ปัญหาการให้สีกราฟแบบแบ็คโบน

Backbone Coloring Problems

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บทคัดย่อ

กำหนดให้ *H* เป็นกราฟย่อยของกราฟ *G* ในคู่อันดับของกราฟ (*G*,*H*) การให้สีกราฟแบบแบ็คโบนของคู่อันดับของกราฟ (*G*,*H*) คือ การให้สีจุดยอดของกราฟ *G* โดยแต่ละจุดยอดที่ประชิดกันมีค่าสีแตกต่างกัน โดยเพิ่มเงื่อนไขจุดยอดที่ประชิดกันในกราฟย่อย *H* ต้องมีค่าสีแตกต่างกันอย่างน้อยที่สุด 2

ในบทความนี้นำเสนอที่มาของการศึกษาปัญหาการให้สีกราฟแบบแบ็คโบนอย่างสังเขปและการให้สีกราฟแบบอื่นๆ ที่เกี่ยวข้อง หลังจากนั้นจะกล่าวถึงงานวิจัยที่เกี่ยวข้องกับปัญหาการให้สีกราฟแบบแบ็คโบน เช่น การกำหนดแบบ $L(2,1)$ การให้สีกราฟแบบ λ -แบ็คโบนและการให้สีกราฟแบบลิสต์แบ็คโบน อีกทั้งยังนำเสนอหัวข้องานวิจัยในอนาคต

ABSTRACT

Let *H* be a subgraph of a graph *G* in a graph pair (*G*,*H*). A backbone coloring for a graph pair (*G*,*H*) is a vertex coloring of *G* such that adjacent vertices receive distinct colors, adding a condition that colors assigned to adjacent vertices in a subgraph *H* must differ by at least 2.

This article presents the brief history of backbone coloring problems and some related colorings. Then some researches of backbone coloring problems is proposed for example, $L(2,1)$-labeling, λ-backbone coloring and list backbone coloring. Moreover, topics of future researches are presented.

คำสำคัญ: การให้สีกราฟแบบแบ็คโบน การกำหนดแบบ $L(2,1)$ การให้สีกราฟแบบลิสต์แบ็คโบน

Keywords: Backbone coloring, $L(2,1)$-labeling, List backbone coloring

1. INTRODUCTION

A graph $G=(V(G), E(G))$ consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each edge consists of two (possibly equal) vertices called its *endpoints*.

Backbone coloring of graphs has its origin from the problem of channel assignment by Hale (1980). In the channel assignment problem, each channel must be assigned to a set of transmitters without interference. Interferences are divided into two types: weak and strong. The channels assigned to two transmitters with weak interference should be distinct while the channels assigned to two transmitters with strong interference should be far apart. A graph G is constructed with the transmitters represented by the vertices of G, and two vertices are adjacent in G when two corresponding transmitters interfere with each another. Let H be a subgraph of G such that an edge of H is formed from two vertices with strong interference. We call (G,H) a *graph pair*, and a subgraph of H a *backbone* of G.

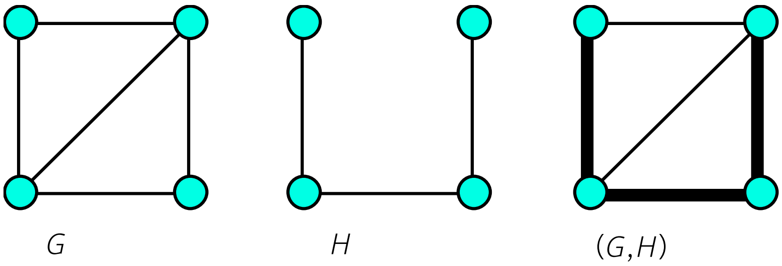

Figure 1. Example of a graph pair (G,H) with a backbone H (bold edges)

2. BACKBONE COLORING OF GRAPHS

In a graph pair (G,H), a vertex coloring $c: V(G) \longrightarrow \{1,2,3,\ldots,k\}$ such that $|c(u)-c(v)| \geq 1$ if $uv \in E(G) \setminus E(H)$, and $|c(u)-c(v)| \geq 2$ if $uv \in E(H)$ is called a *backbone k-coloring* for (G,H). The *backbone chromatic number* of (G,H), denoted by $\chi_{BB}(G,H)$, is the smallest positive integer k for which there exists a backbone k-coloring for (G,H).

