ควอซีนอร์มอลโหมดเชิงวิเคราะห์ของสนามสเกลาร์ไม่มีมวลในปริภูมิเวลาแอนไทเดอซิทเตอร์ที่มีหลุมดำชวารซ์ชิลด์

Analytical Quasinormal Modes of a Massless Scalar Field in Anti-de-Sitter Spacetimes with a Schwarzschild Black Hole

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บทคัดย่อ

หลุมดำในปริภูมิเวลาแอนไทเดอซิทเตอร์ ได้กลายเป็นหัวข้อวิจัยที่แพร่หลาย เนื่องจากหลักการความสอดคล้องกันระหว่างทฤษฎีสัมพัทธภาพทั่วไปในปริภูมิเวลาแอนไทเดอซิทเตอร์ กับ ทฤษฎีสนามคงรูป ที่ทำให้สามารถศึกษาระบบควอนตัมได้ โดยผ่านการศึกษาปริภูมิเวลาแอนไทดิซิทเตอร์ที่มีหลุมดำ งานวิจัยนี้เป็นการศึกษาและคำนวณเชิงวิเคราะห์หา ควอซีนอร์มอลโหมด ซึ่งคือ คลื่นสเกลาร์ที่สอดคล้องกับเงื่อนไขขอบเขตที่ขอบฟ้า กับ ที่ไกลจากหลุมดำ ผลการคำนวณของควอซีนอร์มอลโหมดที่ไม่มีมวล แสดงการลู่ออก ของคลื่นที่ไกลจากหลุมดำ ซึ่งสามารถจำกัดออก ได้ด้วยเงื่อนไขขอบเขตดังกล่าว ทำให้ความถี่ควอซีนอร์มอล มีค่าไม่ต่อเนื่อง ผลการคำนวณเชิงวิเคราะห์กับผลการคำนวณเชิงตัวเลข นั่นสอดคล้องกัน

ABSTRACT

The anti de Sitter black holes have recently become a popular research topic due to the correspondence between the theory of general relativity in the anti de Sitter spacetime and the conformal field theory, which gives us an opportunity to study a quantum system through a corresponding anti de Sitter (AdS) spacetime, containing black holes. In this work we study and analytically approximate the quasinormal modes of massless scalar wave in AdS spacetime, solutions to the wave equation, which satisfy the boundary conditions at the horizon and at distant region. The massless scalar wave at distant region exhibits some divergence. By choosing appropriate boundary conditions, not only the divergence is removed, but the qusinormal modes become polynomials their frequencies are discrete. Our analytical result is compared to the numerical result, which shows some similarity.

คำสำคัญ: ควอซีนอร์มอลโหมด ความถี่ควอซีนอร์มอล หลุมดำแอนไทเดอซิทเตอร์

Keywords: Quasinormal mode, Quasinormal frequency, Anti de Sitter black hole,

INTRODUCTION

Black holes are regions that the gravitational field is so strong that even light cannot escape. Theoretically, every mass has its own critical radius, in a simple case called Schwarzschild radius. When a mass is contracted into a region smaller than its Schwarzschild radius, the gravitational field is getting stronger until the escape velocity is equal to the speed of light at the Schwarzschild radius. Black holes have caused many contractions among physics principles. For example (Hawking, 2005), the information loss paradox had been topics of argument for decades, until Hawking has accepted that the information of particles falling into the black holes might be not kept inside the black holes forever, but it has been released as radiation through quantum proceeds. The assumption used to solve this problem is the correspondence between quantum theory in the d dimensions and the general relativity in the d+1 dimensional anti de Sitter spacetime, AdS, whose cosmological constant is negative, (Maldacena, 1998). The correspondence implies that the general relativity in the d+1 dimensional AdS spacetime is equivalent to the d dimensional quantum theory on the boundary the AdS spacetime. The correspondence has been become topics of interest since it has been proposed in the 90's. Many works on these topics have been studied to gain a better understanding of the nature of quantum gravity.

