การทำสมการเชิงอนุพันธ์สามัญอันดับสามให้เป็นเชิงเส้น โดยการแปลงของซันด์แมนแบบวางนัยทั่วไป

Linearization of Third-order Ordinary Differential Equations by Generalized Sundman Transformation

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บทคัดย่อ

งานวิจัยนี้ศึกษาการประยุกต์ใช้การแปลงของซันด์แมนแบบวางนัยทั่วไป กล่าวคือ

$$\begin{aligned} u(t) &= f(x,y) \\ dt &= g(x,y)dx \end{aligned}$$

สำหรับปัญหาการทำให้เป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับสาม เพื่อหาเงื่อนไขจำเป็นและเพียงพอที่สมบูรณ์สำหรับสมการเชิงอนุพันธ์สามัญอันดับสาม ที่สมมูลกับรูปทั่วไปของสมการเชิงอนุพันธ์สามัญเชิงเส้นอันดับสาม

$$u''' + \beta u'' + \alpha u' + \gamma u = \eta$$

เมื่อ β, α, γ และ η เป็นค่าคงตัว สำหรับกรณี $f_x = 0$

ABSTRACT

This research considers an application of the generalized Sundman transformation, i.e.,

$$\begin{aligned} u(t) &= f(x,y), \\ dt &= g(x,y)dx \end{aligned}$$

to the linearization problem of third-order ordinary differential equations. Complete necessary and sufficient conditions for third-order ordinary differential equations to be linearizable into the general form of a linear third-order ordinary differential equation

$$u''' + \beta u'' + \alpha u' + \gamma u = \eta,$$

where β, α, γ and η are constants, are obtained for the case $f_x = 0$.

คำสำคัญ: ปัญหาการทำให้เป็นเชิงเส้น การแปลงของซันด์แมนแบบวางนัยทั่วไป
สมการเชิงอนุพันธ์สามัญอันดับสาม
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INTRODUCTION

The basic problem in modeling physical phenomena is to find solutions of differential equations. In general, these equations are very difficult to solve explicitly. The method for solving differential equations uses a change of variables that transform a given differential equation into another differential equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there arises the equation where a given differential equation can be equivalent to a linear equation; this problem is called *the linearization problem*. Transformations mostly used for solving the linearization problem are point transformations, contact transformations, reduction of order, tangent transformations and generalized Sundman transformations.

The first linearization problem for ordinary differential equations was solved by Lie (1883). He found the general form of a second-order ordinary differential equation that can be at most cubic in the first-order derivative, and provided a linearization test in terms of its coefficients. Grebot (1997) studied the linearization of a third-order ordinary differential equation by means of a restricted class of point transformations, namely, $t = \varphi(x),\ u = \psi(x, y)$, although the problem was not completely solved. Complete criteria for the linearization by means of point transformations were obtained by Ibragimov and Meleshko (2005).

The generalized Sundman transformation for second-order ordinary differential equations was considered earlier by Duarte et al. (1994) using the Laguerre form. Later, Nakpim and Meleshko (2010b) gave examples which show that the Laguerre form is not sufficient for the linearization problem via the generalized Sundman transformations. Muriel and Romero (2010) studied the class of nonlinear second-order equations that are linearizable by means of generalized Sundman transformations is identified as the class of equations admitting first integrals that are polynomials of first degree in the first-order derivative. Mustafa et al. (2013) considered the linearization problem for nonlinear second-order ordinary differential equations to the Laguerre form by means of generalized Sundman transformations. They gave a new characterization of S-linearizable equations in terms of the coefficients of ordinary differential equations and one auxiliary function. This new criterion is used to obtain the general solutions to the first integral explicitly, providing a direct alternative procedure for constructing the first integrals and Sundman transformations. The generalized Sundman transformation was also applied by Euler et al. (2003) to obtain necessary and sufficient conditions for a third-order ordinary differential equation to be equivalent to the equation $u''' = 0$. Nakpim and Meleshko (2010a) found conditions for a third-order ordinary differential equation to be equivalent to the Laguerre form $u''' + \alpha u' = 0$, where α is a constant.

In this paper, we find the necessary and sufficient conditions which allow a third-order ordinary differential equation to be mapped into a linear equation $u''' + \beta u'' + \alpha u' + \gamma u = \eta$ where $\beta,\ \alpha,\ \gamma$ and η are constants. Complete analysis of the compatibility of arising equations is given for the case $f_x = 0$.

GENERALIZED SUNDMAN TRANSFORMATION

A generalized Sundman transformation is a non-point transformation defined by the formulae

$$u(t) = f(x,y), \qquad dt = g(x,y)dx, \qquad f_y g \neq 0. \tag{1}$$

Let us explain how the generalized Sundman transformation maps one function into another. Assume that $y_0(x)$ is a given function of x. Integrating the second equation of (1), we obtain

$$t = Q(x)$$

for some function $Q(x)$. Using the inverse function theorem, we find that $x = Q^{-1}(t)$. Substituting x into the function $f(x, y_0(x))$, we obtain the transformed function

$$u_0(t) = f\left(Q^{-1}(t), y_0(Q^{-1}(t))\right).$$

Conversely, let $u_0(t)$ be a given function of t. Using the inverse function theorem, we solve the equation

$$u_0(t) = f(x,y)$$

with respect to y, where $y = \emptyset(x,t)$ for some function $\emptyset(x,t)$. Solving the ordinary differential equation

$$\frac{dt}{dx} = g(x, \emptyset(x,t)),$$

we find that $t = H(x)$ for some function $H(x)$. The function $H(x)$ can be written as an action of a functional $H = L(u_0)$. Substituting $t = H(x)$ into the function $\emptyset(x,t)$, the transformed function

$$y_0(x) = \emptyset(x, H(x))$$

is obtained.

Note that for the case $g_y = 0$, the generalized Sundman transformation becomes a point transformation. Hence we shall assume from now on that $g_y \neq 0$.

NECESSARY CONDITIONS FOR LINEARIZATION

First, we find the general form of a third-order ordinary differential equation

$$y''' = H(x, y, y', y''),$$

which can be mapped via a generalized Sundman transformation (1) into the linear equation

$$u''' + \beta u'' + \alpha u' + \gamma u = \eta \tag{2}$$

where $\beta,\ \alpha,\ \gamma$ and η are constants.

