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จำนวนเส้นเชื่อมน้อยที่สุดที่ลบออกจากกราฟเพื่อให้ได้กราฟระนาบ

Minimum Number of Edges whose Removal Gives a Planar Graph

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บทคัดย่อ

กำหนดให้ Ce(G) คือจำนวนเส้นเชื่อมน้อยที่สุดที่ลบออกจากกราฟ G แล้วทำให้กราฟย่อยที่เหลือเป็นกราฟเชิงระนาบ เราศึกษา Ce(G) เมื่อ G เป็นกราฟแบบบริบูรณ์ หรือกราฟ k ส่วนแบบบริบูรณ์

ABSTRACT

Let Ce(G) be the minimum number of edges whose removal from a graph G gives a planar graph. We investigate Ce(G) for complete graphs and complete k-partite graphs.

คำสำคัญ: กราฟ กราฟระนาบ กราฟแบบบริบูรณ์ กราฟ k ส่วนแบบบริบูรณ์

Keywords: Graph, Planar graph, Complete graph, Complete k-partite graph

Introduction

A *graph* G is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. Sometimes, we write G(V(G),E(G)), V(G) and E(G) instead of G, V and E respectively. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We write $H \subseteq G$ and say that "G contains H". A *complete graph* is a graph where every vertex is adjacent to every other vertex. A complete graph on n vertices is denoted by K_n. A *k-partite graph* is a graph whose vertices can be partitioned into k disjoint sets $V_1, \ldots, V_k$ so that no two vertices within the same set are adjacent. We call $V_1, \ldots, V_k$ *partite sets* of G. If $V_1, \ldots, V_k$ are partite sets of a k-partite graph then

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graph is denoted by $G(V_1,\ldots,V_k)$. A graph $G(V_1,\ldots,V_k)$ is a *complete k-partite graph* if $uv \in E(G)$ for each u and v in different partite sets. If $|V_i| = n_i$ for $1 \le i \le k$, then a complete k-partite graph is denoted by $K_{n_1,\ldots,n_k}$. A graph G is *plane* if it is drawn in a plane without edges crossing and a graph G is *planar* if it can be drawn into a plane graph. The following fact about planar graphs is well-known and can be found in standard texts about graph theory such as (West, 2001; Nakprasit, 2011).

Theorem (West, 2001) [Euler's formula] If G is a finite, connected plane graph, then $n(G) - e(G) + f(G) = 2$ where $n(G)$ is the number of vertices, $e(G)$ is the number of edges and $f(G)$ is the number of faces (regions bounded by edges, including the outer, infinitely large region).

If G is a connected plane graph with at least 3 vertices, then $e(G) \le 3n(G) - 6$ and $e(G) = 3n(G) - 6$ if and only if all faces of G are C_3's, where C_3 is a 3-cycle (see, (West,2001)).

If G is a planar bipartite graph with at least 3 vertices, then $e(G) \le 2n(G) - 4$ (see, (West, 2001)).

Corollary A (West, 2001) If G has K_5 or $K_{3,3}$ as a subgraph, then G is not a planar graph.

In this paper, we investigate the minimum number of edges whose removal from a graph gives a planar graph for complete graphs and complete k-partite graphs.

Main Results

Definition 1 Let $Ce(G)$ be the minimum number of edges whose removal from a graph G gives a planar graph.

Obsevation: Let $H \subseteq G$. If $Ce(H) = k$ then $Ce(G) \ge k$.

Theorem 2 The complete bipartite graph $K_{m,n}$ $(m \le n)$ is not a planar graph if and only if $K_{m,n}$ is not isomorphic to $K_{1,n}$ and $K_{2,n}$.

Proof. Let $V_1 = \{x_{11},\ldots,x_{1m}\}$ and $V_2 = \{x_{21},\ldots,x_{2n}\}$ be partite sets of $K_{m,n}$.

($\Rightarrow$) We will prove by contrapositive. Suppose that $K_{m,n}$ is isomorphic to $K_{1,n}$ or $K_{2,n}$. We can draw $K_{m,n}$ in the plane as in Figure 1.

($\Leftarrow$) Let $K_{m,n}$ be not isomorphic to $K_{1,n}$ and $K_{2,n}$. Then $m \ge 3$ and $n \ge 3$. Note that $K_{m,n}$ $(3 \le m \le n)$ has $K_{3,3}$ as a subgraph.

Thus, Corollary A implies that $K_{m,n}$ is not a planar graph. □

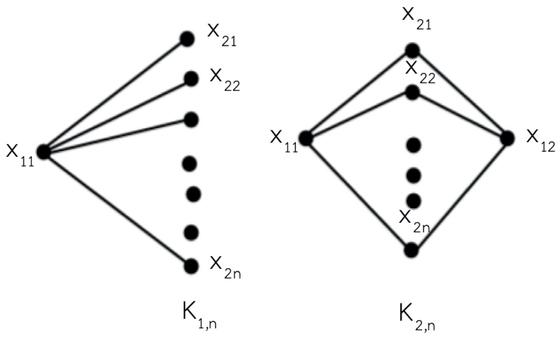

Figure 1. $K_{1,n}$ and $K_{2,n}$

Theorem 3 The complete 3-partite graph K_{n_1,n_2,n_3} $(n_1 \le n_2 \le n_3)$ is not a planar graph if and only if $n_2 \ge 2$ and $n_3 \ge 3$.

