a_3a_5b_1b_6\lambda_4+162\,a_3a_5b_6^2+324\,a_3a_5\lambda_2\lambda_4\\
&-54\,a_3a_5\lambda_3-39\,a_3b_1^3\lambda_4^3+3\,a_3b_1^2b_6\lambda_4^2-63\,a_3b_1b_6^2\lambda_4+81\,a_3b_1\lambda_2\lambda_4^2+63\,a_3b_1\lambda_3\lambda_4+27\,a_3b_6^3\\
&-162\,a_3b_6\lambda_2\lambda_4+81\,a_3b_6\lambda_3+18\,a_5^2b_1^2\lambda_4-162\,a_5^2b_1b_6-162\,a_5^2\lambda_2-486\,a_5a_7b_1\lambda_4^2\\
&+25\,a_5b_1^3\lambda_4^2-150\,a_5b_1^2b_6\lambda_4+405\,a_5b_1b_6^2+54a_5b_1\lambda_1\lambda_4^2-108\,a_5b_1\lambda_2\lambda_4+99a_5b_1\lambda_3\\
&+324\,a_5b_6\lambda_2+729\,a_7\lambda_2\lambda_4^2+21\,b_1^4\lambda_4^3-7b_1^3b_6\lambda_4^2+39\,b_1^2b_6^2\lambda_4+9\,b_1^2\lambda_2\lambda_4^2-39\,\lambda_1^2\lambda_3\lambda_4\\
&-117b_1b_6^3+162\,b_1b_6\lambda_2\lambda_4+27\,b_1b_6\lambda_3+81b_6^2\lambda_2-81\,\lambda_1\lambda_2\lambda_4^2+162\,\lambda_2^2\lambda_4+81\,\lambda_2\lambda_3=0,
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
&162a_{14t}-486a_{16x}-54b_{1tt}\lambda_4-108b_{1t}\lambda_{4t}-54b_{1t}a_5\lambda_4-6\,b_{1t}b_1\lambda_4^2+162\,b_{1x}\lambda_{4t}\lambda_4\\
&+54b_{1x}a_5\lambda_4^2-162\,b_{6tt}-162\,b_{6t}a_5-18\,b_{6t}b_1\lambda_4+486\,b_{6x}\lambda_{4t}+162\,b_{6x}a_5\lambda_4-81\,\lambda_{2x}\lambda_4^2\\
&-54\,\lambda_{3t}-81\,\lambda_{3x}\lambda_4-54\lambda_{4t}a_3b_1\lambda_4+36\,\lambda_{4t}a_5b_1+54\,\lambda_{4t}b_1^2\lambda_4-36\,\lambda_{4t}b_1b_6+54\,a_{14}a_5\\
&+18a_{14}b_1\lambda_4-162\,a_{16}a_3-162\,a_{16}b_1-18a_3a_5b_1\lambda_4^2+27\,a_3\lambda_2\lambda_4^2+18a_5^2b_1\lambda_4+17\,a_5b_1^2\lambda_4^2\\
&-30a_5b_1b_6\lambda_4-27\,a_5b_6^2-54\,a_5\lambda_2\lambda_4-45a_5\lambda_3+b_1^2b_6\lambda_4^2+6\,b_1b_6^2\lambda_4-27\,b_1\lambda_2\lambda_4^2-6b_1\lambda_3\lambda_4\\
&+9b_6^3+54b_6\lambda_2\lambda_4+27b_6\lambda_3=0.
\end{aligned}
\tag{100}
$$

**Case $\boldsymbol{a_1=0}$.**

From (16), we obtain

$$-f_t^x\,b_1^2+f_x^x\,\lambda_5=0, \tag{101}$$

where $\lambda_5=3(3b_{6x}-a_5b_1+b_1b_6+3\lambda_2)$.

Further analysis of the compatibility depends on $b_1$.

**Case $\boldsymbol{b_1\neq 0}$.**

From (101), we obtain

$$f_t^x=\frac{f_x^x\lambda_5}{b_1^2}. \tag{102}$$

Comparing the mixed derivative $(f_t^x)_{xx}=(f_{xx}^x)_t,(f_t^x)_x=f_{tx}^x,$ and $(f_t^x)_{tt}=f_{ttt}^x,$ we find the following conditions:

$$
\begin{aligned}
&-9a_{3t}b_1^4+12\,b_{1t}b_1^4-54\,b_{1xx}b_1\lambda_5+162\,b_{1x}^2\lambda_5-108\,b_{1x}\lambda_{5x}b_1-36\,b_{1x}a_3b_1\lambda_5+36\,b_{1x}b_1^2\lambda_5\\
&+27\,\lambda_{5xx}b_1^2+18\,\lambda_{5x}a_3b_1^2-24\,\lambda_{5x}b_1^3-3a_3a_5b_1^4+3\,a_3b_1^4b_6-7a_3b_1^3\lambda_5+4\,a_5b_1^5+27\,a_7b_1^2\lambda_5
\end{aligned}
$$

$$-4\, b_1^5 b_6 + 4 b_1^4 \lambda_4 - 3 b_1^2 \lambda_1 \lambda_5 = 0, \tag{103}$$

$$-6 b_{1x} \lambda_5 + 3\, \lambda_{5x} b_1 + a_3 b_1 \lambda_5 - a_5 b_1^3 + b_1^3 b_6 - b_1^2 \lambda_5 = 0, \tag{104}$$