By the formulae in (1), we can check that

$$\begin{aligned}
u' &= \frac{1}{g}(f_x + y' f_y),\\
u'' &= \frac{1}{g^3}[f_y g y'' + (f_{yy} g - f_y g_y) y'^2 + (2f_{xy} g - f_x g_y - f_y g_x) y' + f_{xx} g - f_x g_x],\\
u''' &= \frac{1}{g^5}[f_y g^2 y''' + y' y'' (3f_{yy} g^2 - 4 f_y g_y g) + y'' (3 f_{xy} g^2 - f_x g_y g - 3 f_y g_x g)\\
&\quad + y'^3 (f_{yyy} g^2 - 3 f_{yy} g_x g - f_y g_{yy} g + 3 f_y g_y^2) + y'^2 (3 f_x g_y^2 - 6 f_{xy} g_y g - f_x g_{yy} g\\
&\quad - f_{yy} g_x g - 2 f_y g_{xy} g + 6 f_y g_x g_y + 3 f_{xyy} g^2) + y' (3 f_y g_x^2 + 3 f_{xxy} g^2 - 3 f_{xx} g_y g\\
&\quad - 2 f_x g_{xy} g + 6 f_x g_x g_y - f_y g_{xx} g - 6 f_{xy} g_x g) + f_{xxx} g^2 - f_x g_{xx} g - 3 f_{xx} g_x g + 3 f_x g_x^2].
\end{aligned} \tag{3}$$

Substituting $u',\ u''$ and u''' into (2), we have the equation

$$y''' + \lambda_5(x,y) y' y'' + \lambda_4(x,y) y'' + \lambda_3(x,y) y'^3 + \lambda_2(x,y) y'^2 + \lambda_1(x,y) y' + \lambda_0 = 0, \tag{4}$$

where the coefficients $\lambda_i(x,y)$ (for $i = 0,1,2,\dots,5$) are related to the functions f and g in the following way:

$$\lambda_5 = \frac{1}{f_y g}(3 f_{yy} g - 4 f_y g_y), \tag{5}$$

$$\lambda_4 = \frac{1}{f_y g}(3 f_{xy} g - f_x g_y - 3 f_y g_x + \beta f_y g^2), \tag{6}$$

$$\lambda_3 = \frac{1}{f_y g^2}(f_{yyy} g^2 - 3 f_{yy} g_y g - f_y g_{yy} g + 3 f_y g_y^2), \tag{7}$$

$$\begin{aligned}
\lambda_2 = \frac{1}{f_y g^2}[&\beta f_{yy} g^3 + g^2(3 f_{xyy} - \beta f_y g_y) - g(6 f_{xy} g_y + f_x g_{yy} + 3 f_{yy} g_x + 2 f_y g_{xy}) + 3 f_x g_y^2\\
&+ 6 f_y g_x g_y],
\end{aligned} \tag{8}$$

$$\begin{aligned}
\lambda_1 = \frac{1}{f_y g^2}[&\alpha f_y g^4 + 2\beta f_{xy} g^3 + g^2(3 f_{xxy} - \beta f_x g_y - \beta f_y g_x) - g(6 f_{xy} g_x + 3 f_{xx} g_y + 2 f_x g_{xy}\\
&+ f_y g_{xx}) + 6 f_x g_x g_y + 3 f_y g_x^2],
\end{aligned} \tag{9}$$

$$\lambda_0 = \frac{1}{f_y g^2}[g^5(\gamma f - \eta) + \alpha f_x g^4 + \beta f_{xx} g^3 + g^2(f_{xxx} - \beta f_x g_x) - g(3 f_{xx} g_x + f_x g_{xx}) + 3 f_x g_x^2]. \tag{10}$$

Equation (4) presents the necessary form of a third-order ordinary differential equation which can be mapped into a linear equation (2) via a generalized Sundman transformation.

SUFFICIENT CONDITIONS FOR LINEARIZATION

To obtain sufficient conditions, we have to solve the compatibility problem by considering (5)-(10) as an overdetermined system of partial differential equations for the functions f and g with given coefficients $\lambda_i(x,y)$ (for $i = 0,1,2,\dots,5$). Here a complete solution for the case $f_x = 0$ is given.

From equations (5), (7) and (8), we find that

$$f_{yy} = \frac{f_y}{3g}\left(4g_y + g\lambda_5\right), \quad (11)$$
$$g_{yy} = \frac{1}{3g}\left(5g_y^2 + g_y g\lambda_5 - 3\lambda_{5y}g^2 + 9g^2\lambda_3 - g^2\lambda_5^2\right), \quad (12)$$
$$g_{xy} = \frac{1}{6g}\left(9g_x g_y + g_y g\lambda_4 - 3g^2\lambda_2 + g^2\lambda_4\lambda_5\right). \quad (13)$$

Solving equations (6), (9) and (10) for β, α and η, we obtain

$$\beta = \frac{1}{g^2}(3g_x + g\lambda_4), \quad (14)$$
$$\alpha = \frac{1}{g^3}(g_{xx} + g_x\lambda_4 + g\lambda_1), \quad (15)$$
$$\eta = \frac{1}{g^3}\left(\gamma f g^3 - f_y\lambda_0\right). \quad (16)$$

Since β, α and η are constants, differentiating them with respect to x and y and comparing the mixed derivatives $(f_{yy})_x = (f_x)_{yy}$ yields the following equations:

$$g_{xx} = \frac{1}{3g}(6g_x^2 + g_x g\lambda_4 - \lambda_{4x}g^2), \quad (17)$$
$$-3g_x g_y - g_y g\lambda_4 + g^2\left(2\lambda_{4y} - 3\lambda_2 + \lambda_4\lambda_5\right) = 0, \quad (18)$$
$$g_x(6\lambda_{4x} - 18\lambda_1 + 4\lambda_4^2) + g(9\lambda_{1x} - 3\lambda_{4xx} - 4\lambda_{4x}\lambda_4) = 0, \quad (19)$$
$$-g_y[81g_x^2 + g^2(72\lambda_1 - 7\lambda_4^2 - 24\lambda_{4x}) + 54g_x g\lambda_4] + g_x g^2(36\lambda_{4y} - 45\lambda_2$$
$$+15\lambda_4\lambda_5) + g^3(36\lambda_{1y} - 18\lambda_{2x} + 6\lambda_{4x}\lambda_5 + 6\lambda_{5x}\lambda_4 - 21\lambda_2\lambda_4 + 7\lambda_4^2\lambda_5) = 0, \quad (20)$$
$$3g_x\lambda_0 - \lambda_{0x}g = 0, \quad (21)$$
$$\gamma = \frac{1}{3g^4}\left(3\lambda_{0y}g - 5g_y\lambda_0 + g\lambda_0\lambda_5\right), \quad (22)$$
$$6g_x g_y + 2g_y g\lambda_4 + g^2(3\lambda_{5x} - 6\lambda_2 + 2\lambda_4\lambda_5) = 0. \quad (23)$$

Differentiating γ (in (22)) with respect to x and y and comparing the mixed derivatives

$$(g_{yy})_x = (g_{xy})_y, \ (g_{xy})_x = (g_{xx})_y,$$

we obtain the following equations:

$$5g_y(15g_x\lambda_0 - 6\lambda_{0x}g - g\lambda_0\lambda_4) - 18g_x g(3\lambda_{0y} + \lambda_0\lambda_5) + g^2[6(3\lambda_{0xy} + \lambda_{0x}\lambda_5$$
$$+\lambda_{5x}\lambda_0) + 5\lambda_0(3\lambda_2 - \lambda_4\lambda_5)] = 0, \quad (24)$$
$$35g_y^2\lambda_0 - 14g_y g\left(3\lambda_{0y} + \lambda_0\lambda_5\right) + g^2\left(9\lambda_{0yy} + 3\lambda_{0y}\lambda_5 + 18\lambda_{5y}\lambda_0 - 45\lambda_0\lambda_3 + 5\lambda_0\lambda_5^2\right) = 0, \quad (25)$$
$$g_y^2(3g_x + g\lambda_4) + g_x g^2\left(18\lambda_{5y} - 54\lambda_3 + 6\lambda_5^2\right) + g_y g^2[12\lambda_{5x} - 6\lambda_{4y} - 5(3\lambda_2 - \lambda_4\lambda_5)]$$
$$+g^3(18\lambda_{2y} + 108\lambda_{3x} - 6\lambda_{4y}\lambda_5 - 36\lambda_{5xy} - 24\lambda_{5x}\lambda_5 - 6\lambda_2\lambda_5 - 18\lambda_3\lambda_4 + 4\lambda_4\lambda_5^2) = 0, \quad (26)$$
$$-g_y(9g_x^2 + g^2\lambda_4^2 + 6g_x g\lambda_4) - g_x g^2[12\lambda_{4y} - 9(3\lambda_2 - \lambda_4\lambda_5)] + g^3[12\lambda_{4xy} - 18\lambda_{2x}$$
$$+6\lambda_{4x}\lambda_5 + \lambda_4(3\lambda_2 - \lambda_4\lambda_5) + 6\lambda_{5x}\lambda_4] = 0. \quad (27)$$

From equation (18), we have

$$g_x = \frac{g}{3g_y}\left(2\lambda_{4y}g - g_y\lambda_4 - 3g\lambda_2 + g\lambda_4\lambda_5\right). \tag{28}$$

Substituting g_x into (19), (20), (21), (23), (24), (25), (26) and (27) and comparing the mixed derivatives

$g_{xx} = (g_x)_x$ and $g_{xy} = (g_x)_y$, we find that

$$\begin{aligned}
&g_y^3(27\lambda_{1x} - 9\lambda_{4xx} - 18\lambda_{4x}\lambda_4 + 18\lambda_1\lambda_4 - 4\lambda_4^3) + g_y^2 g[6\lambda_{4xy}\lambda_4 - 27\lambda_{2xx} + 9\lambda_{2x}\lambda_4 \\
&+18\lambda_{4xxy} + 9\lambda_{4xx}\lambda_5 + 3\lambda_{4x}(15\lambda_2 - 2\lambda_{4y} - 6\lambda_4\lambda_5) + 16\lambda_{4y}(\lambda_4^2 - 3\lambda_1) \\
&+(18\lambda_1 - 10\lambda_4^2)(3\lambda_2 - \lambda_4\lambda_5) + g_y g^2[9\lambda_{2x}\left(4\lambda_{4y} - 9\lambda_2 + 3\lambda_4\lambda_5\right) + 3\lambda_{4xy}(15\lambda_2 \\
&-6\lambda_{4y} - 5\lambda_4\lambda_5) + 3\lambda_{4x}\lambda_5(9\lambda_2 - 4\lambda_{4y} - 3\lambda_4\lambda_5) + \lambda_{4y}(10\lambda_{4y}\lambda_4 - 57\lambda_2\lambda_4 + 19\lambda_4^2\lambda_5) \\
&+81\lambda_2^2\lambda_4 - 54\lambda_2\lambda_4^2\lambda_5 + 9\lambda_4^3\lambda_5^2] + g^3[14\lambda_{4y}^2(3\lambda_2 - \lambda_4\lambda_5) + 4\lambda_{4y}\lambda_4\lambda_5(6\lambda_2 - \lambda_4\lambda_5) \\
&-12\lambda_{4y}(\lambda_{4y}^2 + 3\lambda_2^2)] = 0,
\end{aligned} \tag{29}$$

$$\begin{aligned}
&2g_y^2(3\lambda_{4x} - 9\lambda_1 + 2\lambda_4^2) + g_y g[9\lambda_{1y} + 9\lambda_{2x} - 9\lambda_{4xy} - 3\lambda_{4x}\lambda_5 - 2\lambda_{4y}\lambda_4 - 9\lambda_2\lambda_4 \\
&+3\lambda_4^2\lambda_5] + g^2[6\lambda_{4y}^2 - 7\lambda_{4y}(3\lambda_2 - \lambda_4\lambda_5) + 18\lambda_2^2 - 12\lambda_2\lambda_4\lambda_5 + 2\lambda_4^2\lambda_5^2] = 0,
\end{aligned} \tag{30}$$

$$g_y\lambda_6 + g\lambda_0\left(2\lambda_{4y} - 3\lambda_2 + \lambda_4\lambda_5\right) = 0, \tag{31}$$

$$\lambda_{5x} = \frac{4}{3}\left(3\lambda_2 - \lambda_4\lambda_5 - \lambda_{4y}\right), \tag{32}$$

$$\begin{aligned}
&5g_y^2\lambda_6 + g_y g[4\lambda_{4y}\lambda_0 - 3\lambda_{6y} - 6\lambda_0\lambda_2 + 2\lambda_0\lambda_4\lambda_5 - \lambda_5\lambda_6] + g^2[(3\lambda_{0y} + \lambda_0\lambda_5)(3\lambda_2 - \lambda_4\lambda_5) \\
&-2\lambda_{4y}(3\lambda_{0y} + \lambda_0\lambda_5)] = 0,
\end{aligned} \tag{33}$$

$$35g_y^2\lambda_0 - 14g_y g(3\lambda_{0y} + \lambda_0\lambda_5) + g^2[9\lambda_{0yy} + 3\lambda_{0y}\lambda_5 + 18\lambda_{5y}\lambda_0 - 5\lambda_0(9\lambda_3 - \lambda_5^2)] = 0, \tag{34}$$

$$\begin{aligned}
&-5g_y^2(2\lambda_{4y} - 3\lambda_2 + \lambda_4\lambda_5) + g_y g[54\lambda_{3x} - 63\lambda_{2y} + 24\lambda_{4yy} + 37\lambda_{4y}\lambda_5 + 21\lambda_{5y}\lambda_4 \\
&-17\lambda_5(3\lambda_2 - \lambda_4\lambda_5)] + g^2[6\lambda_{4y}\lambda_{5y} - 18\lambda_{4y}\lambda_3 + 2\lambda_{4y}\lambda_5^2 + \\
&(9\lambda_3 - 3\lambda_{5y} - \lambda_5^2)(3\lambda_2 - \lambda_4\lambda_5)] = 0,
\end{aligned} \tag{35}$$

$$\begin{aligned}
&g_y[6\lambda_{4xy} - 9\lambda_{2x} + 3\lambda_{4x}\lambda_5 - 2\lambda_{4y}\lambda_4 + 3\lambda_4(3\lambda_2 - \lambda_4\lambda_5)] + g[7\lambda_{4y}(3\lambda_2 - \lambda_4\lambda_5)) - 6\lambda_{4y}^2 \\
&-18\lambda_2^2 + 12\lambda_2\lambda_4\lambda_5 - 2\lambda_4^2\lambda_5^2] = 0,
\end{aligned} \tag{36}$$

$$\begin{aligned}
&g_y^2[10\lambda_{4y} - 15\lambda_2 + 5\lambda_4\lambda_5] + g_y g\left[9\lambda_{2y} - 6\lambda_{4yy} - \lambda_{4y}\lambda_5 - 3\lambda_{5y}\lambda_4 - \lambda_5(3\lambda_2 - \lambda_4\lambda_5)\right] \\
&+g^2[\lambda_{4y}(18\lambda_3 - 6\lambda_{5y} - 2\lambda_5^2) + (3\lambda_{5y} - 9\lambda_3 + \lambda_5^2)(3\lambda_2 - \lambda_4\lambda_5)] = 0,
\end{aligned} \tag{37}$$

where $\lambda_6 = -(\lambda_{0x} + \lambda_0\lambda_4)$. Further analysis of the compatibility depends on λ_6.