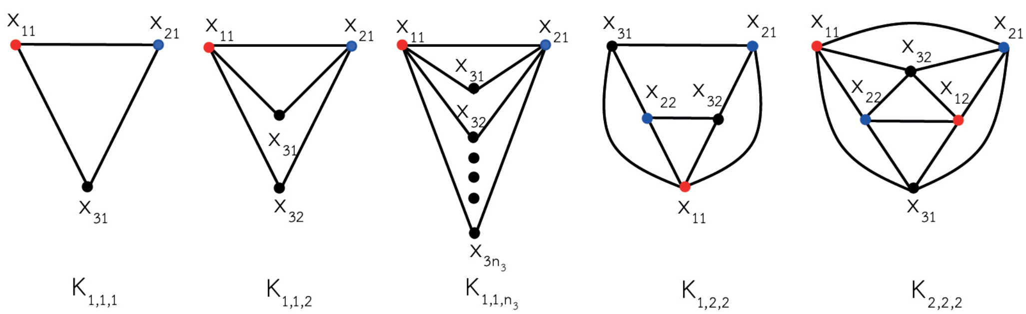

Figure 2. $K_{1,1,1}$, $K_{1,1,2}$, $K_{1,1,n_3}$, $K_{1,2,2}$ and $K_{2,2,2}$

Proof. Let $V_1 = \{x_{11},\ldots,x_{1n_1}\}$, $V_2 = \{x_{21},\ldots,x_{2n_2}\}$ and $V_3 = \{x_{31},\ldots,x_{3n_3}\}$ be partite sets of K_{n_1,n_2,n_3}.

($\Rightarrow$) We will prove by contrapositive. We consider two cases.

Case 1: $n_2 < 2$. We obtain that $K_{1,1,n_3}$ $(n_3 \ge 2)$ is able to be drawn as a plane graph shown in Figure 2.

Case 2: $n_3 < 3$. We obtain that $n_2 = 1,2$ implies that K_{n_1,n_2,n_3} is $K_{1,1,1}, K_{1,1,2}, K_{1,2,2}$ or $K_{2,2,2}$ which is able to be drawn as a plane graph shown in Figure 2.

From case 1 and case 2, K_{n_1,n_2,n_3} where $n_2 < 2$ or $n_3 < 3$ is a planar graph.

($\Leftarrow$) Consider K_{n_1,n_2,n_3} where $n_2 \ge 2$ and $n_3 \ge 3$.

Note that K_{n_1,n_2,n_3} has $K_{3,3}$ as a subgraph. Thus, Corollary A implies that K_{n_1,n_2,n_3} is not a planar graph. □

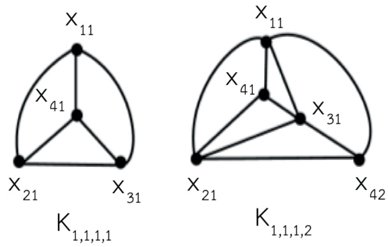

Figure 3. $K_{1,1,1,1}$ and $K_{1,1,1,2}$

Theorem 4 The complete 4-partite graph K_{n_1,n_2,n_3,n_4}, $n_1 \leq n_2 \leq n_3 \leq n_4$ is not a planar graph if and only if K_{n_1,n_2,n_3,n_4} is not isomorphic to $K_{1,1,1,1}$ and $K_{1,1,1,2}$.

Proof. Let $V_1 = \{x_{11},\ldots,x_{1n_1}\}$, $V_2 = \{x_{21},\ldots,x_{2n_2}\}$, $V_3 = \{x_{31},\ldots,x_{3n_3}\}$ and $V_4 = \{x_{41},\ldots,x_{4n_4}\}$ be partite sets of K_{n_1,n_2,n_3,n_4}.

($\Rightarrow$) We will prove by contrapositive. Suppose that K_{n_1,n_2,n_3,n_4} is isomorphic to $K_{1,1,1,1}$ or $K_{1,1,1,2}$. So we can draw $K_{1,1,1,1}$ and $K_{1,1,1,2}$ as a plane graph shown in Figure 3. Therefore K_{n_1,n_2,n_3,n_4} is a planar graph.

($\Leftarrow$) Consider K_{n_1,n_2,n_3,n_4} that is not isomorphic to $K_{1,1,1,1}$ and $K_{1,1,1,2}$. Note that K_{n_1,n_2,n_3,n_4} has $K_{3,3}$ as a subgraph. Thus, Corollary A implies that K_{n_1,n_2,n_3,n_4} is not a planar graph. □

Theorem 5 The complete k-partite graph $K_{n_1,n_2,n_3,\ldots,n_k}$ is not a planar graph where $n_1 \leq n_2 \leq n_3 \leq \cdots \leq n_k$ and $k \geq 5$.

