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Extending the Legacy: A Journey into the Emerging World of Fractional Quantum Calculus

Thanin Sitthiwiratthamm[1,2]

[1]Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, 10300, Thailand
[2]Research Group for Fractional Calculus Theory and Applications, Science and Technology Research Institute, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand

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ABSTRACT

This paper provides a comprehensive and rigorous exposition of **Fractional Quantum Calculus (FQC)**, a sophisticated mathematical framework that systematically extends classical integer-order quantum calculus to the realm of arbitrary fractional orders. We delineate the construction of five interconnected yet distinct types of fractional difference operators: the foundational **Fractional Difference Calculus (FDC)**, **Fractional q-Difference Calculus (FqDC)**, and **Fractional Hahn Difference Calculus (FHDC)**, complemented by the advanced generalizations **Fractional Symmetric Hahn Difference Calculus (FSHC)** and **Fractional (p,q)-Calculus (FpqC)**. The unifying principle undergirding all five frameworks involves the non-integer generalization of iterated summation, leading to the derivation of both Riemann-Liouville and Caputo operators. Fractional quantum calculus furnishes an indispensable analytical instrument for modeling intricate physical and biological phenomena characterized by **inherent non-locality**, **memory-dependent behavior**, and **discrete dynamical evolution**, capabilities that transcend the limitations of conventional integer-order mathematical models.

KEYWORDS: Fractional quantum calculus, Fractional difference calculus, Fractional q-difference calculus, Fractional Hahn difference calculus, Symmetric Hahn difference, (p,q)-calculus, Non-local phenomena, Memory effects, Discrete systems

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1. INTRODUCTION: UNIFYING THE DISCRETE, THE QUANTUM, AND THE FRACTIONAL

The mathematical landscape of **Fractional Calculus (FC)** traces its intellectual lineage to the foundational philosophical inquiries of Leibniz and de l'Hôpital in the waning years of the seventeenth century (Miller & Ross, 1989, Gray & Zhang, 1988). What began as an abstract theoretical speculation has matured into a robust and indispensable analytical tool for elucidating the behavior of complex systems across diverse scientific domains, including viscoelasticity, anomalous diffusion phenomena, and chaotic dynamics (Anastassiou, 2009, Ferreira & Torres, 2011). The distinctive merit of fractional calculus lies in its inherent capacity to encapsulate the temporal history and memory characteristics of dynamical systems, a feature conspicuously absent from the classical integer-order derivative formalism (L'Hôpital, 1695, Leibniz, 1695).

In parallel, the necessity to model phenomena occurring on discrete temporal scales or non-uniformly distributed grids catalyzed the emergence of **Quantum Calculus**, a calculus architecture constructed independently of the limiting process (Al-Salam, 1966, Agarwal, 1969). This framework encompasses several fundamental quantum operators: the h-difference operator (operative on uniformly spaced time scales), the q-difference operator or Jackson derivative (operative on geometric time scales), and the Hahn difference operator (synthesizing both uniform and geometric scaling).

The transformative synthesis emerges from the convergence of these two mathematical paradigms: **Fractional Quantum Calculus**. This novel theoretical structure furnishes a unified analytical vocabulary for investigating systems exhibiting both discrete-time characteristics *and* non-local (memory-dependent) phenomena. This exposition systematically examines the mathematical construction and practical utility of five major classes of fractional quantum calculi, tracing the field's trajectory from its foundational principles to its contemporary, highly generalized instantiations.

2. THE THEORETICAL FOUNDATION: CORE PRINCIPLES OF FRACTIONAL QUANTUM CALCULUS

The rigorous construction of fractional operators within the quantum calculus paradigm rests fundamentally upon two indispensable mathematical constructs: the **Gamma Function** $\Gamma(\cdot)$ and the generalized **Falling Factorial Function**.