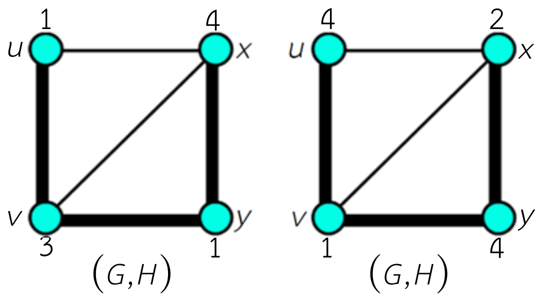

Figure 2. Backbone 4-colorings for (G,H)

For example, two backbone 4-colorings for (G,H) are shown in Figure 2. Thus $\chi_{BB}(G,H) \leq 4$. We claim that $\chi_{BB}(G,H) \geq 4$. To the contrary, suppose that (G,H) is a backbone 3-coloring. It is easy to see that y cannot be colored with color 1 or 3; otherwise we are forced to use the same color for v and x, a clear contradiction. Now suppose that y is colored with color 2. A vertex v is adjacent in H to y, so the colors 1 and 3 are forbidden for v. Then we have no colors available for v, a contradiction. Therefore, $\chi_{BB}(G,H) = 4$.

Backbone coloring of graphs was introduced by Broersma et al. (2003). Denote the chromatic number of a graph G by $\chi(G)$. If $E(H)=\varnothing$, then $\chi_{BB}(G,H)=\chi(G)$. Furthermore, backbone coloring is a generalization of proper coloring. Obviously, $\chi_{BB}(G,H)\leq 2\chi(G)-1$ for any graph pair (G,H). In 2007, Broersma et al. (2007) showed the sharpness of this upper bound that there is a graph G and a spanning tree H of G with $\chi(G)=n$ and $\chi_{BB}(G,H)=2\chi(G)-1$ for any positive integer n. They stated that $\chi_{BB}(G,H)\leq\chi(G)+4$ when G is a chordal graph and H is a Hamiltonian path in G. Next, Miskuf et al. (2009) showed that there is no constant c such that $\chi_{BB}(G,H)\leq\chi(G)+c$ when G is a triangle-free (not necessarily chordal) and H can be any spanning tree. Moreover, Broersma et al. (2007) conjectured that for any planar graph G and any spanning tree H of G, $\chi_{BB}(G,H)\leq 6$. Broersma et al. (2009) proved that $\chi_{BB}(G,H)\leq 6$ when G is planar and H is a perfect matching in G. Later, Campos et al. (2013) proved conjecture of Broersma et al. (2007) for H with diameter at most 4.

Many authors investigated this topic. Miskuf et al. (2010) stated that for a graph G of maximum degree $\Delta(G)$ and a backbone H being a d-degenerated subgraph of G, $\chi_{BB}(G,H)\leq\Delta(G)+d+1$; moreover, if H is a matching then $\chi_{BB}(G,H)\leq\Delta(G)+1$. Later, Wang et al. (2012) gave results for backbone colorings of special graphs such as Halin graphs, complete graphs, wheels, graphs with small maximum average degree. Bu et al. (2013) proved the existence of a graph G with large girth (*i.e.*, the length of shortest cycle in G) such that $\chi_{BB}(G,H)=2\chi(G)-1$ for some spanning tree H. In 2015, Turowski (2015) studied the algorithm for the backbone coloring of split graphs with matching backbone.

A weakened version of the above-mentioned results is to show the existence of a spanning tree T of G such that $\chi_{BB}(G,T)\leq k$, instead of showing that this is true for all spanning trees T of G. For any connected planar graph G there exists a spanning tree T such that $\chi_{BB}(G,T)\leq 4$ if G is C_4-free by Bu and Zhang (2011), or G is C_5-free by Zhang and Bu (2010), or G is C_6-free or C_7-free and without adjacent triangles by Bu and Li (2011), or G is C_8-free or C_9-free without adjacent 4-cycles by Bu and Bao (2015). In 2016, Farzad et al. (2016) showed that for any graph G with $\chi(G)\geq 4$ there exists a spanning tree T of G such that $\chi_{BB}(G,T)=\chi(G)$.