Proof. Note that $K_{n_1,n_2,n_3,\ldots,n_k}$ where $k \geq 5$ has K_5 as a subgraph.
Thus, Corollary A implies that $K_{n_1,n_2,n_3,\ldots,n_k}$ is not a planar graph. □

Observation: It is obvious that $Ce(G) = 0$ if and only if G is a planar graph.

From the definition of Ce(G), Theorem 2, 3 and 4, we obtain the following results (see Figure 4).

1. $Ce(K_n) = 0$ if and only if $n = 1,2,3,4$.
2. $Ce(K_{m,n}) = 0 (m \leq n)$ if and only if $m = 1,2$.
3. $Ce(K_{n_1,n_2,n_3}) = 0$ $(n_1 \leq n_2 \leq n_3)$ if and only if $n_2 < 2$ or $n_3 < 3$.
4. $Ce(K_{n_1,n_2,n_3,n_4}) = 0$ $(n_1 \leq n_2 \leq n_3 \leq n_4)$ if and only if $(n_1,n_2,n_3,n_4) = (1,1,1,1)$ or $(1,1,1,2)$.

Moreover, it is easy to show for complete graphs and complete k-partite graphs G that

5. $Ce(G) = 1$ if and only if G is isomorphic to $K_5, K_{3,3}, K_{1,2,3}, K_{1,1,1,3}$ or $K_{1,1,2,2}$ (see Figure 5).

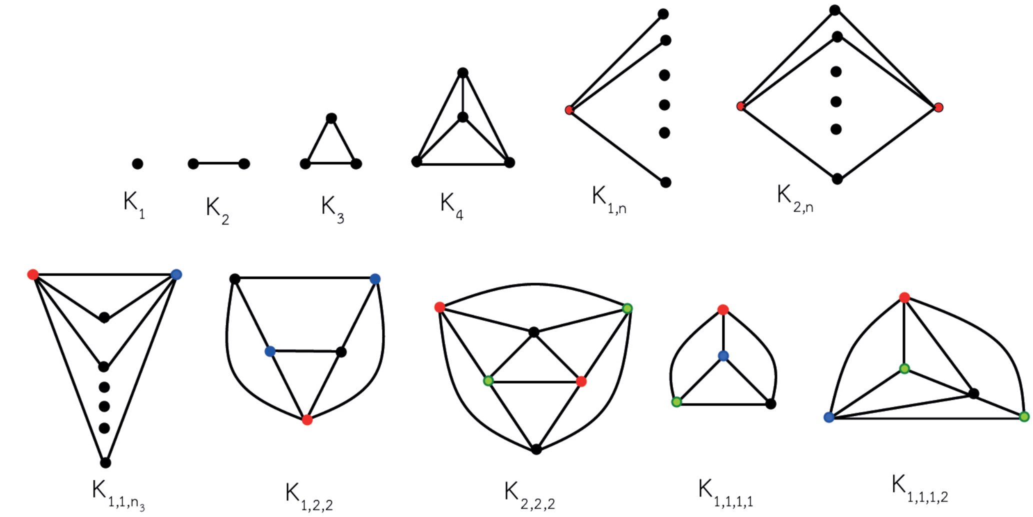

Figure 4. Complete graphs and complete k-partite graphs G with $Ce(G) = 0$

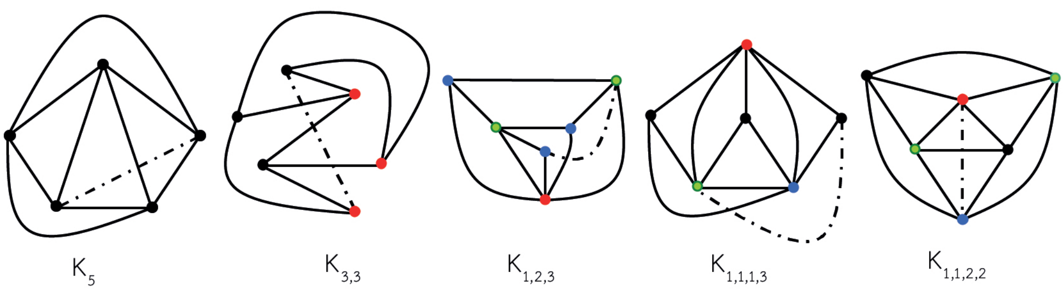

Figure 5. Complete graphs and complete k-partite graphs G with $Ce(G) = 1$

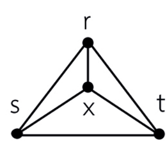

Figure 6. C_3

Lemma 6 For $n \geq 3$, there is a planar graph G with n vertices, $3n-6$ edges and at least one face that is C_3.

proof. We prove by mathematical induction on the number of vertices.

Base. Consider $n = 3$. We have a planar graph $G = C_3$ with 3 vertices, 3 edges and inner face C_3.

Induction step. Let $n \geq 4$. Suppose there is a planar graph G' with $n-1$ vertices, $3(n-1)-6$ edges and there is a face rstr as C_3 shown in Figure 6. Next, we will draw a graph G from G' by adding vertex x in the face rstr and adding edges xr, xs, and xt. Since we add 3 edges, $e(G) = 3(n-1)-6+3 = 3n-6$, G is a plane graph and there is xrtx as a face C_3.