$$\begin{aligned}&-324\, b_{1tt} b_1 \lambda_5 + 972\, b_{1t}^2 \lambda_5 - 648\, b_{1t} \lambda_{5t} b_1 - 216\, b_{1t} a_5 b_1 \lambda_5 - 108\, b_{1t} b_1 b_6 \lambda_5 - 18\, b_{1t} \lambda_5^2 - 54\, b_6\\ &+ b_1^2 \lambda_5 + 162\, \lambda_{5tt} b_1^2 + 108\, \lambda_{5t} a_5 b_1^2 + 54\, \lambda_{5t} b_1^2 b_6 + 54\, a_{14} b_1^2 \lambda_5 - 162\, a_{16} b_1^4 + 18 a_5 b_1^2 b_6 \lambda_5\\ &+ 6 a_5 b_1 \lambda_5^2 - 9 b_1^2 b_6^2 \lambda_5 - 45\, b_1^2 \lambda_5 \lambda_3 - 6 b_1 b_6 \lambda_5^2 - 27 \lambda_2 \lambda_5^2 - \lambda_5^3 = 0.\end{aligned} \tag{105}$$

Equations (7), (11), (12), (14), (15), (17), (18), (69), (70), (67), (68), (71), (72), (75), (76), (78), (79), (81), (82), and (83) become

$$a_2 = 0, \tag{106}$$

$$a_6 = 0, \tag{107}$$

$$9 b_{1x} - 3\, a_3 b_1 + 2 b_1^2 + 9 \lambda_1 = 0, \tag{108}$$

$$-3 a_{3y} + 2\, b_{1y} + 3 a_9 = 0, \tag{109}$$

$$a_{10} = 0, \tag{110}$$

$$a_{12} = 0, \tag{111}$$

$$-6 a_{5y} + 4 b_{6y} + 3\, a_{13} = 0, \tag{112}$$

$$b_{1y} = 0, \tag{113}$$

$$b_{6y} = 0, \tag{114}$$

$$\lambda_{5yy} = 0, \tag{115}$$

$$a_{5yb_1} - 3\, \lambda_{2y} = 0, \tag{116}$$

$$\lambda_{2y} \lambda_5 + \lambda_{3y} b_1^2 = 0, \tag{117}$$

$$\begin{aligned}&-9\, a_{3t} b_1^2 - 9\, b_{1t} a_3 b_1 + 30\, b_{1t} b_1^2 + 9 b_{1x} a_5 b_1 - 18\, b_{1x} b_1 b_6 + 6\, b_{1x} \lambda_5 + 27\, \lambda_{1t} b_1 - 27\, \lambda_{2x} b_1\\ &+ 9\, a_{11} b_1^2 - 3 a_3 a_5 b_1^2 + 3\, a_3 b_1^2 b_6 - a_3 b_1 \lambda_5 - 8 a_5 b_1^3 + 6\, b_1^3 b_6 + 24 b_1^2 \lambda_2 - 2\, b_1^2 \lambda_5\\ &-9\, b_1 b_6 \lambda_1 = 0,\end{aligned} \tag{118}$$