Case 1.1 $\lambda_6 \neq 0$

Suppose $\lambda_6 \neq 0$. From equation (31), it follows that

$$g_y = \frac{g\lambda_0}{\lambda_6}\left(3\lambda_2 - \lambda_4\lambda_5 - 2\lambda_{4y}\right), \tag{38}$$

when $\lambda_0 \neq 0$ and $3\lambda_2 - \lambda_4\lambda_5 - 2\lambda_{4y} \neq 0$.

Substituting g_y into (29), (30), (33), (34), (35), (36), (37) and comparing the mixed derivatives $(g_x)_y = (g_y)_x$, $g_{yy} = (g_y)_y$ and $g_{xy} = (g_y)_x$, we obtain the following conditions:

$$\begin{aligned}
\lambda_{4xx} = \frac{1}{9\lambda_0^3}&[\lambda_0^3(27\lambda_{1x} - 18\lambda_{4x}\lambda_4 + 18\lambda_1\lambda_4 - 4\lambda_4^3) + \lambda_0^2(18\lambda_1\lambda_6 - 24\lambda_4^2\lambda_6 - 18\lambda_{4x}\lambda_6 - 9\lambda_{6xx} \\
&-27\lambda_{6x}\lambda_4) - 27\lambda_{6x}\lambda_0\lambda_6 - 36\lambda_0\lambda_4\lambda_6^2 - 16\lambda_6^3],
\end{aligned} \tag{39}$$

$$\lambda_{1y} = \frac{1}{18\lambda_0\lambda_6}[9\lambda_{2x}\lambda_0\lambda_6 + 3\lambda_{4x}\lambda_0(8\lambda_{4y}\lambda_0 - \lambda_5\lambda_6) + (36\lambda_0^2\lambda_1 - 12\lambda_{4x}\lambda_0^2 - 9\lambda_{6x}\lambda_0 - 8\lambda_0^2\lambda_4^2 - 10\lambda_6^2 - 15\lambda_0\lambda_4\lambda_6)(3\lambda_2 - \lambda_4\lambda_5) + \lambda_{4y}(18\lambda_{6x}\lambda_0 - 72\lambda_0^2\lambda_1 + 16\lambda_0^2\lambda_4^2 + 34\lambda_0\lambda_4\lambda_6 + 21\lambda_6^2)], \quad (40)$$

$$\lambda_{0y} = \frac{\lambda_0}{\lambda_6}(2\lambda_{4y}\lambda_0 + \lambda_{6y} - 3\lambda_0\lambda_2 + \lambda_0\lambda_4\lambda_5), \quad (41)$$

$$\lambda_{4yy} = \frac{1}{3\lambda_0\lambda_6}[9\lambda_{3x}\lambda_0\lambda_6 + 5\lambda_{4y}\lambda_0(\lambda_5\lambda_6 - 4\lambda_{4y}\lambda_0) + (20\lambda_{4y}\lambda_0^2 - 4\lambda_0\lambda_5\lambda_6)(3\lambda_2 - \lambda_4\lambda_5) + 3\lambda_{5y}\lambda_6^2 - 15\lambda_0^2\lambda_2(3\lambda_2 - 2\lambda_4\lambda_5) - 5\lambda_0^2\lambda_4^2\lambda_5^2 - \lambda_6^2(9\lambda_3 - \lambda_5^2)], \quad (42)$$

$$\lambda_{6yy} = \frac{1}{9\lambda_6}[(320\lambda_{4y}\lambda_0^2 + 60\lambda_{6y}\lambda_0 + 20\lambda_0\lambda_5\lambda_6)(3\lambda_2 - \lambda_4\lambda_5) - 40\lambda_{4y}\lambda_0(8\lambda_{4y}\lambda_0 + 3\lambda_{6y} + \lambda_5\lambda_6) - 27\lambda_{5y}\lambda_6^2 - 3\lambda_{6y}\lambda_5\lambda_6 - 720\lambda_0^2\lambda_2^2 + 80\lambda_0^2\lambda_4\lambda_5(6\lambda_2 - \lambda_4\lambda_5) + 8\lambda_6^2(9\lambda_3 - \lambda_5^2)], \quad (43)$$

$$\lambda_{4xy} = \frac{1}{6\lambda_0}[9\lambda_{2x}\lambda_0 - 3\lambda_{4x}\lambda_0\lambda_5 + \lambda_{4y}(2\lambda_0\lambda_4 - 3\lambda_6) + (2\lambda_6 - 3\lambda_0\lambda_4)(3\lambda_2 - \lambda_4\lambda_5)], \quad (44)$$

$$\lambda_{2y} = \frac{1}{9\lambda_0\lambda_6}[18\lambda_{3x}\lambda_0\lambda_6 + \lambda_{4y}\lambda_0(11\lambda_5\lambda_6 - 20\lambda_{4y}\lambda_0 + 20\lambda_{4y}\lambda_0^2(3\lambda_2 - \lambda_4\lambda_5) + 3\lambda_{5y}\lambda_6(\lambda_0\lambda_4 + \lambda_6) - (7\lambda_0\lambda_5\lambda_6 + 15\lambda_0^2\lambda_2)(3\lambda_2 - 2\lambda_4\lambda_5) - 5\lambda_0^2\lambda_4^2\lambda_5^2 - \lambda_6^2(9\lambda_3 - \lambda_5^2)], \quad (45)$$

$$(3\lambda_{6x}\lambda_0 + 4\lambda_0\lambda_4\lambda_6 + 4\lambda_6^2)(3\lambda_2 - \lambda_4\lambda_5) - 2\lambda_{4y}(3\lambda_{6x}\lambda_0 + 4\lambda_0\lambda_4\lambda_6 + 4\lambda_6^2) = 0. \quad (46)$$