By mathematical induction, we have a planar graph G with n vertices, $3n-6$ edges and at least one face that is C_3. □

Theorem 7 For $n \geq 5$, $Ce(K_n) = \binom{n}{2} - (3n-6)$.

Proof. We can eliminate $\binom{n}{2} - (3n-6)$ edges from K_n to obtain a planar graph G in Lemma 6, so

$$Ce(K_n) \leq \binom{n}{2} - (3n-6). \quad (1)$$

The Euler's formula implies that if G is a planar graph then G has at most $3n-6$ edges.

We have to remove at least $\binom{n}{2} - (3n-6)$ edges from K_n, so

$$Ce(K_n) \geq \binom{n}{2} - (3n-6). \quad (2)$$

From (1) and (2), we obtain $Ce(K_n) = \binom{n}{2} - (3n-6)$. □

Lemma 8 For $3 \leq m \leq n$, there is a planar bipartite graph $G(V_1, V_2)$ with $|V_1| = m$, $|V_2| = n$ and $e(G) = 2(m+n) - 4$.

Proof. Let m and n be positive integers with $3 \leq m \leq n$. Let $V_1 = \{x_{11}, \ldots, x_{1m}\}$, $V_2 = \{x_{21}, \ldots, x_{2n}\}$ be partite sets of $G(V_1, V_2)$. We can draw a planar bipartite graph $G(V_1, V_2)$ with $|V_1| = m$, $|V_2| = n$ and $e(G) = 2(m+n) - 4$ as follows:

Step 1. Draw a vertex in a partite set V_1 in the horizontal line by putting $\lceil \frac{m}{2} \rceil$ vertices on the left side and the others vertices are on the right side. Leave some space between a vertex $x_{1\lceil \frac{m}{2} \rceil}$ and a vertex $x_{1(\lceil \frac{m}{2} \rceil + 1)}$.

Step 2. Draw a vertex in a partite set V_2 in the vertical line between a vertex $x_{1\lceil \frac{m}{2} \rceil}$ and a vertex $x_{1(\lceil \frac{m}{2} \rceil + 1)}$ where $\lceil \frac{n}{2} \rceil$ vertices are over vertices in a partite set V_1 and the others vertices are under vertices in a partite set V_1.

Step 3. Draw edges $x_{21}x_{1i}$ and edges $x_{2n}x_{1i}$ for $i \in \{1, 2, 3, \ldots, m\}$. In this step, we have 2m edges.

Step 4. Draw edges $x_{1\lceil \frac{m}{2} \rceil} x_{2j}$ and edges $x_{1(\lceil \frac{m}{2} \rceil + 1)} x_{2j}$ for $j \in \{2, 3, \ldots, n-1\}$. In this step, we have $2(n-2)$ edges.

Thus, we have a planar bipartite graph $G(V_1, V_2)$ with $e(G) = 2(m + n) - 4$, as shown in Figure 7.

□

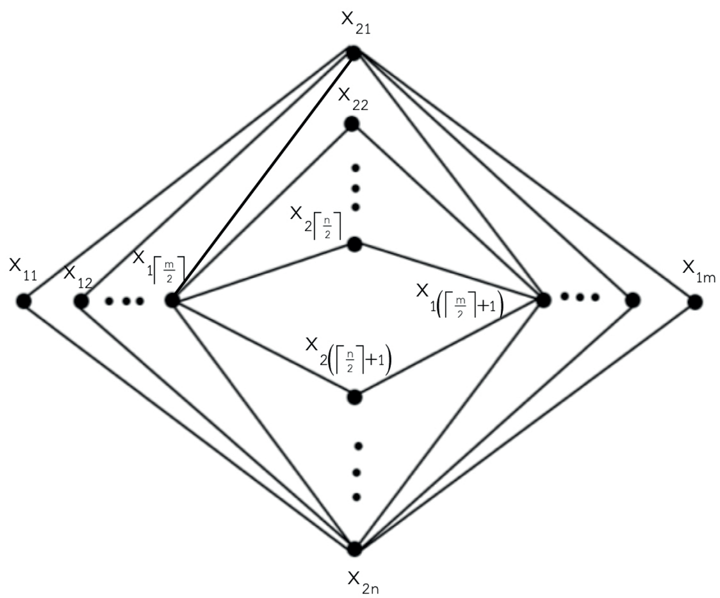

Figure 7. $K_{m,n}$

Theorem 9 For $2 \le m \le n$, $Ce(K_{m,n}) = (m-2)(n-2)$.

Proof. We can remove $mn - [2(m+n) - 4]$ edges from $K_{m,n}$ to obtain a planar bipartite graph G in Lemma 8, so

$$Ce(K_{m,n}) \le mm - 2(m+n) + 4 = (m-2)(n-2). \quad (3)$$

The Euler's formula implies that if G is a planar bipartite graph then G has at most $2(m+n) - 4$ edges. We have to remove at least $mm - 2(m+n) + 4$ edges from $K_{m,n}$.