The wave and particle behaviors in the AdS spacetime with black holes have been studied numerically in many works e.g. (Horowitz, et al., 2000; Cardoso, et al., 2003). For analytical work, a method, called monodromy, has been proposed (Motl, et al., 2003), which maps the variable, radius *r*, to be a complex variable, and takes certain contour paths as boundary conditions in order to solve for quasinormal frequencies of scalar field in Schwarzschild spacetime. Its approximated result is the same as those in the numerical work (Hod, S., 1998). The quasinormal modes and frequencies in the large AdS Schwarzschild black hole spacetime have been perturbatively calculated by monodromy method in many research e.g., (Cardoso, et al., 2004; Musiri, et al., 2005). However, in the non-extremal cases and the massless scalar field in the AdS Schwarzschild spacetime, the monodromy method could not give the solutions as in (Siopsis, 2007), in which contrary with the numerical result in this model that has given the frequencies, (Cardoso, et al., 2003). Also there are some analytical solutions of a scalar field but only for an extremal de-Sitter Schwarzschild spacetime, e.g. (Cardoso, et al., 2003). In this work, with no exploiting monodromy method and non-extremal black hole case, we analytically study the massless scalar field in the 4 dimensional AdS Schwarzschild black hole spacetime by taking out the singularities and matching boundary conditions. Then we compare our result to the numerical from other research work. The difficulty of the massless problem is the divergence of the wave solution at the two boundaries, one at the black hole surface, called the horizon and the other boundary is the very far away from the black holes, (Siopsis, 2007). The wave solutions, called the quasinormal modes, satisfy the two boundary conditions (Horowitz, et al., 2000), one is the only ingoing wave into the black hole at horizon, and the other one is ether the outgoing wave or a decaying wave at the very far away from the black hole depend on the potential in the regions. (in this case, decaying wave). The frequencies of the scalar wave are also called quasinormal frequencies. We will show later in this work the finite scalar wave solutions and compare the frequencies to those from numerical work.

OBJECTIVES

The correspondence between general relativity and quantum theory has allowed us to solve the problems by the relativity theory methods and then take the results to the quantum theory. In this work we solve the wave equation of the massless scalar wave under the influence of the Schwarzschild black in the 4 dimensional AdS spacetimes. However most of the problems in general relativity could not be solved exactly, and there are number of numerical calculation on these problems, possibly due to divergences from the analytic calculation of the massless paticle wave function at the horizon and at the far away zone. Nevertheless, in this work we can obtain a finite massless wave solution and its frequencies. The scalar wave in this curved spacetime has to satisfy two boundary conditions, (Horowitz, et al., 2000). The first condition is, only ingoing wave at the horizon falling into the black holes. The second is either only outgoing wave at the far away region from the black holes, or only decaying wave allowed if the potential at the far away zone is so great. In this research, the potential far away from is diverged, and then we take the decaying scalar wave as the far away boundary condition. The wave that satisfies these boundary conditions is called quasinormal modes and its frequencies, quasinormal frequencies. The quasinormal modes and frequencies are discrete, due to the boundary conditions.

In this paper has been arranged in the next section as the following

Section Methods: We start from the Lagrangian density in the anti de Sitter spacetime with a scalar wave. We write down the metric and Hawking temperature at the horzon and the wave equation of the massless scalar wave, which we will attempt to solve. To simplify the problem we change variable from radius *r* to new variable defined by $x = r_1 / r$, where r_1 is the horizon radius of the black holes. We separate the solution to the second order partial differential equation, and obtain the radial part of the separated solution needed solved next. The differential equation of the radial scalar wave could not be exactly solved; therefore, we take the approximated ingoing wave solution at the horizon, as the hypergometric function. Then we expand the hypergeometric function to the far away zone and keep only the decaying wave in this region. The two boundary conditions cause the solutions and frequencies discrete.

Results: We write down the quasinormal modes and their frequencies and compare them to a numerical result.