Case 1.2 $\lambda_6 = 0$

If $\lambda_6 = 0$, then equations (29), (30), (31), (33), (35), (36) and (37) become

$$g_y^3[1296(3\lambda_{1x} - \lambda_{4xx}) - 2592\lambda_4(\lambda_{4x} - \lambda_1) - 576\lambda_4^3] + g_y^2 g[(432\lambda_1 - 144\lambda_{4x} - 96\lambda_4^2 - 36\lambda_{9x} - 12\lambda_4\lambda_9)(\lambda_8 - \lambda_9) + 144(3\lambda_{7x} + 2\lambda_4\lambda_7) + 36\lambda_7(\lambda_8 + 3\lambda_9)] + g^3[(\lambda_8^2 - \lambda_9^2)(\lambda_8 - 7\lambda_9)] = 0, \quad (47)$$

$$48g_y^2(3\lambda_{4x} - 9\lambda_1 + 2\lambda_4^2) + g_y g[216\lambda_{1y} - 36\lambda_{9x} + 6\lambda_4\lambda_8 - 54\lambda_4\lambda_9 - 36\lambda_7] + g^2[\lambda_8^2 - \lambda_9^2] = 0, \quad (48)$$

$$\lambda_0(2\lambda_{4y} - \lambda_9) = 0, \quad (49)$$

$$\lambda_{0y}(2\lambda_{4y} - \lambda_9) = 0, \quad (50)$$

$$5g_y^2(\lambda_8 - \lambda_9) + g_y g[324\lambda_{3x} - 12\lambda_{8y} - 42\lambda_{9y} - 8\lambda_5\lambda_8 - 46\lambda_5\lambda_9] + g^2[(9\lambda_3 - 3\lambda_{5y})(\lambda_8 - \lambda_9) - \lambda_5^2(\lambda_8 + \lambda_9)] = 0, \quad (51)$$

$$24g_y\lambda_7 - g(\lambda_8^2 - \lambda_9^2) = 0, \quad (52)$$

$$-5g_y^2(\lambda_8 - \lambda_9) + g_y g[3\lambda_{8y} - 3\lambda_{9y} - \lambda_5(\lambda_8 - \lambda_9)] + g^2[(3\lambda_{5y} - 9\lambda_3 + \lambda_5^2)(\lambda_8 - \lambda_9)] = 0, \quad (53)$$

Where $\lambda_7 = -9\lambda_{2x} + 6\lambda_{4xy} + 3\lambda_{4x}\lambda_5 - 2\lambda_{4y}\lambda_4 + 3\lambda_4(3\lambda_2 - \lambda_4\lambda_5)$,

$\lambda_8 = -12\lambda_{4y} + 21\lambda_2 - 7\lambda_4\lambda_5$ and $\lambda_9 = 3\lambda_2 - \lambda_4\lambda_5$.

Note that (50) can be omitted as a result of (49). Further analysis of the compatibility depends on λ_7.

Case 1.2.1 $\lambda_7 \neq 0$

Suppose $\lambda_7 \neq 0$. From equation (52), we have

$$g_y = \frac{g}{24\lambda_7}(\lambda_8^2 - \lambda_9^2) \quad (54)$$

where $\lambda_8^2 - \lambda_9^2 \neq 0$.

Substituting g_y into (47), (48), (34), (51) and (53) and comparing the mixed derivatives $(g_x)_y = (g_y)_x$, $g_{yy} = (g_y)_y$ and $g_{xy} = (g_y)_x$, we find the following conditions:

$$\begin{aligned}
&(27\lambda_{1x} - 9\lambda_{4xx} + 18\lambda_1\lambda_4)(\lambda_8^2 - \lambda_9^2)^3 - (18\lambda_{4x}\lambda_4 + 4\lambda_4^3)(\lambda_8^6 - \lambda_9^6) \\
&+(12\lambda_4^3\lambda_8^2\lambda_9^2 + 54\lambda_{4x}\lambda_4\lambda_8^2\lambda_9^2)(\lambda_8^2 - \lambda_9^2) + (48\lambda_{4x}\lambda_7\lambda_8^2\lambda_9^2 - 24\lambda_{4x}\lambda_7\lambda_8^4 - 24\lambda_{4x}\lambda_7\lambda_9^4 \\
&+72\lambda_1\lambda_7\lambda_8^4 - 144\lambda_1\lambda_7\lambda_8^2\lambda_9^2 + 72\lambda_1\lambda_7\lambda_9^4 - 16\lambda_4^2\lambda_7\lambda_8^4 + 32\lambda_4^2\lambda_7\lambda_8^2\lambda_9^2 - 16\lambda_4^2\lambda_7\lambda_9^4 + 48\lambda_4\lambda_7^2\lambda_8^3 \\
&-48\lambda_4\lambda_7^2\lambda_8\lambda_9^2 + 240\lambda_7^3\lambda_8^2 - 240\lambda_7^3\lambda_9^2)(\lambda_8 - \lambda_9) + 72\lambda_{7x}\lambda_7(\lambda_8^4 - 2\lambda_8^2\lambda_9^2 + \lambda_9^4) - 144\lambda_{9x}\lambda_7^2(\lambda_8^3 \\
&-\lambda_8^2\lambda_9 - \lambda_8\lambda_9^2 + \lambda_9^3) = 0, \quad (55)
\end{aligned}$$

$$\begin{aligned}
&\left(108\lambda_{1y}\lambda_7 - 18\lambda_{9x}\lambda_7 - 27\lambda_4\lambda_7\lambda_9 + 3\lambda_4\lambda_7\lambda_8 - 6\lambda_7^2\right)(\lambda_8^2 - \lambda_9^2) \\
&+(3\lambda_{4x} - 9\lambda_1 + 2\lambda_4^2)(\lambda_8^2 - \lambda_9^2)^2 = 0, \quad (56)
\end{aligned}$$

$$\begin{aligned}
&1728\lambda_7^2(3\lambda_{0yy} + \lambda_{0y}\lambda_5) - (1008\lambda_{0y}\lambda_7 + 336\lambda_0\lambda_5\lambda_7)(\lambda_8^2 - \lambda_9^2) + 10368\lambda_{5y}\lambda_0\lambda_7^2 \\
&-2880\lambda_0\lambda_7^2(9\lambda_3 - \lambda_5^2) + 35\lambda_0(\lambda_8^4 + \lambda_9^4) - 70\lambda_0\lambda_8^2\lambda_9^2 = 0, \quad (57)
\end{aligned}$$

$$\begin{aligned}
&\left(7776\lambda_{3x}\lambda_7 - 288\lambda_{8y}\lambda_7 - 1008\lambda_{9y}\lambda_7 - 1104\lambda_5\lambda_7\lambda_9 - 192\lambda_5\lambda_7\lambda_8\right)(\lambda_8^2 - \lambda_9^2) \\
&+(5184\lambda_3\lambda_7^2 - 1728\lambda_{5y}\lambda_7^2 - 576\lambda_5^2\lambda_7^2 + 5\lambda_8^4 - 10\lambda_8^2\lambda_9^2 + 5\lambda_9^4)(\lambda_8 - \lambda_9) = 0, \quad (58)
\end{aligned}$$

$$\begin{aligned}
&(1728\lambda_{5y}\lambda_7^2 - 5184\lambda_3\lambda_7^2 + 576\lambda_5^2\lambda_7^2 - 24\lambda_5\lambda_7\lambda_8^2 + 24\lambda_5\lambda_7\lambda_9^2 - 5\lambda_8^4 + 10\lambda_8^2\lambda_9^2 - 5\lambda_9^4)(\lambda_8 - \lambda_9) \\
&+\left(72\lambda_{8y}\lambda_7 - 72\lambda_{9y}\lambda_7\right)(\lambda_8^2 - \lambda_9^2) = 0, \quad (59)
\end{aligned}$$