2.1. The Generalization Principle

The transition from integer order m to fractional order α is universally effected through the generalization of the formula for the m-fold iterated sum (or integral) using the Gamma function. Within a discrete system, the fractional sum of order α is defined as the foundational operator, with the fractional difference serving as its inverse operator.

2.2. The Two Principal Operators

All fractional quantum operators adhere to two canonical formulations, which determine how the temporal history of the function is incorporated into the operator:

- **Riemann-Liouville (R-L) Fractional Difference:** This operator is defined as an integer-order difference applied to a fractional-order sum. It is predominantly employed in theoretical investigations and exhibits dependence on the initial summation point.

- **Caputo Fractional Difference:** Defined as a fractional-order sum of an integer-order difference, the Caputo formulation possesses a marked advantage in applied mathematical modeling: it admits initial conditions structurally analogous to those employed in classical integer-order calculus (namely, the specification of the function and its integer-order derivatives at an initial time), thereby facilitating application to boundary value problems and initial value problems.

3. THE FOUNDATIONAL TRIUMVIRATE: PRIMARY FRACTIONAL DIFFERENCE OPERATORS

The early development phase of fractional quantum calculus centered upon the fractional generalizations of three primary quantum operators.

3.1. Fractional Difference Calculus (FDC)

This constitutes the cornerstone of discrete fractional analysis, constructed upon the h-difference operator defined as $\Delta_h f(t) = \frac{f(t+h)-f(t)}{h}$ (with $h = 1$ for the classical discrete difference). Figure 1 shows the shift in $f(t)$-value and the shift in t-value of Fractional Difference Calculus.

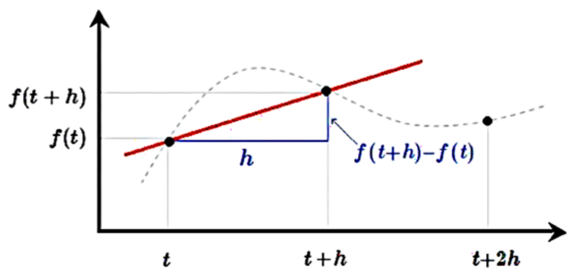

Figure 1 Displays the shift in $f(t)$-value and the shift in t-value of Fractional Difference Calculus (as shown in (Sitthiwirattham, 2020)).

Construction: The theoretical edifice rests upon the discrete Gamma function and the associated h-falling factorial, denoted $t^{(\alpha,h)}$.

Applications: This framework proves indispensable for modeling phenomena evolving on **uniform temporal grids** where memory effects play a significant role, exemplified in financial mathematics, discrete-time control systems, and difference equations arising from the calculus of variations (Ferreira, 2011).

3.2. Fractional q-Difference Calculus (FqDC)

This calculus is constructed upon the **Jackson** q-difference operator, which operates on a **geometric time scale** characterized by the progression $(t, qt, q^2t, \dots)$. Figure 2 demonstrates the shift in $f(t)$-value and the shift in t-value of q-Difference Calculus.

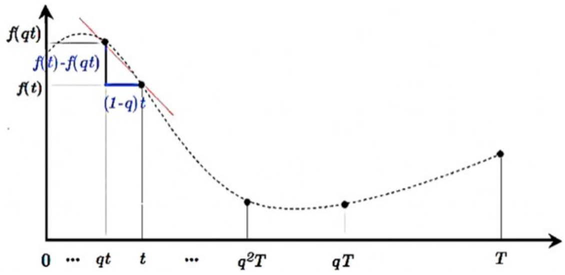

Figure 2 Displays the shift in $f(t)$-value and the shift in t-value of Fractional q-Difference Calculus (as shown in (Sitthiwirattham, 2020)).

Construction: The theoretical framework employs the q-analogue of the Gamma function and the fractional$q$-Falling Factorial. The foundational fractional q-Integral (Kac & Cheung, 2002, Ernst, 2012) constitutes the precursor to both the R-L and Caputo q-difference operators.