Thus, we obtain that

$$Ce(K_{m,n}) \ge mm - 2(m+n) + 4 = (m-2)(n-2). \quad (4)$$

From (3) and (4), we have $Ce(K_{m,n}) = (m-2)(n-2)$. □

Lemma 10 For $1 \le n_2 \le n_3$, there is a planar 3-partite graph $G(V_1, V_2, V_3)$ with $|V_1| = 1$, $|V_2| = n_2$, $|V_3| = n_3$, and $e(G) = 3n_2 + 2n_3 - 2$.

Proof. Let n_2 and n_3 be positive integers with $1 \le n_2 \le n_3$.

Let $V_1 = \{x_{11}\}, V_2 = \{x_{21}, \ldots, x_{2n_2}\}$, and $V_3 = \{x_{31}, \ldots, x_{3n_3}\}$ be partite sets of $G(V_1, V_2, V_3)$. We can draw a planar 3-partite graph $G(V_1, V_2, V_3)$ with $|V_1| = 1, |V_2| = n_2, |V_3| = n_3$, as follows:

Step 1. Draw a vertex x_{11} and a vertex x_{21} in the horizontal line and leave some space between a vertex x_{11} and a vertex x_{21}.

Step 2. Draw vertices $x_{31},\dots,x_{3n_3}$ in the vertical line between a vertex x_{11} and a vertex x_{21}.

Step 3. For each $i \in \{2,3,\dots,n_2\}$, draw a vertex x_{2i} between vertices $x_{3(i-1)}$ and x_{3i}.

Step 4. Draw edges $x_{11}x_{2i}$ for all $i \in \{1,2,3,\dots,n_2\}$ and draw edges $x_{11}x_{3j}$ for all $j \in \{1,2,3,\dots,n_3\}$. In this step, we have $n_2 + n_3$ edges.

Step 5. Draw edges $x_{2i}x_{3(i-1)}$ and edges $x_{2i}x_{3i}$ for all $i \in \{2,3,\dots,n_2\}$. In this step, we have $2(n_2 - 1)$ edges.

Step 6. Draw edges $x_{21}x_{3j}$ for all $j \in \{1,2,3,\dots,n_3\}$. In this step, we have n_3 edges.

Therefore, we obtain a planar 3-partite graph $G(V_1,V_2,V_3)$ with $e(G) = n_2 + n_3 + n_3 + 2(n_2 - 1) = 3n_2 + 2n_3 - 2$ as shown in Figure 8. □

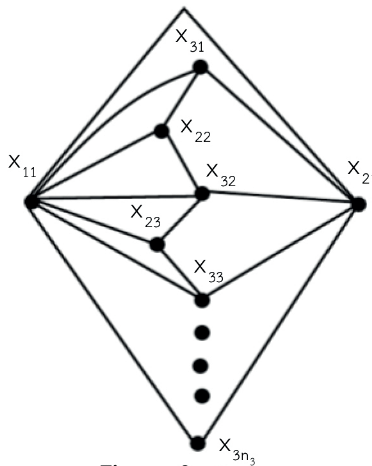

Figure 8. K_{1,n_2,n_3}

Theorem 11 For $1 \le n_2 \le n_3$, $Ce(K_{1,n_2,n_3}) = (n_2 - 1)(n_3 - 2)$.

Proof. We can remove $n_2n_3 - 2n_2 - n_3 + 2$ edges from K_{1,n_2,n_3} to obtain a planar 3-partite graph G in Lemma 10, so

$$Ce(K_{1,n_2,n_3}) \le n_2n_3 - 2n_2 - n_3 + 2 = (n_2 - 1)(n_3 - 2). \quad (5)$$

Note that K_{1+n_2,n_3} is a subgraph of K_{1,n_2,n_3} and $Ce(K_{1+n_2,n_3}) = (n_2 - 1)(n_3 - 2)$. So

$$Ce(K_{1,n_2,n_3}) \ge (n_2 - 1)(n_3 - 2). \quad (6)$$

From (5) and (6), we have $Ce(K_{1,n_2,n_3}) = (n_2 - 1)(n_3 - 2)$. □

Lemma 12 For $2 \leq n_2 \leq n_3$, there is a planar 3-partite $G(V_1, V_2, V_3)$ with $|V_1| = 2$, $|V_2| = n_2$, $|V_3| = n_3$, and $e(G) = 4n_2 + 2n_3$.

Proof. Let n_2 and n_3 be positive integers with $2 \leq n_2 \leq n_3$.

Let $V_1 = \{x_{11}, x_{12}\}, V_2 = \{x_{21}, \ldots, x_{2n_2}\}$, and $V_3 = \{x_{31}, \ldots, x_{3n_3}\}$ be partite sets of $G(V_1, V_2, V_3)$. We can draw a planar 3-partite graph $G(V_1, V_2, V_3)$ with $|V_1| = 2$, $|V_2| = n_2$, $|V_3| = n_3$, and $e(G) = 4n_2 + 2n_3$ as follows:

Step 1. Draw a vertex x_{11} and a vertex x_{12} in the horizontal line and leave some space between a vertex x_{11} and a vertex x_{12}.