$$\begin{aligned}&54\, a_{14x} b_1^5 - 54\, a_{3tt} b_1^5 - 108\, a_{3t} b_{1t} b_1^4 + 36\, a_{3t} b_1^5 b_6 - 18\, b_{1tx} b_1^3 \lambda_5 - 54\, b_{1tt} a_3 b_1^4 + 180\, b_{1tt} b_1^5\\ &+ 180\, b_{1t}^2 b_1^4 - 324\, b_{1t} b_{1xx} b_1 \lambda_5 + 216\, b_{1t} b_{1x} a_3 b_1 \lambda_5 - 36 b_{1t} b_{1x} b_1^2 \lambda_5 - 72\, b_{1t} a_3^2 b_1^2 \lambda_5\\ &+ 36 b_{1t} a_3 b_1^4 b_6 + 12 b_{1t} a_3 b_1^3 \lambda_5 - 24\, b_{1t} a_5 b_1^5 + 324\, b_{1t} a_7 b_1^2 \lambda_5 - 96 b_{1t} b_1^5 b_6 - 36\, b_{1t} b_1^2 \lambda_1 \lambda_5\\ &+ 162\, b_{1xx} \lambda_{5t} b_1^2 + 54\, b_{1xx} a_5 b_1^2 \lambda_5 + 18\, b_{1x}^2 \lambda_5^2 - 108\, b_{1x} b_{6t} b_1^4 - 108\, b_{1x} \lambda_{5t} a_3 b_1^2 + 108\, b_{1x} a_{14} b_1^4\\ &- 36\, b_{1x} a_3 a_5 b_1^2 \lambda_5 - 12\, b_{1x} a_3 b_1 \lambda_5^2 - 90 b_{1x} a_5 b_1^4 b_6 + 60 b_{1x} a_5 b_1^3 \lambda_5 + 81\, b_{1x} b_1^4 b_6^2 - 9 b_{1x} b_1^4 \lambda_3\\ &- 60 b_{1x} b_1^3 b_6 \lambda_5 - 108\, b_{1x} b_1^2 \lambda_2 \lambda_5 + 27 b_{1x} b_1^2 \lambda_5^2 + 36 b_{6t} a_3 b_1^5 + 6 b_{6t} b_1^6 - 54 b_{6t} b_1^4 \lambda_1 + 162\, \lambda_{1tt} b_1^4\\ &- 108\, \lambda_{1t} b_1^4 b_6 + 54\, \lambda_{2t} b_1^5 - 81\, \lambda_{2xx} b_1^2 \lambda_5 + 54\, \lambda_{2x} a_3 b_1^2 \lambda_5 - 54\, \lambda_{2x} a_5 b_1^4 + 216 \lambda_{2x} b_1^4 b_6\\ &- 54 \lambda_{2x} b_1^3 \lambda_5 + 36 \lambda_{5t} a_3^2 b_1^3 - 162\, \lambda_{5t} a_7 b_1^3 - 6\, \lambda_{5t} b_1^5 + 18\, \lambda_{5t} b_1^3 \lambda_1 - 18 \lambda_{5x} b_1^4 b_6 + 6 \lambda_{5x} b_1^3 \lambda_5\\ &- 81\, \lambda_{3xx} b_1^4 + 36\, \lambda_{3x} b_1^5 - 72\, a_{11} b_1^5 b_6 - 27\, a_{11} b_1^4 \lambda_2 + 18\, a_{11} b_1^4 \lambda_5 - 18\, a_{14} a_3 b_1^5 - 36\, a_{14} b_1^6\\ &+ 12\, a_3^2 a_5 b_1^3 \lambda_5 - 18\, a_3^2 b_1^2 \lambda_2 \lambda_5 + 2 a_3^2 b_1^2 \lambda_5^2 + 30\, a_3 a_5 b_1^5 b_6 + 18\, a_3 a_5 b_1^4 \lambda_2 - 20\, a_3 a_5 b_1^4 \lambda_5\\ &- 6 a_3 b_1^5 b_6^2 + 6 a_3 b_1^5 \lambda_3 - 9\, a_3 b_1^4 b_6 \lambda_2 + 14\, a_3 b_1^4 b_6 \lambda_5 + 36\, a_3 b_1^3 \lambda_2 \lambda_5 - 8\, a_3 b_1^3 \lambda_5^2 - 12\, a_5^2 b_1^6\\ &- 54\, a_5 a_7 b_1^3 \lambda_5 + 90\, a_5 b_1^6 b_6 + 126\, a_5 b_1^5 \lambda_2 - 16\, a_5 b_1^5 \lambda_5 + 6\, a_5 b_1^3 \lambda_1 \lambda_5 + 81\, a_7 b_1^2 \lambda_2 \lambda_5 - 65\, b_1^6 b_6^2\\ &+ 3 b_1^6 \lambda_3 - 246\, b_1^5 b_6 \lambda_2 + 32 b_1^5 b_6 \lambda_5 + 18\, b_1^4 b_6^2 \lambda_1 - 180\, b_1^4 \lambda_2^2 + 51\, b_1^4 \lambda_2 \lambda_5 - 3 b_1^4 \lambda_5^2\\ &- 9\, b_1^2 \lambda_1 \lambda_2 \lambda_5 = 0,\end{aligned} \tag{119}$$

$$\begin{aligned}&-108\, b_{1t} b_{1x} \lambda_5 + 36\, b_{1t} a_3 b_1 \lambda_5 - 18\, b_{1t} a_5 b_1^3 + 36\, b_{1t} b_1^3 b_1 - 6\, b_{1t} b_1^2 \lambda_5 + 54\, b_{1x} \lambda_{5t} b_1\\ &+ 18\, b_{1x} a_5 b_1 \lambda_5 + 18\, b_{6t} b_1^4 + 54\, \lambda_{2t} b_1^3 - 27 \lambda_{2x} b_1 \lambda_5 - 18\, \lambda_{5t} a_3 b_1^2 - 6\, \lambda_{5t} b_1^3 - 27\, \lambda_{3x} b_1^3\\ &- 6\, a_3 a_5 b_1^2 \lambda_5 + 9\, a_3 b_1 \lambda_2 \lambda_5 + 6\, a_5^2 b_1^4 - 18\, a_5 b_1^4 b_6 - 9\, a_5 b_1^3 \lambda_2 + 6 a_5 b_1^3 \lambda_5 + 15\, b_1^4 b_6^2 + 9\, b_1^4 \lambda_3\\ &+ 45\, b_1^3 b_6 \lambda_2 - 10\, b_1^3 b_6 \lambda_5 - 18\, b_1^2 \lambda_2 \lambda_5 + 3\, b_1^2 \lambda_5^2 = 0,\end{aligned} \tag{120}$$

$$9 b_{1t} - 3\, a_5 b_1 + 3\, b_1 b_6 + 9 \lambda_2 - \lambda_5 = 0, \tag{121}$$

$$\begin{aligned}&18\, b_{1tt} b_1^3 + 36\, b_{1t} b_{1x} \lambda_5 - 12\, b_{1t} a_3 b_1 \lambda_5 - 12\, b_{1t} b_1^3 b_6 + 2\, b_{1t} b_1^2 \lambda_5 - 18\, b_{1x} \lambda_{5t} b_1 - 6 b_{1x} a_5 b_1 \lambda_5\\ &+ 9\, \lambda_{2x} b_1 \lambda_5 + 6\, \lambda_{5t} a_3 b_1^2 + 9\, \lambda_{3x} b_1^3 - 6\, a_{14} b_1^4 + 2\, a_3 a_5 b_1^2 \lambda_5 - 3\, a_3 b_1 \lambda_2 \lambda_5 + 8\, a_5 b_1^4 b_6\end{aligned}$$