$$\begin{aligned}
&3\lambda_{7x}(\lambda_8^4 - \lambda_9^4) + 6\lambda_{7x}\lambda_8\lambda_9(\lambda_8^2 - \lambda_9^2) - 96\lambda_{7y}\lambda_7^2(\lambda_8 + \lambda_9) + (6\lambda_{9x}\lambda_7\lambda_9 - 6\lambda_{8x}\lambda_7\lambda_8)(\lambda_8^2 + \lambda_9^2) \\
&+96\lambda_7^3(\lambda_{8y} + \lambda_{9y}) - 12\lambda_{8x}\lambda_7\lambda_8\lambda_9(\lambda_8 - \lambda_9) + 2\lambda_7^2\lambda_8^2(\lambda_8 - 5\lambda_9) - 2\lambda_7^2\lambda_9^2(13\lambda_8 + 7\lambda_9) = 0, \quad (60)
\end{aligned}$$

$$\begin{aligned}
&-288\lambda_7^2\left(3\lambda_{5y} - 9\lambda_3 + \lambda_5^2\right) + \left(12\lambda_5\lambda_7 + 36\lambda_{7y}\right)(\lambda_8^2 - \lambda_9^2) - 72\lambda_7\left(\lambda_{8y}\lambda_8 - \lambda_{9y}\lambda_9\right) \\
&+(\lambda_8^2 - \lambda_9^2)^2 = 0, \quad (61)
\end{aligned}$$

$$3\lambda_{7x}(\lambda_8^2 - \lambda_9^2) - 6\lambda_{8x}\lambda_7(\lambda_8 - \lambda_9) - 2\lambda_7^2(\lambda_8 + 5\lambda_9) = 0. \quad (62)$$

Case 1.2.2 $\lambda_7 = 0$

If $\lambda_7 = 0$, then (52) yields

$$\lambda_8^2 - \lambda_9^2 = 0. \quad (63)$$

From equations (47) and (48), we have

$$\begin{aligned}
&-6g_y^2(3\lambda_{10x} + 2\lambda_{10}\lambda_4) + g_y g(2\lambda_{10}\lambda_9 - 2\lambda_{10}\lambda_8) + g^2[9\lambda_{1y}\lambda_9 - 9\lambda_{1y}\lambda_8 \\
&-\lambda_{10y}\lambda_8 + \lambda_{10y}\lambda_9 + 2\lambda_4\lambda_8\lambda_9 - 2\lambda_4\lambda_9^2] = 0, \quad (64)
\end{aligned}$$

$$2g_y\lambda_{10} - g\lambda_{10y} = 0, \quad (65)$$

where $\lambda_{10} = 3\lambda_{4x} - 9\lambda_1 + 2\lambda_4^2$. Further analysis of the compatibility depends on λ_{10}.

Case 1.2.2.1 $\lambda_{10} \neq 0$

Suppose $\lambda_{10} \neq 0$. From equation (65), it follows that

$$g_y = \frac{g}{2\lambda_{10}}\lambda_{10y} \quad (66)$$

and $\lambda_{10y} \neq 0$.

Substituting g_y into (64), (34), (51) and (53) and comparing the mixed derivatives $(g_x)_y = (g_y)_x$, $g_{yy} = (g_y)_y$ and $g_{xy} = (g_y)_x$, we find the following conditions:

$$\left(4\lambda_{10}^2\lambda_4\lambda_9 - 18\lambda_{1y}\lambda_{10}^2 - 4\lambda_{10y}\lambda_{10}^2\right)(\lambda_8 - \lambda_9) - 9\lambda_{10x}\lambda_{10y}^2 - 6\lambda_{10y}^2\lambda_{10}\lambda_4 = 0, \quad (67)$$

$$\lambda_{0yy} = \frac{1}{36\lambda_{10}^2}[12\lambda_{0y}\lambda_{10}(7\lambda_{10y} - \lambda_{10}\lambda_5) - 7\lambda_{10y}\lambda_0(5\lambda_{10y} - 4\lambda_{10}\lambda_5) - 72\lambda_{5y}\lambda_0\lambda_{10}^2 + 20\lambda_0\lambda_{10}^2(9\lambda_3 - \lambda_5^2)], \quad (68)$$

$$(5\lambda_{10y}^2 - 12\lambda_{5y}\lambda_{10}^2 + 36\lambda_{10}^2\lambda_3 - 4\lambda_{10}^2\lambda_5^2)(\lambda_8 - \lambda_9) + 4\lambda_{10y}\lambda_{10}(162\lambda_{3x} - 6\lambda_{8y} - 21\lambda_{9y} - 4\lambda_5\lambda_8 - 23\lambda_5\lambda_9) = 0, \quad (69)$$

$$\left(12\lambda_{5y}\lambda_{10}^2 - 5\lambda_{10y}^2 - 2\lambda_{10y}\lambda_{10}\lambda_5 - 36\lambda_{10}^2\lambda_3 + 4\lambda_{10}^2\lambda_5^2\right)(\lambda_8 - \lambda_9) + 6\lambda_{10y}\lambda_{10}\left(\lambda_{8y} - \lambda_{9y}\right) = 0, \quad (70)$$

$$\lambda_{10xy} = \frac{1}{54\lambda_{10y}^2\lambda_{10}}[54\lambda_{10x}\lambda_{10y}^3 + 7\lambda_{10y}^2\lambda_{10}^2\lambda_8 - 25\lambda_{10y}^2\lambda_{10}^2\lambda_9 + (4\lambda_{10y}\lambda_{10}^3\lambda_5 - 24\lambda_{5y}\lambda_{10}^4 + 72\lambda_{10}^4\lambda_3 - 8\lambda_{10}^4\lambda_5^2)(\lambda_8 - \lambda_9) - 12\lambda_{10y}\lambda_{10}^3(\lambda_{8y} - \lambda_{9y})], \quad (71)$$

$$\lambda_{10yy} = \frac{1}{3\lambda_{10}}\left[4\lambda_{10y}^2 + \lambda_{10y}\lambda_{10}\lambda_5 - 6\lambda_{5y}\lambda_{10}^2 + 2\lambda_{10}^2(9\lambda_3 - \lambda_5^2)\right]. \quad (72)$$

Case 1.2.2.2 $\lambda_{10} = 0$

If $\lambda_{10} = 0$, then (64) yields

$$(2\lambda_4\lambda_9 - 9\lambda_{1y})(\lambda_8 - \lambda_9) = 0. \quad (73)$$

From equation (34), we find that

$$35g_y^2\lambda_0 - 14g_y g\lambda_{11} + g^2\lambda_{12} = 0, \quad (74)$$

where $\lambda_{11} = 3\lambda_{0y} + \lambda_0\lambda_5$ and $\lambda_{12} = 3\lambda_{11y} + 15\lambda_{5y}\lambda_0 - 45\lambda_0\lambda_3 + 5\lambda_0\lambda_5^2$.

Further analysis of the compatibility depends on λ_0.