Step 2. Draw vertices $x_{31}, \ldots, x_{3n_3}$ in the vertical line between a vertex x_{11} and a vertex x_{12}.

Step 3. For each $i \in \{1, 2, 3, \ldots, n_2\}$, draw a vertex x_{2i} between vertices x_{3i} and $x_{3(i+1)}$.

Step 4. Draw edges $x_{11}x_{2i}$ for all $i \in \{1, 2, 3, \ldots, n_2\}$ and draw edges $x_{11}x_{3j}$ for all $j \in \{1, 2, 3, \ldots, n_3\}$. In this step, we have $n_2 + n_3$ edges.

Step 5. Draw edges $x_{12}x_{2i}$ for all $i \in \{1, 2, 3, \ldots, n_2\}$ and draw edges $x_{12}x_{3j}$ for all $j \in \{1, 2, 3, \ldots, n_3\}$. In this step, we have $n_2 + n_3$ edges.

Step 6. Draw edges $x_{2i}x_{3i}$ and edges $x_{2i}x_{3(i+1)}$ for all $i \in \{1, 2, 3, \ldots, n_2\}$. In this step, we have $2n_2$ edges.

Therefore, we obtain a planar 3-partite $G(V_1, V_2, V_3)$ with $e(G) = (n_2 + n_3) + (n_2 + n_3) + 2n_2 = 4n_2 + 2n_3$ as shown in Figure 9. □

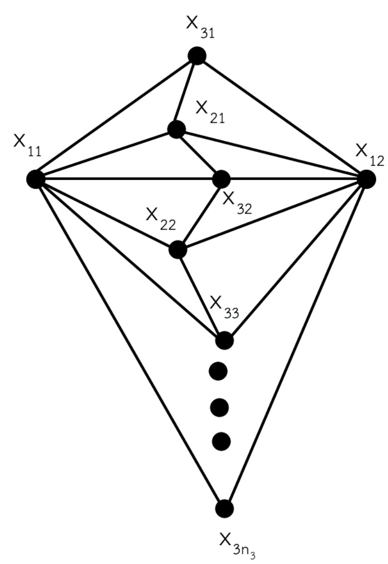

Figure 9. K_{2,n_2,n_3}

Theorem 13 For $2 \leq n_2 \leq n_3$, $Ce(K_{2,n_2,n_3}) = n_2(n_3 - 2)$.

Proof. We can remove $n_2(n_3 - 2)$ edges from K_{2,n_2,n_3} to obtain a planar 3-partite graph G in Lemma 12, so

$$Ce(K_{2,n_2,n_3}) \leq n_2(n_3 - 2). \tag{7}$$

Note that K_{2+n_2,n_3} is a subgraph of K_{2,n_2,n_3} and $Ce(K_{2+n_2,n_3}) = n_2(n_3 - 2)$. So

$$Ce(K_{2,n_2,n_3}) \geq n_2(n_3 - 2). \tag{8}$$

From (7) and (8), we have $Ce(K_{2,n_2,n_3}) = n_2(n_3 - 2)$. □

Lemma 14 For a positive integer n, there is a planar 3-partite graph $G(V_1, V_2, V_3)$ with $|V_1| = |V_2| = |V_3| = n$, $e(G) = 9n - 6$ and $f(G) = 6n - 4$.

Proof. Let n be a positive integer, $V_1 = \{x_{11}, \ldots, x_{1n}\}, V_2 = \{x_{21}, \ldots, x_{2n}\}$, and $V_3 = \{x_{31}, \ldots, x_{3n}\}$ be partite sets of $G(V_1, V_2, V_3)$. We can draw a planar 3-partite graph $G(V_1, V_2, V_3)$ with $|V_1| = |V_2| = |V_3| = n$, $e(G) = 9n - 6$ and $f(G) = 6n - 4$ as follows:

Step 1. Draw a vertex x_{11} and a vertex x_{21} in the horizontal line and leave some space between a vertex x_{11} and a vertex x_{21}.

Step 2. Draw vertices $x_{31}, \ldots, x_{3n}$ in the vertical line between a vertex x_{11} and a vertex x_{21}.

Step 3. For each $j \in \{2, 3, \ldots, n\}$, draw a vertex x_{2j} between vertices $x_{3(j-1)}$ and x_{3j} on the left side.

Step 4. For each $i \in \{2, 3, \ldots, n\}$, draw a vertex x_{1i} between vertices $x_{3(i-1)}$ and x_{3i} on the right side.

Step 5. Draw edges $x_{11}x_{2i}$ and edges $x_{11}x_{3i}$ for all $i \in \{1, 2, 3, \ldots, n\}$. In this step, we have $2n$ edges.

Step 6. Draw edges $x_{21}x_{1i}$, for all $i \in \{2, 3, \ldots, n\}$ and edges $x_{21}x_{3k}$ for all $k \in \{1, 2, 3, \ldots, n\}$. In this step, we have $(n - 1) + n = 2n - 1$ edges.