$$+3\, a_5 b_1^3 \lambda_2 - 2a_5 b_1^3 \lambda_5 - 7\, b_1^4 b_6^2 - b_1^4 \lambda_3 - 21\, b_1^3 b_6 \lambda_2 + 4\, b_1^3 b_6 \lambda_5 + 6\, b_1^2 \lambda_2 \lambda_5 - b_1^2 \lambda_5^2 = 0, \quad (122)$$

$$36\, b_{1t} \lambda_5 + 18\, b_{6t} b_1^2 - 18\, \lambda_{5t} b_1 - 6\, a_5 b_1 \lambda_5 + 3\, b_1^2 b_6^2 + 9\, b_1^2 \lambda_3 + 9\, \lambda_2 \lambda_5 + \lambda_5^2 = 0, \quad (123)$$

$$9a_{3t} b_1^2 - 12\, b_{1t} b_1^2 + 3\, b_{1x} \lambda_5 - 9\, a_{11} b_1^2 + 3\, a_3 a_5 b_1^2 - a_3 b_1 \lambda_5 + 4a_5 b_1^3 - 4\, b_1^3 b_6 - 6b_1^2 \lambda_2$$
$$+2\, b_1^2 \lambda_5 = 0, \quad (124)$$

$$486\, a_{14t} a_3 b_1^6 - 648\, a_{14t} b_1^7 + 1458\, a_{14x} b_1^6 b_6 + 486\, a_{14x} b_1^5 \lambda_5 - 4374\, a_{16xx} b_1^6 - 2916\, a_{16x} a_3 b_1^6$$
$$+ 3888\, a_{16x} b_1^7 + 1458\, a_{3ttt} b_1^6 + 1458\, a_{3tt} a_5 b_1^6 - 1458\, a_{3tt} b_1^6 b_6 - 486\, a_{3tt} b_1^5 \lambda_5$$
$$+486\, a_{3t} b_{1t} b_1^4 \lambda_5 - 1458\, a_{3t} b_{6t} b_1^6 - 486\, a_{3t} \lambda_{5t} b_1^5 + 1458\, a_{3t} a_{14} b_1^6 - 972\, a_{3t} a_5 b_1^6 b_6$$
$$-324\, a_{3t} a_5 b_1^5 \lambda_5 + 486\, a_{3t} b_1^6 b_6^2 - 486\, a_{3t} b_1^6 \lambda_3 + 324\, a_{3t} b_1^5 b_6 \lambda_5 + 54\, a_{3t} b_1^4 \lambda_5^2$$
$$- 486\, b_{1tx} b_1^4 b_6 \lambda_5 - 162\, b_{1tx} b_1^3 \lambda_5^2 - 1944\, b_{1ttt} b_1^6 - 162\, b_{1tt} a_3 b_1^4 \lambda_5 - 1944\, b_{1tt} a_5 b_1^6$$
$$+ 1944\, b_{1tt} b_1^6 b_6 + 864\, b_{1tt} b_1^5 \lambda_5 + 648\, b_{1t}^2 a_3 b_1^3 \lambda_5 - 1512\, b_{1t}^2 b_1^4 \lambda_5 - 2916\, b_{1t} b_{1xx} b_1 \lambda_5^2$$
$$+ 1944\, b_{1t} b_{1x} a_3 b_1 \lambda_5^2 - 4860\, b_{1t} b_{1x} a_5 b_1^3 \lambda_5 + 5832\, b_{1t} b_{1x} b_1^3 b_6 \lambda_5 - 2268\, b_{1t} b_{1x} b_1^2 \lambda_5^2$$
$$+ 1944\, b_{1t} b_{6t} b_1^6 + 8748\, b_{1t} \lambda_{2x} b_1^3 \lambda_5 - 324\, b_{1t} \lambda_{5t} a_3 b_1^4 + 1080\, b_{1t} \lambda_{5t} b_1^5 - 972\, b_{1t} \lambda_{5x} b_1^3 \lambda_5$$
$$- 3888\, b_{1t} a_{11} b_1^4 \lambda_5 - 1944\, b_{1t} a_{14} b_1^6 - 648\, b_{1t} a_3^2 b_1^2 \lambda_5^2 + 1566\, b_{1t} a_3 a_5 b_1^4 \lambda_5 - 594\, b_{1t} a_3 b_1^4 b_6 \lambda_5$$
$$+ 648\, b_{1t} a_5 b_1^6 b_6 + 1692\, b_{1t} a_5 b_1^5 \lambda_5 + 2916\, b_{1t} a_7 b_1^2 \lambda_5^2 + 468\, b_{1t} a_3 b_1^3 \lambda_5^2 + 648\, b_{1t} b_1^6 \lambda_3$$
$$- 1260\, b_{1t} b_1^5 b_6 \lambda_5 - 3564\, b_{1t} b_1^4 \lambda_2 \lambda_5 + 348\, b_{1t} b_1^4 \lambda_5^2 - 324\, b_{1t} b_1^2 \lambda_1 \lambda_5^2 + 1458\, b_{1xx} \lambda_{5t} b_1^2 \lambda_5$$
$$+ 486\, b_{1xx} a_5 b_1^2 \lambda_5^2 + 162\, b_{1x}^2 \lambda_5^3 - 972\, b_{1x} b_{6t} b_1^4 \lambda_5 - 972\, b_{1x} \lambda_{5t} a_3 b_1^2 \lambda_5 + 2430\, b_{1x} \lambda_{5t} a_5 b_1^4$$
$$- 2916\, b_{1x} \lambda_{5t} b_1^4 b_6 + 972\, b_{1x} \lambda_{5t} b_1^3 \lambda_5 + 972\, b_{1x} a_{14} b_1^4 \lambda_5 + 1944\, b_{1x} a_{16} b_1^6 - 324\, b_{1x} a_3 a_5 b_1^2 \lambda_5^2$$
$$- 108\, b_{1x} a_3 b_1 \lambda_5^3 + 810\, b_{1x} a_5^2 b_1^4 \lambda_5 - 1458\, b_{1x} a_5 b_1^4 b_6 \lambda_5 + 702\, b_{1x} a_5 b_1^3 \lambda_5^2 + 243\, b_{1x} b_1^4 b_6^2 \lambda_5$$