3. λ-BACKBONE COLORING OF GRAPHS

In a graph pair (G,H), given an integer $\lambda\geq 2$, a vertex coloring $c:V(G)\rightarrow\{1,2,3,\ldots,k\}$ such that $|c(u)-c(v)|\geq 1$ if $uv\in E(G)\setminus E(H)$, and $|c(u)-c(v)|\geq\lambda$ if $uv\in E(H)$ is called a λ-*backbone k-coloring* for (G,H). In case $\lambda=2$, a backbone coloring is a special case of a λ-backbone coloring.

This topic has been investigated rather extensively in recent years. Broersma et al. (2009) gave results for λ-backbone coloring for (G,H) when G is a split graph and H is a matching backbone or H is a star backbone. Later, Havet et al. (2014) showed results for λ-backbone coloring for (G,H) when G is planar. Then, Janczewski and Turowski (2015) studied λ-backbone coloring of complete graphs with bipartite backbones.

4. $L(2,1)$-LABELING OF GRAPHS

An $L(2,1)$-*labeling* of a graph G is a function $c:V(G)\rightarrow\mathbb{N}\cup\{0\}$ such that $|c(u)-c(v)|\geq 2$ if $d(u,v)=1$ and $|c(u)-c(v)|\geq 1$ if $d(u,v)=2$, where $d(u,v)$ is the length of the shortest uv-path in a graph G. For a nonnegative integer k, a k-$L(2,1)$-*labeling* is an $L(2,1)$-labeling such that no label is greater than k. The $L(2,1)$-

labeling number of *G*, denoted by $\lambda(G)$, is the smallest number *k* such that *G* has a *k*-*L*(2,1)-labeling. For example, an *L*(2,1)-labeling of C_5 is shown in Figure 3.

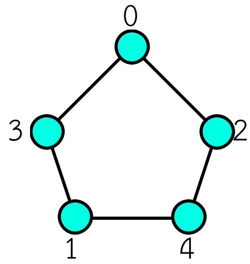

Figure 3. An *L*(2,1)-labeling of C_5

For any graph pair (*G*,*H*), backbone coloring is a generalization of *L*(2,1)-labeling by setting $G=H^2$, that is $V(G)=V(H)$ and $uv\in E(G)$ when $uv\in E(H)$ or there is a path with length 2 from *u* to *v* in *H*. For example, a backbone coloring for (C_5^2, C_5) is an *L*(2,1)-labeling of C_5 as shown in Figure 4.

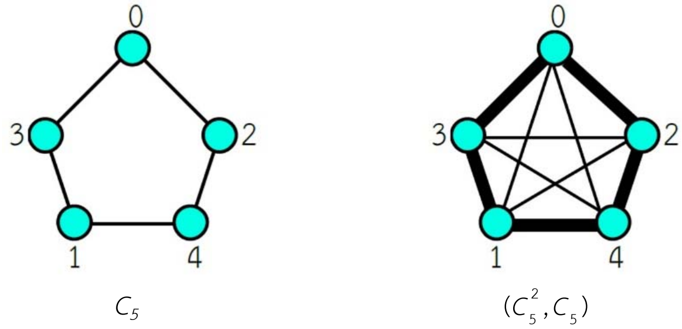

Figure 4. A backbone coloring is a generalization of an *L*(2,1)-labeling

A mapping *L* is called a *list assignment* of a graph *G* if *L* assigns a set *L*(*v*) of positive integers to each vertex *v*. A *k-assignment L* of a graph *G* is a list assignment *L* such that $|L(v)|=k$ for all vertices *v* in a graph *G*. For a list assignment *L** of a graph *G*, *G* is *L*-L*(2,1)-*colorable* if there exists an *L*(2,1)-labeling *c* of *G* such that $c(v)\in L(v)$ for each $v\in V(G)$. Such labeling *c* is called a *L*-L*(2,1)-*labeling* of *G*. If *G* is *L*-L*(2,1)-colorable for every list assignment *L** with $|L^*(v)|\geq k$ for all $v\in V(G)$, then *G* is said to be *k-L*(2,1)-*choosable*.