Case 1.2.2.2.1 $\lambda_0 \neq 0$

Since $\lambda_0 \neq 0$, it follows from (74) that

$$g_y^2 = \frac{g}{35\lambda_0}\left(14g_y\lambda_{11} - g\lambda_{12}\right). \quad (75)$$

Substituting ${g_y}^2$ into (51) and (53), we obtain

$$14g_y\lambda_{13} + g[\lambda_{14}(7\lambda_0\lambda_{15} + \lambda_{12})] = 0, \quad (76)$$

$$7g_y\left(\lambda_0\lambda_{14}\lambda_5 - 3\lambda_{14y}\lambda_0 + 2\lambda_{11}\lambda_{14}\right) - g\lambda_{14}(7\lambda_0\lambda_{15} + \lambda_{12}) = 0, \quad (77)$$

Where $\lambda_{13} = 162\lambda_{3x}\lambda_0 - 6\lambda_{8y}\lambda_0 - 21\lambda_{9y}\lambda_0 - 4\lambda_0\lambda_5\lambda_8 - 23\lambda_0\lambda_5\lambda_9 + \lambda_{11}\lambda_8 - \lambda_{11}\lambda_9$,

$\lambda_{14} = -\lambda_8 + \lambda_9$ and $\lambda_{15} = 3\lambda_{5y} - 9\lambda_3 + \lambda_5^2$.

Further analysis of the compatibility depends on λ_{13}.

Case 1.2.2.2.1.1 $\lambda_{13} \neq 0$

Suppose $\lambda_{13} \neq 0$. From equation (76), it follows that

$$g_y = -\frac{g\lambda_{14}}{14\lambda_{13}}(7\lambda_0\lambda_{15} + \lambda_{12}), \quad (78)$$

where $7\lambda_0\lambda_{15} + \lambda_{12} \neq 0$.

Substituting g_y into (77) and comparing the mixed derivatives $(g_x)_y = (g_y)_x$, $g_{yy} = (g_y)_y$ and $g_{xy} = (g_y)_x$ yield the following conditions:

$$(42\lambda_{14y}\lambda_{13} - 14\lambda_{13}\lambda_{14}\lambda_5)(7\lambda_0\lambda_{15} + \lambda_{12}) + 7\lambda_0\lambda_{14}\lambda_{15}(35\lambda_0\lambda_{14}\lambda_{15} + 10\lambda_{12}\lambda_{14}) + 5\lambda_{12}^2\lambda_{14}^2 - 196\lambda_{13}^2\lambda_{15} = 0, \quad (79)$$

$$42\lambda_{12y}\lambda_{13} - 42\lambda_{12}(\lambda_{13y} + \lambda_0\lambda_{14}\lambda_{15}) - 294\lambda_0\lambda_{13y}(\lambda_{15} - \lambda_{15y}\lambda_{13}) - 147\lambda_0^2\lambda_{14}\lambda_{15}^2 - 98\lambda_{13}\lambda_{15}(\lambda_0\lambda_5 - \lambda_{11}) - 3\lambda_{12}^2\lambda_{14} = 0, \quad 80)$$

$$21\lambda_{13}\lambda_{14}(7\lambda_{15y}\lambda_0 + \lambda_{12y}) + (21\lambda_{14y}\lambda_{13} - 21\lambda_{13y}\lambda_{14})(7\lambda_0\lambda_{15} + \lambda_{12}) + 49\lambda_0\lambda_{14}\lambda_{15}(\lambda_0\lambda_{14}\lambda_{15} - 2\lambda_{13}\lambda_5) - 14\lambda_{15}(7\lambda_{13}^2 - \lambda_0\lambda_{12}\lambda_{14}^2) + 7\lambda_{13}\lambda_{14}(7\lambda_{11}\lambda_{15} - \lambda_{12}\lambda_5) + \lambda_{12}^2\lambda_{14}^2 = 0, \quad (81)$$

$$18\lambda_{12x}\lambda_{13}\lambda_{14} + (18\lambda_{14x}\lambda_{13} - 18\lambda_{13x}\lambda_{14})(7\lambda_0\lambda_{15} + \lambda_{12}) + 126\lambda_0\lambda_{13}\lambda_{14}(\lambda_{15x} - \lambda_{15}\lambda_4) - 7\lambda_{13}^2(6\lambda_9 - \lambda_{14}) = 0. \quad (82)$$

Case 1.2.2.2.1.2 $\lambda_{13} = 0$

From equation (76) and (77), we obtain

$$\lambda_{14}(7\lambda_0\lambda_{15} + \lambda_{12}) = 0, \quad (83)$$

$$\lambda_0\lambda_{14}\lambda_5 - 3\lambda_{14y}\lambda_0 + 2\lambda_{11}\lambda_{14} = 0. \quad (84)$$

Case 1.2.2.2.2 $\lambda_0 = 0$

From equation (51) and (53), we obtain

$$5g_y^2\lambda_{16} + 2g_y g\lambda_{17} + g^2\lambda_{16}\lambda_{13} = 0, \quad (85)$$

$$-10g_y^2\lambda_{16} + g_y g\left[162\lambda_{3x} - 27\lambda_{9y} - 6\lambda_{16}\lambda_5 - \lambda_{17} - 27\lambda_5\lambda_9\right] - 2g^2\lambda_{16}\lambda_{18} = 0, \quad (86)$$

Where $\lambda_{16} = \lambda_8 - \lambda_9$,

$\lambda_{17} = -6\lambda_{16y} + 162\lambda_{3x} - 27\lambda_{9y} - 4\lambda_{16}\lambda_5 - 27\lambda_5\lambda_9$,

$\lambda_{18} = -3\lambda_{5y} + 9\lambda_3 - \lambda_5^2$.

Further analysis of the compatibility depends on λ_{16}.

Case 1.2.2.2.2.1 $\lambda_{16} \neq 0$

Suppose $\lambda_{16} \neq 0$. From equation (85), we have

$$g_y^2 = -\frac{g}{5\lambda_{16}}\left(2g_y\lambda_{17} + g\lambda_{16}\lambda_{18}\right). \quad (87)$$

Substituting ${g_y}^2$ into (86), we obtain the condition

$$54\lambda_{3x} - 9\lambda_{9y} - 2\lambda_{16}\lambda_5 + \lambda_{17} - 9\lambda_5\lambda_9 = 0. \quad (88)$$

Case 1.2.2.2.2.2 $\lambda_{16} = 0$

If $\lambda_{16} = 0$, then we obtain from (85) and (86) the conditions

$$\lambda_{17} = 0, \quad (89)$$

$$\lambda_{3x} = \frac{1}{6}\left(\lambda_{9y} + \lambda_5\lambda_9\right). \quad (90)$$

CONCLUSIONS

We are now ready to state our main theorems.

Theorem 1. *Any third-order ordinary differential equation*

$$y''' = H(x,y,y',y'') \tag{91}$$

which yields a linear equation $u''' + \beta u'' + \alpha u' + \gamma u = \eta$ *via a generalized Sundman transformation*

$$u = f(x,y), \qquad dt = g(x,y)dx, \qquad f_y g \neq 0 \tag{92}$$

has to be of the form

$$y''' + \lambda_5(x,y)y'y'' + \lambda_4(x,y)y'' + \lambda_3(x,y)y'^3 + \lambda_2(x,y)y'^2 + \lambda_1(x,y)y' + \lambda_0(x,y) = 0. \tag{93}$$

Theorem 2. *Sufficient conditions for equation* (93) *to be linearizable via a generalized Sundman transformation with* $f_x = 0$ *are as follows:*

(a) If $\lambda_6 \neq 0$, then the conditions are (32), (39), (40), (41), (42), (43), (44), (45) and (46).