$$-243\, b_{1x} b_1^4 \lambda_5 \lambda_3 - 324\, b_{1x} b_1^3 b_6 \lambda_5^2 - 1134\, b_{1x} b_1^2 \lambda_2 \lambda_5^2 + 225\, b_{1x} b_1^2 \lambda_5^3 - 486\, b_{6tt} a_3 b_1^6$$
$$+ 648\, b_{6tt} b_1^7 - 486\, b_{6t} a_3 a_5 b_1^6 + 270\, b_{6t} a_3 b_1^5 \lambda_5 - 648\, b_{6t} a_5 b_1^7 + 1944\, b_{6t} b_1^7 b_6$$
$$+2916\, b_{6t} b_1^6 \lambda_2 - 360\, b_{6t} b_1^6 \lambda_5 + 1458\, \lambda_{2t} b_1^6 b_6 + 486\, \lambda_{2t} b_1^5 \lambda_5 - 729\, \lambda_{2xx} b_1^2 \lambda_5^2$$
$$-4374\, \lambda_{2x} \lambda_{5t} b_1^4 + 486\, \lambda_{2x} a_3 b_1^2 \lambda_5^2 - 2430\, \lambda_{2x} a_5 b_1^4 \lambda_5 + 972\, \lambda_{2x} b_1^4 b_6 \lambda_5 - 324\, \lambda_{2x} b_1^3 \lambda_5^2$$
$$+486\, \lambda_{5t} \lambda_{5x} b_1^4 + 1944\, \lambda_{5t} a_{11} b_1^5 + 324\, \lambda_{5t} a_3^2 b_1^3 \lambda_5 - 864\, \lambda_{5t} a_3 a_5 b_1^5 + 378\, \lambda_{5t} a_3 b_1^5 b_6$$
$$- 162\, \lambda_{5t} a_3 b_1^4 \lambda_5 - 846\, \lambda_{5t} a_5 b_1^6 - 1458\, \lambda_{5t} a_7 b_1^3 \lambda_5 + 522\, \lambda_{5t} b_1^6 b_6 + 2268\, \lambda_{5t} b_1^5 \lambda_2$$
$$- 378\, \lambda_{5t} b_1^5 \lambda_5 + 162\, \lambda_{5t} b_1^3 \lambda_1 \lambda_5 + 162\, \lambda_{5x} a_5 b_1^4 \lambda_5 - 162\, \lambda_{3t} a_3 b_1^6 + 216\, \lambda_{3t} b_1^7 - 729\, \lambda_{3xx} b_1^4 \lambda_5$$
$$- 486\, \lambda_{3x} a_5 b_1^6 + 648\, a_{11} a_5 b_1^5 \lambda_5 - 729\, a_{11} b_1^6 b_6^2 - 243\, a_{11} b_1^6 \lambda_3 - 324\, a_{11} b_1^5 b_6 \lambda_5$$
$$- 486\, a_{11} b_1^4 \lambda_2 \lambda_5 - 27\, a_{11} b_1^4 \lambda_5^2 + 162\, a_{14} a_3 a_5 b_1^6 - 270\, a_{14} a_3 b_1^5 \lambda_5 + 1080\, a_{14} a_5 b_1^7$$
$$-1944\, a_{14} b_1^7 b_6 - 2916\, a_{14} b_1^6 \lambda_2 + 360\, a_{14} b_1^6 \lambda_5 + 1134\, a_{16} a_3 b_1^7 - 4374\, a_{16} a_7 b_1^6 - 648\, a_{16} b_1^8$$
$$+ 486\, a_{16} b_1^6 \lambda_1 + 108\, a_3^2 a_5 b_1^3 \lambda_5^2 - 162\, a_3^2 b_1^2 \lambda_2 \lambda_5^2 + 18\, a_3^2 b_1^2 \lambda_5^3 - 270\, a_3 a_5^2 b_1^5 \lambda_5$$
$$+ 162\, a_3 a_5 b_1^6 b_6^2 - 54\, a_3 a_5 b_1^6 \lambda_3 + 342\, a_3 a_5 b_1^5 b_6 \lambda_5 + 324\, a_3 a_5 b_1^4 \lambda_2 \lambda_5 - 174\, a_3 a_5 b_1^4 \lambda_5^2$$
$$+ 27\, a_3 b_1^6 b_6^3 + 81 a_3 b_1^6 b_6 \lambda_3 - 63\, a_3 b_1^5 b_6^2 \lambda_5 + 63\, a_3 b_1^5 \lambda_5 \lambda_3 - 162\, a_3 b_1^4 b_6 \lambda_2 \lambda_5 + 111\, a_3 b_1^4 b_6 \lambda_5^2$$
$$+ 405\, a_3 b_1^3 \lambda_2 \lambda_5^2 - 75\, a_3 b_1^3 \lambda_5^3 - 648\, a_5^2 b_1^7 b_6 - 162\, a_5^2 b_1^6 \lambda_2 - 90\, a_5^2 b_1^6 \lambda_5 - 486\, a_5 a_7 b_1^3 \lambda_5^2$$
$$+ 1296\, a_5 b_1^7 b_6^2 - 144\, a_5 b_1^7 \lambda_3 + 2268\, a_5 b_1^6 b_6 \lambda_2 - 366\, a_5 b_1^6 b_6 \lambda_5 - 108\, a_5 b_1^5 \lambda_2 \lambda_5$$
$$+ 142\, a_5 b_1^5 \lambda_5^2 + 54\, a_5 b_1^3 \lambda_1 \lambda_5^2 + 729\, a_7 b_1^2 \lambda_2 \lambda_5^2 - 360\, b_1^7 b_6^3 + 432\, b_1^7 b_6 \lambda_3 - 648\, b_1^6 b_6^2 \lambda_2$$
$$+ 282\, b_1^6 b_6^2 \lambda_5 + 810\, b_1^6 \lambda_2 \lambda_3 - 66 b_1^6 \lambda_5 \lambda_3 + 972\, b_1^5 b_6 \lambda_2 \lambda_5 - 232\, b_1^5 b_6 \lambda_5^2 + 1134\, b_1^4 \lambda_2^2 \lambda_5$$
$$- 612\, b_1^4 \lambda_2 \lambda_5^2 + 78\, b_1^4 \lambda_5^3 - 81\, b_1^2 \lambda_1 \lambda_2 \lambda_5^2 = 0, \quad (125)$$