RESEARCH METHODODOLOGY

From (Gubser, 2008), the Lagrangian density , □, of the Anti de sitter Schwarzschild spacetime with a scalar field Φ is

$$\square = R - \frac{6}{L^2} - \left|\nabla_\mu \Phi\right|^2 - m^2 \left|\Phi\right|^2 \qquad (1)$$

R is the scalar curvature, L is the anti de sitter radius, and m is the mass of the scalar field, however in our work we consider for the massless case, m=0, in order to be able to compare our analytical results to the numerical results in (Horowitz, et al., 2000). From the Lagrangian density in eq(1), the metric of the AdS Schwarzschild black hole in the 4 dimensions is (Horowitz, et al., 2000)

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (2)$$

where the function $f(r)$ is

$$f(r) = \frac{r^2}{L^2} + 1 - \frac{2M}{r} \tag{3}$$

L is the Anti de Sitter radius which is defined from the negative cosmological constant (Λ), $L = \sqrt{-3/\Lambda}$ *M* is the black hole mass. The Hawking Temperature is calculated by (Uchikata, el at, 2011)

$$T_H = \frac{1}{4\pi}\left(\frac{d}{dr}f(r)\right) = \frac{1}{4\pi}\left(\frac{3r_+}{L^2} + \frac{1}{r_+}\right) \tag{4}$$

There are three solutions of $f(r) = 0$ in eq(2), which are called r_1, r_2 and r_3. One can choose one of the solution r_1 to be positive real number, called the horizon, and the rest two solutions are complex number. Because we interest only the behavior of the scalar wave outside the black holes, $r_1 \le r \le \infty$, it is convenient to define a new variable and parameters

$$x = \frac{r_1}{r}, \quad x_1 = \frac{r_1}{r_1} = 1, \quad x_2 = \frac{r_1}{r_2}, \quad x_3 = \frac{r_1}{r_3}, \tag{5}$$

The range of x is

$$0 \le x \le 1 \tag{6}$$

The function $f(r)$ can be changed to be a function of *x* as

$$\begin{aligned} f(r) &= \frac{1}{L^2 r}(r - r_1)(r - r_2)(r - r_3) \\ &= -\frac{2M}{r_1}\frac{1}{x^2}(x - x_1)(x - x_2)(x - x_3) \\ &= -\frac{2M}{r_1}\frac{1}{x^2}\left[x^3 - \frac{r_1}{2M}x^2 - \frac{r_1^3}{2ML^2}\right] \end{aligned} \tag{7}$$

From eq(7) one can fine the relation equations of the solutions of $f(r)$ as

$$x^3 - \frac{r_1}{2M}x^2 - \frac{r_1^3}{2MR^2} = 0 \tag{8}$$

There are also three solutions to eq(8) from eq(5) $x_1 = 1$ or

$$1 - \frac{r_1}{2M} - \frac{r_1^3}{2MR^2} = 0, \text{ or } \frac{r_1}{2M} = \left(1 + \frac{r_1^2}{L^2}\right)^{-1} \tag{9}$$

and the others two solutions are

$$x_2, x_3 = -\frac{r_1^2 / L^2}{2(1 + r_1^2 / L^2)} \pm i\frac{r_1 / L}{2(1 + r_1^2 / L^2)}\sqrt{4 + 3r_1^2 / R^2} \tag{10}$$

and the relation between x_1, x_2, x_3 and parameters r_1 and M are

$$x_1 + x_2 + x_3 = \frac{r_1}{2M} \ , \quad x_1x_2 + x_1x_3 + x_2x_3 = 0 \ , \quad x_1x_2x_3 = \frac{r_1^3}{2ML^2} \tag{11}$$

The wave equation of the scalar field can be obtained from the Lagrangian density eq(1), as

$$\nabla_{\mu}\nabla^{\mu}\Phi = 0 \tag{12}$$

To solve eq(12) one can separate the variables (t,r,θ,ϕ) as

$$\Phi(t,r,\theta,\phi) = e^{-i\omega t}Y_{lm}(\theta,\phi)\Psi(r) \tag{13}$$

ω is the frequency of the scalar field, $Y_{im}(\theta,\phi)$ is the harmonic function and $\Psi(r)$ is the radial part function of the scalar field in which will be investigated. After substitute Φ eq(13) into the wave equation eq(12), one can obtain the differential equation of the function $\Psi(r)$ as

$$f(r)\frac{d}{dr}\left(f(r)\frac{d\Psi(r)}{dr}\right) + \left[\omega^2 - V(r)\right]\Psi(r) = 0 \tag{14}$$

$$V(r) = f(r)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + \frac{2}{L^2}\right] = \left(\frac{r^2}{L^2} + 1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r3} + \frac{2}{L^2}\right] \tag{15}$$