The topic of *L*(2,1)-labeling was introduced by Griggs and Yeh (1992). Kuo and Yan (2004) studied *L*(2,1)-labeling numbers of Cartesian products of paths and cycles. Later, Lin and Lam (2008) gave results for *L*(2,1)-labeling numbers of direct products of complete graphs, paths, and cycles. In 2012, Zhou et al. (2012) showed that all cycles are 5-*L*(2,1)-choosable.

5. LIST BACKBONE COLORING OF GRAPHS

A coloring *c* is a *list coloring* or an *L-coloring* of a graph *G* if $c(v)\in L(v)$ for each $v\in V(G)$ and no two adjacent vertices have the same color. If a graph *G* has an *L*-coloring, then we say that *G* is *L-colorable*. A graph *G* is *k-choosable* if every *k*-assignment of *G* gives a list coloring. The *choice number* of a graph *G*, denoted by $ch(G)$, is the smallest positive integer *k* such that *G* is *k*-choosable. For convenience, we usually write a list of each vertex without any bracket.

For example, a complete bipartite graph $K_{2,4}$ is not 2-choosable as shown in Figure 5. Then $ch(K_{2,4})\geq 3$.

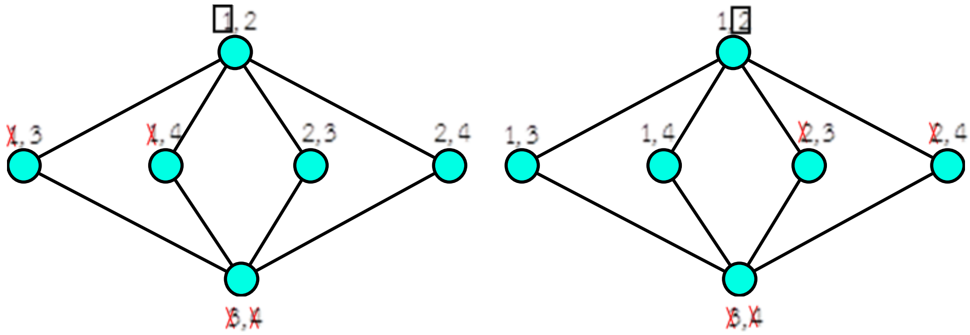

Figure 5. $K_{2,4}$ is not 2-choosable

To prove a complete bipartite graph $K_{2,4}$ is 3-choosable, we consider any 3-assignment L of $K_{2,4}$. Let U and V be a partite set of $K_{2,4}$ such that $U=\{u_1,u_2,u_3,u_4\}$ and $V=\{v_1,v_2\}$. First, we choose $a\in L(v_1)$ and $b\in L(v_2)$. Then we cannot choose two colors a and b from $L(x)$ for all $x\in U$. Since $|L(x)|=3$ for all $x\in U$, there is at least one color in $L(x)\setminus\{a,b\}$ for all $x\in U$. Then $K_{2,4}$ is 3-choosable. Therefore $ch(K_{2,4})=3$.

The concept of list coloring was introduced independently by Vizing (1976) and Erdős et al. (1979). One of the most interesting results of list coloring is showed by Thomassen (1994). Thomassen proved that every planar graph is 5-choosable. In 2005, Wang and Lih (2005) studied list colorings of Halin graphs.

List backbone coloring of graphs is the list coloring version of backbone coloring of graphs. We say that c is a *backbone L-coloring* of (G,H) if c is a backbone coloring of (G,H) such that $c(v)\in L(v)$ for each $v\in V(G)$. If (G,H) has a backbone L-coloring, then we say that (G,H) is *backbone L-colorable*. We say that (G,H) is *backbone k-choosable* if every k-assignment L of G gives a backbone L-coloring. The *backbone choice number* of (G,H), denoted by $ch_{BB}(G,H)$, is the smallest positive integer k such that G is backbone k-choosable. Obviously, $ch_{BB}(G,H)=ch(G)$ where $E(H)=\varnothing$.