$$162 a_{14t} b_1^5 - 486\, a_{16x} b_1^5 - 54\, b_{1tt} b_1^3 \lambda_5 + 216\, b_{1t}^2 b_1^2 \lambda_5 - 324\, b_{1t} b_{1x} \lambda_5^2 - 108\, b_{1t} \lambda_{5t} b_1^3$$
$$+ 108\, b_{1t} a_3 b_1 \lambda_5^2 - 450\, b_{1t} a_5 b_1^3 \lambda_5 + 396\, b_{1t} b_1^3 b_6 \lambda_5 + 972\, b_{1t} b_1^2 \lambda_2 \lambda_5 - 222\, b_{1t} b_1^2 \lambda_5^2$$
$$+162\, b_{1x} \lambda_{5t} b_1 \lambda_5 + 54\, b_{1x} a_5 b_1 \lambda_5^2 - 162\, b_{6tt} b_1^5 - 162\, b_{6t} a_5 a_1^5 - 18\, b_{6t} b_1^4 \lambda_5 - 81\, \lambda_{2x} b_1 \lambda_5^2$$
$$- 54\, \lambda_{5t} a_3 b_1^2 \lambda_5 + 198\, \lambda_{5t} a_5 b_1^4 - 198\, \lambda_{5t} b_1^4 b_6 - 486\, \lambda_{5t} b_1^3 \lambda_2 + 108\, \lambda_{5t} b_1^3 \lambda_5 - 54\, \lambda_{3t} b_1^5$$
$$- 81\, \lambda_{3x} b_1^3 \lambda_5 + 54\, a_{14} a_5 b_1^5 + 18\, a_{14} b_1^4 \lambda_5 - 162\, a_{16} a_5 b_1^5 + 162\, a_{16} b_1^6 - 18\, a_3 a_5 b_1^2 \lambda_5^2$$
$$+ 27\, a_3 b_1 \lambda_2 \lambda_5^2 + 72\, a_5^2 b_1^4 \lambda_5 - 27\, a_5 b_1^5 b_6^2 - 45\, a_5 b_1^5 \lambda_3 - 84\, a_5 b_1^4 b_6 \lambda_5 - 216\, a_5 b_1^3 \lambda_2 \lambda_5$$
$$+ 35\, a_5 b_1^3 \lambda_5^2 + 9\, b_1^5 b_6^3 + 27\, b_1^5 b_6 \lambda_3 + 6\, b_1^4 b_6^2 \lambda_5 - 6\, b_1^4 \lambda_5 \lambda_3 + 54\, b_1^3 b_6 \lambda_2 \lambda_5 + b_1^3 b_6 \lambda_5^2$$
$$- 27\, b_1^2 \lambda_2 \lambda_5^2 = 0. \quad (126)$$

**Case $\boldsymbol{b_1 = 0}$.**

In this case, equations (12), (20), (23), (30), (31), (68), (76)-(79), (101), and (81)-(83) become