Applications: This formalism possesses considerable relevance in theoretical physics, particularly in quantum field theory and molecular physics, and is applicable to any system exhibiting geometric progression or inherent scaling symmetry. Furthermore, it proves instrumental in solving advanced boundary value problems for fractional q-differential equations (Čermák & Nechvatál, 2010).

3.3. Fractional Hahn Difference Calculus (FHDC)

The Hahn operator $D_{q,\omega}$ represents a synthesis of the h-difference and q-difference operators, thereby creating a more versatile **affine time scale**. The shift in $f(t)$-value and the shift in t-value of Fractional Hahn Difference Calculus is shown in Figure 3.

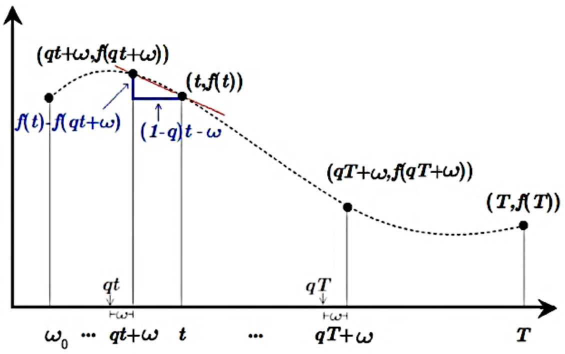

Figure 3 Displays the shift in $f(t)$-value and the shift in t-value of Fractional Hahn Difference Calculus (as shown in (Sitthiwirattham, 2020)).

Construction: Introduced by Brikshavana and Sitthiwirattham in 2017 (Brikshavana & Sitthiwirattham, 2017), this calculus develops its integral and differential operators (Riemann-Liouville and Caputo forms) utilizing the q-Gamma function and the fractional (q, ω)-falling factorial.

Applications: The fractional Hahn difference calculus serves as a **unifying theoretical platform**, offering a systematic approach to generalize results from both FDC and FqDC It has proven instrumental in the investigation of complex nonlocal boundary value problems incorporating fractional Hahn integral boundary conditions (Brikshavana & Sitthiwirattham, 2018).

4. THE ADVANCING FRONTIER: ADVANCED GENERALIZATIONS AND CONTEMPORARY EXTENSIONS

Recent scholarly developments have substantially enlarged the scope of fractional quantum calculus by introducing progressively more general or specialized difference operators, thereby significantly amplifying the field's analytical capabilities.

4.1. Fractional Symmetric Hahn Difference Calculus (FSHC)

The **Symmetric Hahn Difference Operator**, denoted $\widetilde{D}_{q,\omega}$, represents a sophisticated extension of the classical Hahn operator, engineered to simplify analytical procedures through the manifestation of distinct symmetrical properties.

Construction: Developed by Patanarapeelert and Sitthiwirattham in 2019 (Patanarapeelert & Sitthiwirattham, 2019), this calculus formulates the **fractional symmetric Hahn integral** alongside the corresponding Riemann-Liouville and Caputo symmetric difference operators.

Applications: This theoretical framework proves essential for addressing highly intricate nonlocal boundary value problems of symmetric character, specifically those involving sequential Caputo fractional Hahn integrodifference equations and nonlocal Robin-type boundary conditions (Patanarapeelert & Sitthiwirattham, 2018, Soontharanon & Sitthiwirattham, 2022). The symmetric character of the operator substantially simplifies the analytical techniques employed for establishing existence and uniqueness results.

4.2. Fractional (p, q)-Calculus (FpqC)

The (p, q)-calculus, characterized by the parameter constraint $0 < q < p \leq 1$, represents the most contemporary and maximally generalized formulation, inherently containing standard q-calculus as a particular limiting case (when $p = 1$).

Construction: Introduced by Soontharanon and Sitthiwirattham in 2020 (Soontharanon & Sitthiwirattham, 2020), this calculus is grounded in the (p, q)-difference operator and defines the **fractional (p, q)-integral** with corresponding Riemann-Liouville and Caputo operators. The theoretical development necessitates employment of the generalized (p,q)-Gamma and (p,q)-falling factorial functions.