Step 7. Draw edges $x_{2j}x_{3(j-1)}$ and edges $x_{2j}x_{3j}$ for all $j \in \{2, 3, \ldots, n\}$. Draw edges $x_{1i}x_{3(i-1)}$ and edges $x_{1i}x_{3i}$ for all $i \in \{2, 3, \ldots, n\}$. Draw edges $x_{1i}x_{2i}$ for all $i \in \{2, 3, \ldots, n\}$. In this step, we have $2(n - 1) + 2(n - 1) + (n - 1) = 5n - 5$ edges.

We obtain a planar 3-partite graph $G(V_1, V_2, V_3)$ with $e(G) = 2n + (2n - 1) + (5n - 5) = 9n - 6$ as shown in Figure 10. The Euler's formula implies that

$f(G) = 2 - n(G) + e(G) = 2 - 3n + 9n - 6 = 6n - 4$. □

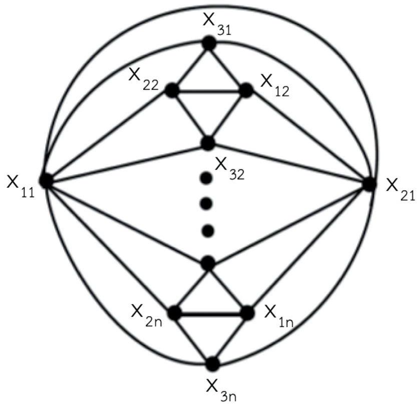

Figure 10. $K_{n,n,n}$

Theorem 15 For $n \geq 1$, $Ce(K_{n,n,n}) = 3n^2 - (9n - 6)$.

Proof. We can remove $3n^2 - (9n - 6)$ edges from $K_{n,n,n}$ to obtain a planar 3-partite graph G in Lemma 14, so

$$Ce(K_{n,n,n}) \leq 3n^2 - (9n - 6). \qquad (9)$$

The Euler's formula implies that if G is a planar graph then G has at most $9n - 6$ edges.
So we have to remove at least $3n^2 - (9n - 6)$ edges, and obtain that

$$Ce(K_{n,n,n}) \geq 3n^2 - (9n - 6). \qquad (10)$$

From (9) and (10), we have $Ce(K_{n,n,n}) = 3n^2 - (9n - 6)$. □

Lemma 16 For $n \geq 1, 1 \leq r \leq 6n - 4$, there is a planar 4-partite graph $G(V_1, V_2, V_3, V_4)$ with $|V_1| = |V_2| = |V_3| = n, |V_4| = r$, $e(G) = 9n + 3r - 6$ and $f(G) = 6n + 2r - 4$.

Proof. Lemma 14 implies that $K_{n,n,n}$ have $\tilde{G} = \tilde{G}(V_1, V_2, V_3)$ as a subgraph with $|V_1| = |V_2| = |V_3| = n$, $e(\tilde{G}) = 9n - 6$, all $6n - 4$ faces are C_3 and each face contains a vertex of each partite sets V_1, V_2, V_3.

Let $\{f_1, f_2, \ldots, f_{6n-4}\}$ be the set of faces C_3 of $\tilde{G}$.

For each $i \in \{1, 2, \ldots, r\}$, draw a vertex x_{4i} in a face f_i, and draw edges joining a vertex x_{4i} and all vertices in a face C_3. For one vertex added, we add 3 edges and 2 faces. If we add r vertices, then we add 3r edges, and 2r faces. So, $e(G) = 9n + 3r - 6$ where $1 \leq r \leq 6n - 4$ and $f(G) = 6n + 2r - 4$. □

Theorem 17 For $1 \leq n \leq r$, $Ce(K_{n,n,n,r}) = 3n^2 + 3nr - (9n + 3r - 6)$.

Proof. We can remove $3n^2+3nr-(9n+3r-6)$ edges from $K_{n,n,n,r}$ to obtain a graph G in Lemma 16 which is a planar graph, so

$$Ce(K_{n,n,n,r}) \leq 3n^2+3nr-(9n+3r-6). \tag{11}$$

The Euler's formula implies that if G is a planar graph then G has at most $9n+3r-6$ edges. So we have to remove at least $3n^2+3nr-(9n+3r-6)$ edges.
We obtain that