$$\lambda_1 = 0, \tag{127}$$

$$a_{15} = 0, \tag{128}$$

$$b_2 = 0, \tag{129}$$

$$b_9 = 0, \tag{130}$$

$$b_{10} = 0, \tag{131}$$

$$\lambda_{3y} = 0, \tag{132}$$

$$\lambda_2 = 0, \tag{133}$$

$$b_{6x} = 0, \tag{134}$$

$$\lambda_{3x} = 0, \tag{135}$$

$$\lambda_5 = 0, \tag{136}$$

$$\lambda_6 = 0, \tag{137}$$

$$3a_{3t} - 3\,a_{11} + a_3a_5 = 0, \tag{138}$$

$$\begin{aligned}&18a_{14t}a_3 + 54\,a_{14x}b_6 - 162\,a_{16xx} - 108\,a_{16x}a_3 + 54\,a_{3ttt} + 54\,a_{3ttt} + 54\,a_{3tt}a_5 - 54\,a_{3tt}b_6\\ &+54\,a_{3t}a_{14} - 36\,a_{3t}a_5b_6 + 27\,a_{3t}b_6^2 + 9\,a_{3t}\lambda_3 + 9a_{3t}\lambda_6 + 3\lambda_{6t}a_3 + 9\lambda_{6x}b_6 - 27a_{11}b_6^2 - 9a_{11}\lambda_3\\ &+6\,a_{14}a_3a_5 - 162\,a_{16}a_7 + 9\,a_3a_5b_6^2 + 7a_3a_5\lambda_3 + 3a_3a_5\lambda_6 - a_3b_6\lambda_6 = 0,\end{aligned} \tag{139}$$

$$18\,a_{14t} - 54a_{16x} + 3\,\lambda_{3t} + 3\lambda_{6t} + 6a_{14}a_5 - 18a_{16}a_3 + 4a_5\lambda_3 + 3a_5\lambda_6 = 0, \tag{140}$$

where $\lambda_6 = -6b_{6t} - b_6^2 - 3\,\lambda_3$.

All obtained results can be summarized into the following theorems.

**Theorem 1.** Any system of two third-order ordinary differential equations

$$\begin{aligned}x''' &= F(t,x,y,x',y',x'',y''),\\ y''' &= G(t,x,y,x',y',x'',y'')\end{aligned} \tag{141}$$

which yields the system of two linear third-order ordinary differential equations

$$u''' = 0,\ v''' = 0 \tag{142}$$

via a generalized Sundman transformation

$$u = f^x(t,x,y),\ v = f^y(t,x,y),\ d\,\tau = g(t,x,y)dt, \tag{143}$$

where $g \neq 0$ and $f_x^x\,f_y^y - f_y^x\,f_x^y \neq 0$, has to be of the form

$$\begin{aligned}&x''' + a_1x'y'' + a_2x''y' + a_3x''x' + a_4y''y' + a_5x'' + a_6y'' + a_7x'^3 + a_8y'^3 + a_9x'^2y'\\ &\quad + a_{10}y'^2x' + a_{11}x'^2 + a_{12}y'^2 + a_{13}x'y' + a_{14}x' + a_{15}y' + a_{16} = 0,\\ &y''' + b_1x'y'' + b_2x''y' + b_3x''x' + b_4y''y' + b_5x'' + b_6y'' + b_7x'^3 + b_8y'^3 + b_9x'^2y'\\ &\quad +b_{10}y'^2x' + b_{11}x'^2 + b_{12}y'^2 + b_{13}x'y' + b_{14}x' + b_{15}y' + b_{16} = 0.\end{aligned} \tag{144}$$

**Theorem 2.** Sufficient conditions for the system (144) to be linearizable via a generalized Sundman transformation (143) with $f_y^x = 0, f_t^y = 0,$ and $f_x^y = 0$ are as follows:

(a) If $a_1 \neq 0,$ then the conditions are (39), (41), (44), (45), (46), (51), (52), (54), (56)-(66), (69), (70), (73), (74), (80), (86)-(99), and (100).

(b) If $a_1 = 0$ and $b_1 \neq 0,$ then the conditions are (39), (41), (44), (45), (46), (51), (52), (54), (56)-(66), (69), (70), (73), (74), (80), (103)-(125), and (126).

(c) If $a_1 = 0$ and $b_1 = 0$, then the conditions are (39), (41), (44), (45), (46), (51), (52), (54), (56)-(66), (69), (70), (73), (74), (80), (127)-(139), and (140).

**Theorem 3.** Provided that the sufficient conditions in Theorem 2 are satisfied, the transformation that maps the system (144) into the system of linear equations (143) is obtained by solving one of the following compatible system of equations for the functions $g, f^x$, and $f^y$:

(a) (38), (40), (50), (53), (55), and (85);

(b) (38), (40), (50), (53), (55), and (102);

(c) (38), (40), (42), (49), (50), (53), and (55).

## EXAMPLES

**Example 1.** Consider a system of two nonlinear third-order ordinary differential equations

$$\begin{aligned} x''' + m_1 y^{k_1} x'y'' + m_2 y^{k_2} x''y' + m_3 y^{k_3} y'' &= 0, \\ y''' + m_4 y^{k_4} x'y'' + m_5 y^{k_5} x''y' + m_6 y^{k_6} y''y' &= 0, \end{aligned} \tag{145}$$

where $k_i$ and $m_i$ (for $i = 1,2,3,\ldots,6$) are arbitrary constants.

Note that the system (145) is of the form (144) with coefficients

$$a_1 = m_1 y^{k_1}, \quad a_2 = m_2 y^{k_2}, \quad a_6 = m_3 y^{k_3}, \quad b_1 = m_4 y^{k_4}, \quad b_2 = m_5 y^{k_5}, \quad b_4 = m_6 y^{k_6}.$$

If $m_2 = 3m_1,\ m_4 = m_5 = 0$, and $m_1,\ m_3,\ m_6 \neq 0$, then $a_1,\ a_2,\ a_6,\ b_4 \neq 0$.