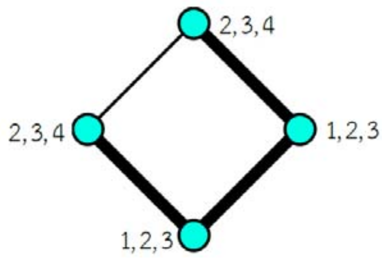

Figure 6. A 3-assignment that prevents (G,H) from being backbone 3-choosable

For example, we consider a graph pair (G,H) as shown in Figure 6. Then (G,H) is not backbone 3-choosable. Hence $ch_{BB}(G,H)\geq 4$. Is it true that $ch_{BB}(G,H)=4$? If (G,H) is backbone 4-choosable, then $ch_{BB}(G,H)=4$. To prove (G,H) is backbone 4-choosable, we need to characterize all 4-assignment L and prove that it is backbone L-colorable. Then this problem is complicated.

The concept of list backbone of graphs was first studied by Bu et al. (2014). They provided several upper bounds for $ch_{BB}(G,H)$ in terms of the choice number of a k-choosable graph G and the structure of H in the following results:

$$ch_{BB}(G,H) \leq \begin{cases} 2k, & \text{if } H \text{ is a matching;} \\ 2k+1, & \text{if } H \text{ is a disjoint union of paths with length at most 2;} \\ 3k, & \text{if each component of } H \text{ contains at most one cycle.} \end{cases}$$

From these results, if G is a path or a cycle then $ch_{BB}(G,H) \leq 9$. Pimpasalee and Nakprasit (2016) obtained $ch_{BB}(G,H) \leq 5$ by showing exact values of $ch_{BB}(G,H)$ where G is a path or a cycle in all possible structures of subgraphs H of G. They showed that $ch_{BB}(G,H) = 4$ where G is a path or a cycle and H is a spanning subgraph of G such that H contains a path with length at least 3. The answer for Figure 6 is $ch_{BB}(G,H) = 4$. Recently, Pimpasalee and Nakprasit (2018) showed that $ch_{BB}(G,T) = 5$ where $G = T \cup C$ is a Halin graph. Moreover, they proved that if L is 4-assignment for a connected graph G which is outerplanar or with the maximum average degree less than 8/3, then there is a spanning tree T of G such that (G,T) is backbone L-colorable. Note that a spanning tree T in which (G,T) is L-colorable depends on L for a connected graph G which is an outerplanar or has the maximum average degree less than 8/3. One may ask for the existence of a spanning tree T in which (G,T) is L-colorable for all 4-assignments of G. In other words, the existence of T in which $ch_{BB}(G,T) \leq 4$. They showed that the answer is negative by giving a counterexample in Figure 7.

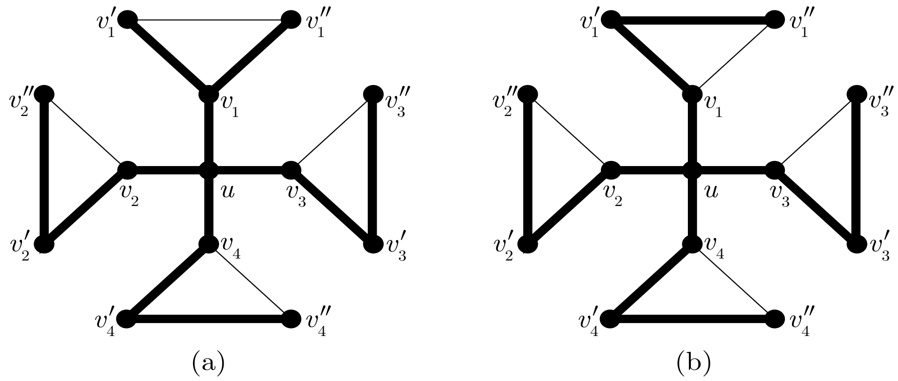

Figure 7. A graph G with spanning tree T (bold edges) of G such that $ch_{BB}(G,T) \geq 5$

6. CONCLUSIONS

This article collects many results related to backbone coloring problems; for example, $L(2,1)$-labeling, λ-backbone coloring and list backbone coloring. Many problems related to backbone colorings are open. In Section 2, the conjecture of Broersma et al. (2007) is still open such that for any planar graph G and any spanning tree H of G, $\chi_{BB}(G,H) \leq 6$. In Section 3, an open problem is to investigate results of λ-backbone coloring by using graphs in results of Section 2. In Section 4, an open problem is to investigate results of $L(2,1)$-labeling in terms of the backbone chromatic number. In Section 5, an open problem is to prove the existence of a spanning tree T of a graph G in which (G,T) is L-colorable for all 5-assignments of G.

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