$V(r)$ can be consider as an effective potential, where for large r, $V(r\to\infty)\to\infty$, V is diverged, which means that the scalar Ψ has to decay in the far away region.

$$\Psi\ (r\to\infty)\to 0 \tag{16}$$

The anther boundary condition is the only ingoing wave at the horizon ($r = r_1$ or $x_1 = 1$). The boundary condition can be considered as following. Start from defining a new variable

$$f(r)\frac{d}{dr} = \frac{d}{dr_*} \text{ or } r_* = \int\frac{1}{f(r)}dr \tag{17}$$

One can perform the integration in eq(17) and would obtain the result as

$$\frac{r_*}{r_1} = \frac{r_1^2}{2M}\ln(x-x_1)^{1/a_1}(x-x_2)^{1/a_2}(x-x_3)^{1/a_3} \tag{18}$$

Then the wave equation eq(14) would change to

$$\frac{d^2\Psi}{dr_*^2} + \left[\omega^2 - V(r)\right]\Psi(r) = 0 \tag{19}$$

At the horizon, $x_1 = 1$ or $r = r_1$, $V(r = r_1) = 0$, therefore solution to eq(19) can be approximated to

$$\Psi(r\to r_+)\sim e^{-i\omega r_*} = (x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2}(x-x_3)^{\alpha_3} \tag{20}$$

$$\alpha_1 = -i\frac{\omega r_1^2/(2M)}{(x_1-x_2)(x_1-x_3)} \tag{21.1}$$

$$\alpha_2 = \pm i\frac{\omega r_1^2/(2M)}{(x_2-x_1)(x_2-x_3)} \tag{21.2}$$

$$\alpha_3 = \pm i\frac{\omega r_1^2/(2M)}{(x_3-x_1)(x_3-x_2)} \tag{21.3}$$

The only ingoing to the horizon corresponds to the negative sign of the parameter α_1 in eq(21.1), however there are choices of the signs ($\pm$) for the parameter α_2 and α_3 in eq(21.2) and eq(21.3)

To simplify the problem let change the variable radius *r* to *x* in the wave equation eq(14) and rearrange the equation to

$$x^2(x-x_1)(x-x_2)(x-x_3)\frac{d}{dx}(x-x_1)(x-x_2)(x-x_3)\frac{d\Psi}{dx}$$
$$+\frac{r_1^2(\omega r_1)^2}{2M}x^2\Psi+(x-x_1)(x-x_2)(x-x_3)\left\{l(l+1)+\frac{r_1^2}{2M}x^2+x^3+\frac{r_1^3}{ML^2}\right\}\Psi=0 \quad (22)$$

At the far way region $r\to\infty$, or $x\to\infty$, there are two independent solutions which are proportional to $\Psi(x\to 0)\approx x^2,\ \ x^{-1}$. However from the behavior of the function $\psi(x)$ in eq(15) and eq(16) has to vanish in this region. Therefore, the only solution that survives in this zone is

$$\Psi(x\to 0)\approx x^2+.... \quad (23)$$

To solve eq(22), one reduce the problem by assuming the solution as

$$\Psi(x)=x^2(x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2}(x-x_3)^{\alpha_3}K(x) \quad (24)$$