We can check that the coefficients obey the conditions in Case (a) of Theorem 2. Thus, the system of equations

$$\begin{aligned} x''' + m_1 y^{k_1} x'y'' + 3m_1 y^{k_2} x''y' + m_3 y^{k_3} y'' &= 0, \\ y''' + m_6 y^{k_6} y''y' &= 0 \end{aligned} \tag{146}$$

is linearizable by a generalized Sundman transformation.

To find the functions $f^x, f^y$, and $g$, we have to solve the overdetermined system of partial differential equations

$$f_t^x = \frac{y^{k_3} f_x^x m_3}{y^{k_1} m_1}, \tag{147}$$

$$f_{xx}^x = 0, \tag{148}$$

$$f_{yy}^y = \frac{f_y^y}{3}(-4y^{k_1} m_1 + k^{k_6} m_6), \tag{149}$$

$$g_t = 0, \tag{150}$$

$$g_x = 0, \tag{151}$$

$$g_y = -y^{k_1} g\, m_1. \tag{152}$$

For example, if $k_1 = k_2 = k_3 = k_6 = -1$, then we take the simplest solution of the system of equations (147)-(152), i.e.,

$$f^x = m_3\, t + m_1\, x,\ f^y = \frac{3\sqrt[3]{y^{m_6 - 4m_1 + 3}}}{m_6 - 4m_1 + 3},\ g = \frac{1}{y^{m_1}}. \tag{153}$$

Then the generalized Sundman transformation becomes

$$u = m_3\, t + m_1\, x,\ v = \frac{3\sqrt[3]{y^{\{m_6 - 4m_1 + 3\}}}}{m_6 - 4m_1 + 3},\ d\tau = \frac{1}{y^{m_1}} dt. \tag{154}$$

Hence, the system of equations

$$\begin{aligned} x''' + \frac{m_1}{y} x'y'' + \frac{3m_1}{y} x''y' + \frac{m_3}{y} y'' &= 0, \\ y''' + \frac{m_6}{y} y''y' &= 0 \end{aligned} \tag{155}$$

is mapped by the generalized Sundman transformation (154) into the system of two linear third-order ordinary differential equations

$$u''' = 0,\ v''' = 0. \tag{156}$$

The general solution of (156) is

$$\begin{aligned} u &= c_1\,\tau^2 + c_2\,\tau + c_3, \\ v &= c_4\,\tau^2 + c_5\,\tau + c_6, \end{aligned} \tag{157}$$

where $c_1, c_2, c_3, c_4, c_5,$ and $c_6$ are arbitrary constants. Applying the generalized Sundman transformation (154) to the system (157), we obtain the general solution of the system (155) as

$$\begin{aligned} x &= \frac{c_1 t^2}{2\,m_1 y^{2m_1}} + \frac{c_2 t}{m_1 y^{m_1}} + \frac{c_3}{m_1} - \frac{m_3 t}{m_1}, \\ \frac{3\sqrt[3]{y^{m_6-4m_1+3}}}{m_6-4m_1+3} &= \frac{c_4 t^2}{2y^{2m_1}} + \frac{c_5 t}{y^{m_1}} + c_6, \end{aligned}$$

where $c_1, c_2, c_3, c_4, c_5,$ and $c_6$ are arbitrary constants.

**Example 2.** Consider the system of two nonlinear third-order ordinary differential equations

$$\begin{aligned} x''' - \frac{1}{6}x'^3 - \frac{x^2x'^3}{18} &= 0, \\ y''' + \frac{3x\,x'y''}{2} + \frac{x\,x''y'}{2} + y\,y''y' + \frac{(3+y^2)y'^3}{9} + \frac{(1+x^2)x'^2y'}{2} + \frac{xyy'^2x'}{2} &= 0. \end{aligned} \tag{158}$$

Note that the system (158) is of the form (144) with coefficients

$$a_7 = -\frac{1}{6} - \frac{x^2}{18}, \quad b_1 = \frac{3x}{2}, \quad b_2 = \frac{x}{2}, \quad b_4 = y,$$
$$b_8 = \frac{3+y^2}{9}, \quad b_9 = \frac{1+x^2}{2}, \quad b_{10} = \frac{xy}{2}.$$

We can check that the coefficients obey the condition in Case (b) of Theorem 2. Thus, the system (158) is linearizable via a generalized Sundman transformation.

To find the functions $f^x, f^y,$ and $g,$ we have to solve the overdetermined system of partial differential equations

$$f_t^x = 0, \tag{159}$$
$$f_{xx}^x = -\frac{2xf_x^x}{3}, \tag{160}$$
$$f_{yy}^y = \frac{yf_y^y}{3}, \tag{161}$$
$$g_t = 0, \tag{162}$$
$$g_x = -\frac{gx}{2}, \tag{163}$$
$$g_y = 0. \tag{164}$$

Choosing the simplest solution to the system (159)-(164), i.e.,

$$f^x = \int e^{-x^{\frac{2}{3}}}\,dx, \quad f^y = \int e^{y^{\frac{2}{6}}}\,dy, \quad g = e^{-\frac{x^2}{4}},$$

we obtain the linearizing generalized Sundman transformation

$$u = \int e^{-x^{\frac{2}{3}}}\,dx,\ v = \int e^{y^{\frac{2}{6}}}\,dy,\ d\tau = e^{-\frac{x^2}{4}}dt. \tag{165}$$

Hence, the system (158) can be mapped by the transformation (165) into the system of linear equations

$$u''' = 0, \quad v''' = 0.$$

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