Applications: The fractional (p,q)-calculus framework provides two-parameter flexibility, conferring substantial advantages in both approximation theory and operator-theoretic investigations. Its principal applications encompass demonstrating convergence properties for iterates of (p,q)-Bernstein operators, and establishing a diverse array of inequalities, for example Opial-type integral, trapezoid and midpoint-type inequalities. These developments yield tighter analytical bounds for discrete systems (Nasiruzzaman et al., 2019, Neang et al., 2021). Furthermore, applications to elastic beam analysis and related structural mechanics problems underscore the practical utility of this framework in engineering contexts (Etemad et al., 2024).

5. THE MATHEMATICAL TOOLKIT: COMPARATIVE ANALYSIS OF FQC

The significance of fractional quantum calculus transcends the boundaries of pure mathematics; it furnishes a high-fidelity analytical lens for examining discrete, memory-laden phenomena in the natural world.

Table 1 Comparative framework of five major fractional quantum calculus types (Sitthiwirattham, 2021)

FQC Type	Governing Time Scale / Parameters	Core Analytical Advantage
FDC	Uniform $t, t+h, t+2h, \ldots$	Discrete memory on regular grids
FqDC	Geometric $t, qt, q^2t, \ldots$	Non-uniform evolution, scaling symmetry
FHDC	Affine $\sigma_{q,\omega}^k(t)$	Unified FDC/FqDC framework
FSHC	Affine/Symmetric $\sigma_{q,\omega}^k(t) - \rho_{q,\omega}^k(t)$	Symmetric BVP analysis
FpqC	Two-Parameter Affine $p^k q^{n-k} t$	Maximal parameter flexibility

where $\sigma_{q,\omega}^k(t) := q^{kt} + \omega[k]_q$ and $\rho_{q,\omega}^k(t) := \frac{t-\omega[k]_q}{q^k}$.

The paramount contribution of fractional quantum calculus across these diverse applications manifests in its capacity to accurately incorporate the **non-local kernel** inherent to fractional operators, thereby transforming frequently inadequate integer-order models into sophisticated analytical instruments that respect the complete temporal history and intrinsic mathematical structure of the system.

6. CONCLUSIONS AND FUTURE TRAJECTORIES

The field of Fractional Quantum Calculus has progressed beyond its nascent phase to embrace increasingly sophisticated multi-parameter generalizations. The foundational unifying principle, the systematic generalization of the iterated sum or integral has demonstrated remarkable robustness, constituting the structural foundation for all advanced operators, spanning from the elementary FDC to the sophisticated FpqC.

The trajectory for future mathematical inquiry is substantial and vital for the field's continued maturation:

- **Stability and Asymptotic Analysis:** The foundational theory pertaining to the **stability characteristics and asymptotic behavior** of solutions to generalized fractional quantum calculus remains incompletely developed. Rigorous mathematical investigation in this domain constitutes a prerequisite for dependable application in engineering and control systems.
- **General Solution Methodologies:** Whereas existence and uniqueness theorems proliferate extensively, comprehensive and generally applicable **analytical techniques for solving** the diverse classes of fractional quantum

differential equations remain severely constrained. The development of advanced transformation methods and series solution techniques will prove essential for realizing the complete potential of these mathematical models.

- **FSHC and FpqC Expansion and Specialization:** Additional exploration of the distinctive advantages inherent to the two-parameter in domains extending beyond approximation theory, such as non-local dynamical systems and fractional integral transforms, represents a particularly promising avenue for investigation.

The classical inheritance of traditional calculus is undergoing a profound transformation, being forged into an innovative mathematical apparatus precisely calibrated to represent the fundamental discreteness and memory characteristics intrinsic to contemporary scientific inquiry.

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