Substitute eq(24) in to eq(22) and divide the eq(22) by $x^3(x-x_1)^{\alpha_1+1}(x-x_2)^{\alpha_2+1}(x-x_3)^{\alpha_3+1}$ and after some algebra, one would obtain the

$$x(x-x_1)\frac{d^2K}{dx^2}$$
$$+\left\{4(x-x_1)+(1+2\alpha_1)x+(x-x_1)x\left[\frac{(1+2\alpha_2)}{x-x_2}+\frac{(1+2\alpha_3)}{x-x_3}\right]\right\}\frac{dK}{dx}$$
$$+2\left\{1+2\alpha_1+(x-x_1)\left[\frac{(1+2\alpha_2)}{x-x_2}+\frac{(1+2\alpha_3)}{x-x_3}\right]\right\}K \quad (25)$$
$$+\frac{x}{(x-x_2)(x-x_3)}\left\{[l^2+l-2]\frac{r_1}{2M}+3x\right\}K$$
$$+\frac{(\alpha_1+\alpha_2+\alpha_3)^2}{(x-x_2)(x-x_3)}x^2K+\sum_{i\neq j\neq k=1}^{3}\frac{\left[\alpha_i^2-(\alpha_j+\alpha_k)^2\right]}{(x-x_2)(x-x_3)}x_ixK=0$$

where

$$\sum_{j\neq k=1}^{3}(1+2\alpha_i)(x-x_j)(x-x_k)=(1+2\alpha_1)(x-x_2)(x-x_3)+(1+2\alpha_2)(x-x_1)(x-x_3)$$
$$+(1+2\alpha_3)(x-x_1)(x-x_2)$$

$$\sum_{i\neq j\neq k=1}^{3}\left[\alpha_i^2-(\alpha_j+\alpha_k)^2\right]x_i=\left(\alpha_1^2-(\alpha_2+\alpha_3)^2\right)x_1+\left(\alpha_2^2-(\alpha_1+\alpha_3)^2\right)x_2+\left(\alpha_3^2-(\alpha_1+\alpha_2)^2\right)$$

After we have taken out the singularities at $x=1$, $x=x_2$, $x=x_3$, and $x=0$ from eq(24). Now we can approximate the solution at the horizon, $x=1$, where only going wave equation is allowed. Near *x*=1 the solution can be approximated as, where we change the variable x to

$$y\equiv 1-x \quad (26)$$

Where $y=0$ at the horizon. Therefore near the horizon eq(25) would be approximated to

$$y(1-y)\frac{d^2K}{dy^2}+\{1+2\alpha_1-[5+2\alpha_1]y\}\frac{dK}{dy}$$
$$-\left\{2(1+2\alpha_1)+\frac{1}{a_1}\left[(l^2+l-2)\frac{r_1}{2M}+3\right]\right\}K \tag{27}$$
$$-\frac{1}{a_1}(\alpha_1+\alpha_2+\alpha_3)^2K-\sum_{i\neq j\neq k=1}^{3}\left[\alpha_i^2-(\alpha_j+\alpha_k)^2\right]\frac{x_i}{a_1}K=0$$

where $a_1=(1-x_2)(1-x_3)$

Eq(27) is actually the hypergeometric functions, (Abramowitz, et al., 1972), which is of the form

$$y(1-y)\frac{d^2F(y)}{dy^2}+\{c-(1+a+b)y\}\frac{dF(y)}{dy}-abF(y)=0 \tag{28}$$

where

$$F(y)={}_2F_1(a,b;c;y)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{y^j}{j!}=1+\frac{ab}{c}y+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{y^2}{2!}+\ldots \tag{29}$$
$$(a)_0=1,\ \ (a)_1=a,\ \ (a)_2=a(a+1),\ \ (a)_n=a(a+1)\ldots(a+n-1)$$

For our case the parameters a , b and c are

$$a,b=2+\alpha_1\pm\sqrt{(2+\alpha_1)^2-J} \tag{30.1}$$
$$c=a+b-3 \tag{30.2}$$
$$J=\left\{2(1+2\alpha_1)+\frac{1}{a_1}\left[(l^2+l-2)\frac{r_1}{2M}+3\right]\right\}$$
$$+\frac{1}{a_1}(\alpha_1+\alpha_2+\alpha_3)^2+\sum_{i\neq j\neq k=1}^{3}\left[\alpha_i^2-(\alpha_j+\alpha_k)^2\right]\frac{x_i}{a_1} \tag{30.3}$$