$$Ce(K_{n,n,n,r}) \geq 3n^2+3nr-(9n+3r-6). \tag{12}$$

From (11) and (12), we have $Ce(K_{n,n,n,r})=3n^2+3nr-(9n+3r-6)$. □

Lemma 18 For $n\geq 1, 1\leq n_k \leq |V_1|+|V_2|+\cdots+|V_{k-1}|, k\geq 4,$, there is a planar k-partite graph $G(V_1,V_2,\ldots,V_k)$ with $|V_1|=|V_2|=|V_3|=n, |V_4|=n_4,\ldots,|V_k|=n_k$, $e(G)=9n+3(n_4+\cdots+n_k)-6$ and $f(G)=6n+2(n_4+\cdots+n_k)-4$.
Proof. We prove by mathematical induction on the number of partite sets.
Base. Consider $k=4$. By Lemma 17, we have a planar 4-partite graph $G(V_1,V_2,V_3,V_4)$ with $|V_1|=|V_2|=|V_3|=n, |V_4|=n_4$, $e(G)=9n+3n_4-6$ and $f(G)=6n+2n_4-4$.
Induction step. Let $k\geq 5$. Suppose there is a planar (k-1)-partite graph $G(V_1,V_2,\ldots,V_{k-1})$ with $3n+n_4+\cdots+n_{k-1}$ vertices, $9n+3(n_4+\cdots+n_{k-1})-6$ edges and $6n+2(n_4+\cdots+n_{k-1})-4$ faces.
Next, we will draw a planar k-partite graph $G(V_1,V_2,\ldots,V_k)$ where each face contains 3 vertices from different partite sets $V_1,V_2,\ldots,V_{k-1}$.
Let $\{f_1,f_2,\ldots,f_{n_k}\}$ be the set of faces C_3 of $\tilde{G}$ where $n_k\leq 6n+2(n_4+\cdots+n_{k-1})-4$. For each $i\in\{1,2,\ldots,n_k\}$, draw a vertex x_{ki} in a face f_i by drawing edges joining a vertex x_{ki} and all vertices in face C_3. For one vertex we added, there are 3 edges and 2 faces added. If we add n_k vertices, then we added $3n_k$ edges and $2n_k$ faces.

Therefore, we obtain a planar k-partite graph $G(V_1,V_2,\ldots,V_k)$ with $e(G)=9n+3(n_4+\cdots+n_k)-6$ and $f(G)=6n+2(n_4+\cdots+n_k)-4$ where $1\leq n_k\leq 6n+3(n_4+\cdots+n_k)-4$.

By mathematical induction, we have a planar k-partite graph $G(V_1,V_2,\ldots,V_k)$ with $|V_1|=|V_2|=|V_3|=n, |V_4|=n_4,\ldots,|V_k|=n_k$, $e(G)=9n+3(n_4+\cdots+n_k)-6$ and $f(G)=6n+2(n_4+\cdots+n_k)-4$. □

Theorem 19 Let $k \geq 5$. For $1 \leq n \leq n_4 \leq \cdots \leq n_k$,

$Ce(K_{n,n,n,n_4,\ldots,n_k})$

$= 3n^2 + 3nn_4 + n_5(3n + n_4) + \cdots + n_k(3n + n_4 + \cdots + n_{k-1}) - (9n + 3(n_4 + \cdots + n_k) - 6).$

Proof. We can remove

$3n^2 + 3nn_4 + n_5(3n + n_4) + \cdots + n_k(3n + n_4 + \cdots + n_{k-1}) - (9n + 3(n_4 + \cdots + n_k) - 6)$ edges from $K_{n,n,n,n_4,\ldots,n_k}$ to obtain a planar k-partite graph G in Lemma 18, so

$$Ce(K_{n,n,n,n_4,\ldots,n_k}) \leq 3n^2 + 3nn_4 + n_5(3n + n_4) + \cdots + n_k(3n + n_4 + \cdots + n_{k-1}) - (9n + 3(n_4 + \cdots + n_k) - 6). \quad (13)$$

The Euler's formula implies that if G is a planar graph then G has at most $9n + 3(n_4 + \cdots + n_k) - 6$ edges. We have to remove at least $3n^2 + 3nn_4 + n_5(3n + n_4) + \cdots + n_k(3n + n_4 + \cdots + n_{k-1}) - (9n + 3(n_4 + \cdots + n_k) - 6)$ edges. So

$$Ce(K_{n,n,n,n_4,\ldots,n_k}) \geq 3n^2 + 3nn_4 + n_5(3n + n_4) + \cdots + n_k(3n + n_4 + \cdots + n_{k-1}) - (9n + 3(n_4 + \cdots + n_k) - 6). \quad (14)$$

From (13) and (14), we obtain that

$$Ce(K_{n,n,n,n_4,\ldots,n_k}) = 3n^2 + 3nn_4 + n_5(3n + n_4) + \cdots + n_k(3n + n_4 + \cdots + n_{k-1}) - (9n + 3(n_4 + \cdots + n_k) - 6).$$

□

Conclusion

We conclude our results in Table 1.

Table 1. Summary results for Ce(G) where G is a complete graph or a complete k-partite graph

Graph G	Ce(G)
K_n $(n \geq 5)$	$\binom{n}{2} - (3n - 6)$
$K_{m,n}$ $(2 \leq m \leq n)$	$(m-2)(n-2)$
K_{1,n_2,n_3} $(1 \leq n_2 \leq n_3)$	$(n_2 - 1)(n_3 - 2)$
K_{2,n_2,n_3} $(2 \leq n_2 \leq n_3)$	$n_2(n_3 - 2)$
$K_{n,n,n}$ $(n \geq 1)$	$3n^2 - (9n - 6)$
$K_{n,n,n,r}$ $(1 \leq n \leq r)$	$3n^2 + 3nr - (9n + 3r - 6)$
$K_{n,n,n,n_4,\ldots,n_k}$ $(1 \leq n \leq n_4 \leq \cdots \leq n_k, k \geq 5)$	$3n^2 + 3nn_4 + n_5(3n + n_4) + \cdots + n_k(3n + n_4 + \cdots + n_{k-1}) - (9n + 3(n_4 + \cdots + n_k) - 6)$

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