(b) For $\lambda_6 = 0$,

(b.1) if $\lambda_7 \neq 0$, then the conditions are (32), (49), (55), (56), (57), (58), (59), (60), (61) and (62);

(b.2) if $\lambda_7 = 0$ and $\lambda_{10} \neq 0$, then the conditions are (32), (49), (63), (67), (68), (69), (70), (71) and (72);

(b.3) if $\lambda_7 = 0$, $\lambda_{10} = 0$, $\lambda_0 \neq 0$ and $\lambda_{13} \neq 0$, then the conditions are (32), (49), (63), (73), (79), (80), (81) and (82);

(b.4) if $\lambda_7 = 0$, $\lambda_{10} = 0$, $\lambda_0 \neq 0$ and $\lambda_{13} = 0$, then the conditions are (32), (49), (63), (73), (83) and (84);

(b.5) if $\lambda_7 = 0$, $\lambda_{10} = 0$, $\lambda_0 = 0$, $\lambda_{16} \neq 0$, then the conditions are (32), (49), (63), (73) and (88);

(b.6) if $\lambda_7 = 0$, $\lambda_{10} = 0$, $\lambda_0 = 0$, $\lambda_{16} = 0$, then the conditions are (32), (49), (63), (73), (89) and (90).

Theorem 3. *Provided that the sufficient conditions in Theorem 2 are satisfied, the transformation (92) mapping equation (93) into a linear equation* $u''' + \beta u'' + \alpha u' + \gamma u = \eta$ *is obtained by solving the following compatible system of equations for the functions* $f(y)$ *and* $g(x,y)$*:*

(a) For $\lambda_6 \neq 0$, (11), (28) and (38);

(b) For $\lambda_6 = 0$,

(b.1) (11), (28) and (54);

(b.2) (11), (28) and (66);

(b.3) (11), (28) and (78);

(b.4) (11), (28) and (75);

(b.5) (11), (28) and (87);

(b.6) (11) and (28).

Examples

Example 1. Consider the nonlinear third-order ordinary differential equation

$$y''' + \frac{2}{y}y'y'' - \frac{3}{x}y'' - \frac{1}{y^2}y'^3 - \frac{2}{xy}y'^2 + \frac{3}{x^2}y' + x^3y = 0. \tag{94}$$

Note that equation (94) is of the form (93) with the coefficients

$$\lambda_5 = \frac{2}{y}, \quad \lambda_4 = -\frac{3}{x}, \quad \lambda_3 = -\frac{1}{y^2}, \quad \lambda_2 = -\frac{2}{xy}, \quad \lambda_1 = \frac{3}{x^2}, \quad \lambda_0 = x^3y. \tag{95}$$

We can check that the coefficients in (95) satisfy the conditions for the case (b.4) in Theorem 2. Thus, equation (94) is linearizable via a generalized Sundman transformation.

To find the functions f and g, we have to solve the system of equations

$$\begin{aligned} g_x &= \frac{g}{18g_y}(g\lambda_{14} - 6g_y\lambda_4), \\ g_y^2 &= \frac{g}{35\lambda_0}(14g_y\lambda_{11} - g\lambda_{12}), \\ f_{yy} &= \frac{f_y}{3g}(4g_y + g\lambda_5). \end{aligned} \tag{96}$$

Here we take the simplest solution which satisfies (96), that is, $f = \frac{y^3}{3}$ and $g = xy$. Thus, the generalized Sundman transformation for linearization is

$$u = \frac{y^3}{3}, \quad dt = xydx. \tag{97}$$

From equations (14), (15), (16) and (22), we find that $\beta = 0$, $\alpha = 0$, $\eta = -1$ and $\gamma = 0$. Hence, equation (94) is mapped by the transformation (97) into the linear equation

$$u''' + 1 = 0. \tag{98}$$

The general solution to equation (98) is

$$u(t) = -\frac{t^3}{6} + C_1\frac{t^2}{2} + C_2t + C_3, \tag{99}$$

where C_1, C_2 and C_3 are constants. Applying the generalized Sundman transformation (98) to equation (99), we obtain that the general solution to equation (94) is

$$y(x) = \left(-\frac{\phi^3}{2} + C_1\frac{\phi^2}{2} + C_2\phi + C_3\right)^{\frac{1}{3}},$$

where the function $t = \phi(x)$ is a solution to the equation

$$\frac{dt}{dx} = x\left(-\frac{t^3}{2} + C_1\frac{t^2}{2} + C_2t + C_3\right)^{\frac{1}{3}}.$$

Example 2. Consider the nonlinear third-order ordinary differential equation

$$y''' + \frac{2}{y}y'y'' - \frac{1}{y^2}y'^3 - \frac{y^2}{9}y' + y = 0. \quad (100)$$

Note that equation (100) is of the form (93) with the coefficients

$$\lambda_5 = \frac{2}{y}, \quad \lambda_4 = 0, \quad \lambda_3 = -\frac{1}{y^2}, \quad \lambda_2 = 0, \quad \lambda_1 = -\frac{y^2}{9}, \quad \lambda_0 = y. \quad (101)$$

We can check that the coefficients in (101) do not satisfy the conditions of linearizability by point and contact transformations (Ibragimov and Meleshko, 2005). Nevertheless, they obey the conditions for the case (b.2) in Theorem 2. Thus, equation (100) is linearizable via a generalized Sundman transformation.

To find the functions f and g, we have to solve the system of equations

$$\begin{aligned} g_x &= \frac{g}{9\lambda_{10y}}\left(\lambda_{10}\lambda_9 - 3\lambda_{10y}\lambda_4 - \lambda_{10}\lambda_8\right), \\ g_y &= \frac{g}{2\lambda_{10}}\lambda_{10y}, \\ f_{yy} &= \frac{f_y}{3\lambda_{10}}\left(2\lambda_{10y} + \lambda_{10}\lambda_5\right). \end{aligned} \quad (102)$$

Here we take the simplest solution which satisfies (102), that is, $f = \frac{y^3}{3}$ and $g = y$.

Thus, the generalized Sundman transformation for linearization is

$$u = \frac{y^3}{3}, \quad dt = ydx. \quad (103)$$

From equations (14), (15), (16) and (22), we find that $\beta = 0$, $\alpha = -\frac{1}{9}$, $\eta = -1$ and $\gamma = 0$.

Hence, equation (100) is mapped by the transformation (103) into the linear equation

$$u''' - \frac{1}{9}u' + 1 = 0. \quad (104)$$

The general solution to equation (104) is

$$u(t) = C_1 + C_2 e^{-\frac{t}{3}} + C_3 e^{\frac{t}{3}} + 9t, \quad (105)$$

where C_1, C_2 and C_3 are constants. Applying the generalized Sundman transformation (104) to equation (105), we obtain that the general solution to equation (100) is

$$y(x) = \left(C_1 + C_2 e^{-\frac{\varphi}{3}} + C_3 e^{\frac{\varphi}{3}} + 27\varphi\right)^{\frac{1}{3}},$$

where the function $t = \varphi(x)$ is a solution to the equation

$$\frac{dt}{dx} = \left(C_1 + C_2 e^{-\frac{t}{3}} + C_3 e^{\frac{t}{3}} + 27t\right)^{\frac{1}{3}}.$$

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