At the horizon we approximate the solution as the hypergeometric function

$$\Psi(x)\approx x^2(x-1)^{\alpha_1}(x-x_2)^{\alpha_2}(x-x_3)^{\alpha_3}K(x)$$
$$\approx x^2(x-1)^{\alpha_1}(x-x_2)^{\alpha_2}(x-x_3)^{\alpha_3}\,{}_2F_1(a,b;a+b-3;y) \tag{31}$$

From the property of the hypergeometric functions, (Abramowitz, et al.,1972), which expand the function between two regions at the horizon $y\approx 0$ and at the far away region $x=1-y\approx 0$. At the horizon we approximate the solution as the hypergeometric function

$${}_2F_1(a,b;a+b-3;y)=\frac{\Gamma(a+b-3)}{\Gamma(a)\Gamma(b)}(1-y)^{-3}\sum_{j=0}^{3-1}\frac{(a-3)_j(b-3)_j}{j!(1-3)_j}(1-y)^j$$
$$-\frac{(-1)^m\Gamma(a+b-3)}{\Gamma(a-3)\Gamma(b-3)\Gamma(3)}\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{j!(j+3)!}(1-y)^j\begin{bmatrix}\ln(1-y)\\-\psi(j+1)-\psi(j+3+1)\\+\psi(a+j)+\psi(b+j)\end{bmatrix} \tag{32}$$

where $\Gamma(a)$ is the Gamma function and $\psi(a)$ is the Digamma function, (Abramowitz, et al., 1972)

$$\psi(a+j)=\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+a+j-1}\right]-0.5772156649... \quad (33)$$

From eq(32) $\psi(x)$ in the far away region is proportional to

$$\Psi(x)\approx x^2(x-1)^{\alpha_1}K(x)\approx x^{-1}...+x^2.....$$

where the lowest power of x in the first term in eq(32) is x^{-1} while the lowest power of x in the second term in eq(32) is x^2. From the behavior of the function $\Psi(x\approx 0)\approx x^2$, eq(23), then one needs to eliminate the first term in eq(32) by setting

$$a=-n \quad (34)$$

where $n=0,1,2,3,...$, which makes the Gamma function $\Gamma(-n)\rightarrow\infty$ diverge. However there are two terms in the second of eq(32) also diverged, one in the denominator and the other one in the dominator

$$\Gamma(a-3)=\Gamma(-n-3) \quad (35)$$

$$\psi(a+j)=\psi(-n+j)\approx\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k-n+j-1}\right]-0.5772156649... \quad (36)$$

The summation inside $\psi(-n+j)$ diverges when

$$k=n-j+1 \quad (37)$$

Actually the second term in eq(32) is finite, which can be calculated by the L'Hospital's rule method

$$\frac{\psi(-n+j)}{\Gamma(-n-3)}=(-1)^{n+3-1}(n+3)! \quad (38)$$

Our analytical solution of the scalar wave, quasinormal modes, is approximated as

$$\Psi(x)\approx x^2(x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2}(x-x_3)^{\alpha_3}\,{}_2F_1(-n,b,4,x) \quad (39)$$

The function ${}_2F_1(-n,b;4;x)$ is therefore a polynomial degree n. Finally from eq(34) and eq(30), one can calculate the frequency ω, i.e.,

$$\frac{i\omega r_1}{1+r_1^2}=-\frac{B}{2A}+\frac{1}{2A}\sqrt{B^2-4A\left(n^2+4n+2+\frac{\left[(l^2+l-2)/(1+r_1^2)+3\right]}{a_1}\right)} \quad (40)$$

$$A=\frac{1}{a_1}\left(\frac{s_1}{a_1}+\frac{s_2}{a_2}+\frac{s_3}{a_3}\right)^2+\frac{1}{a_1}\sum_{i\neq j\neq k=1}^{3}x_i\left[\frac{s_i^2}{a_i^2}-\left(\frac{s_j^2}{a_j^2}+\frac{s_k^2}{a_k^2}\right)\right],\quad B=2(n+2)\frac{s_1}{a_1}$$

$$a_1=(1-x_2)(1-x_3)\ ,\quad a_2=(x_2-1)(x_2-x_3)\ ,\quad a_1=(x_3-1)(x_3-x_2)$$

$$s_1=-1,\quad s_2=\pm1,\ \ s_3=\pm1, \quad (41)$$

s_1, s_2, and s_3 are the signs of the frequencies, α_1, α_2 and α_3 respectively, where s_1 has to be negative sign due to the boundary condition at the horizon, only the going wave travelling into the black hole, while s_2, and s_3 still have choices either to be positive or negative.

RESULTS

To compare the frequencies we parameterize the quantity $r_1\rightarrow r_1/L$ and $r_1^3/2ML^2\rightarrow(r_1^3/L^3)/(2M/L^2)$ to be dimensionless numbers. In Table 1, the numerical results, (Cardoso, et al., 2003), for the case $r_1/L=1$ and $l=0$. There are many choices of signs, s_1, s_2, and s_3, which give different results. The closest analytical result to the numerical frequency has the signs to be $s_1=-1$, $s_2=1$, and $s_3=-1$, therefore we choose those signs in the comparison as the following.

Table 1 Comparison the frequencies between the numerical work (Cardoso, et al., 2003) and our analytical results

n	The Numerical Results (ω)	n	The Analytical Results (ω)
0	2.7982-2.6712i	0	1.78154-2.5573i
		1	3.2374-4.11876i
1	4.79849-5.03757i	2	4.56443-5.60847i
		3	5.84587-7.07548i
2	6.71927-7.39449i	4	7.10558-8.53208i
3	8.46153-9.74852i	5	8.35316-9.98299i
		6	9.59327-11.4304i
4	10.6467-12.1012i	7	10.8284-12.8756i
5	12.6121-14.4533i	8	12.0602-14.31921i
		9	13.2894-15.7616i
6	14.5782-16.8049i	10	14.5168-17.2032i
		11	15.7428-18.6442i
7	16.5449-19.1562i	12	16.9676-20.0846i
8	18.5119-21.5073i	13	18.1916-21.5246i
		14	19.4148-22.9643i
9	20.4792-23.8583i	15	20.6374-24.4038i
		16	21.8595-25.843i
10	22.44671-26.20913i	17	23.0812-27.282i

The pattern of both results has some similarities, but only the *n* number different. The frequencies are proportional to the number *n*, when *n* is large. Both results can be represent in graph as

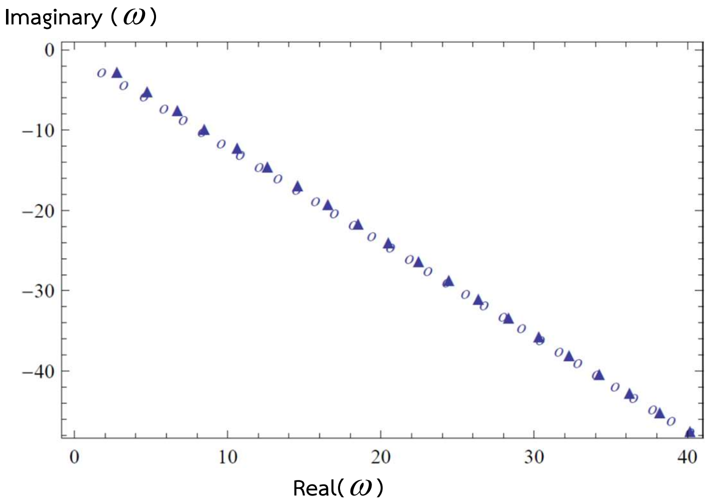

Figure 1 show the plot of the frequencies with $r_1/L=1$ and $l=0$. The horizontal axis represents the real part of frequency, while the vertical axis represents the imaginary part of frequency. The symbol □ represents the numerical frequencies and the symbol *0* represents our analytical results.

From Table 1 and Figure 1, one would notice that the analytical results have more modes than the numerical works do. The reason might be that in our analytical works we have more boundary conditions at the outer and inner horizons, $x = x_1$, $x = x_2$ and $x = x_3$ to satisfy than in the numerical works which only considers at the ingoing wave at the outer horizon, $x = x_1$. Also in the zero mode ($\mathrm{n} = 0$) there is some discrepancy differences between the analytical and the numerical frequencies. These might be due to the differential equation eq(25) can not been analytically exactly solved. Therefore, we approximate its solutions near the horizon and at far distant regions and, then match them together by the use of the hypergeometric function properties. These approximations would make the results not exactly in agreement with the numerical results. If there are some frequency differences, it should be either at small modes, e.g. $\mathrm{n} = 0$, or at large modes depending on the input parameters (in this research, i.e., r_1 / L and l). As one could see in Figure 2, we take $r_1 / L = 100$, one would expect more differences at large modes n.

The next picture, the frequency is plotted for the parameters $r_1 / L = 100$ and $l = 0$.

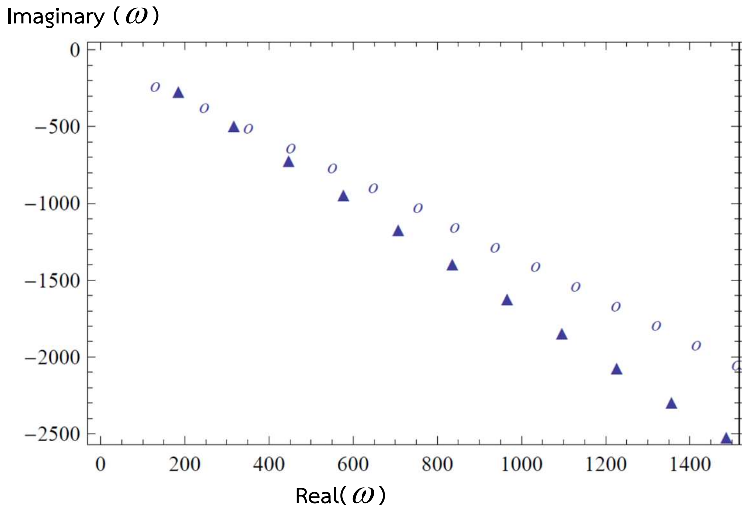

Figure 2 show the plot of the frequency with $r_1 / L = 100$ and $l = 0$. The horizontal axis represents the real part of frequency, while the vertical axis represents the imaginary part of frequency. The symbol □ represents the numerical frequencies and the symbol *0* represents our analytical results. The number $\mathrm{n} = 0,\ldots,15$.

From Figure 2, when the parameters, $r_1 / L = 100$, and the large number *n* get larger, the difference of numerical and analytical frequencies go bigger as we are expected.

DISCUSSION

1. We study the massless scalar wave in the anti de Sitter spacetime with the Schwarzschild black holes. We analytically calculate the solution, quasinormal modes and its frequencies by applying the boundary conditions, i.e. only ingoing wave at the horizon and the decaying wave far away from the black hole.

2. The approximated solution at the horizon is the hypergeometric function, eq(28). We then expand this solution to the far away zone, eq(32). Due to its massless property, the far away solution diverges. To eliminate the divergence, the allowed wave frequencies become discrete, eq(34) and eq(40).

3. We compare our analytical frequencies to the numerical result. However, there are choices of either ingoing wave or outgoing wave near singularizes other than the horizon, eq(21.2) and eq(21.3). Each choice corresponds different set of discrete frequencies. For example, the parameter $r_1/L=1$, our choice of signs at the three singularities are $s_1=-1$, $s_2=1$, and $s_3=-1$, whose analytical frequencies are very good in agreement with the numerical frequencies as in Figure 1. We also compare for the case $r_1/L=100$ as in the Figure 2, which seems having less agreement when r_1/L and the number *n* get bigger.

4. Our analytical solution is approximated at the horizon and seems working well at small r_1/L. It would be very interesting to take this approximated solution as the zero order perturbation, and then perform the first order perturbation and then compare this new result to numerical work again. If this method has good result, it would mean that one can take these solutions and its analytical method to study other kind of curved spacetimes.

CONCLUSIONS

Our result has show the similarity to the numerical work as in table 1. However, when the number n gets large the difference goes bigger. It would be interesting to take perturbation methods to collect all those vanishing terms at the horizon but not anywhere else, to see our result would get closer to the numerical work or not.

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