Journal of Applied Science and Emerging Technology (JASET)
Published by Faculty of Applied Science, KMUTNB, Thailand
doi: 10.14416/JASET.KMUTNB.2025.02.001
Vol. 24, No. 2 [2025]: e900006

Modern Calculus

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Received 20 August 2025; Revised 28 August 2025; Accepted 29 August 2025

ABSTRACT

Calculus, since its inception by Newton and Leibniz, has served as a foundational tool for understanding change, motion, and accumulation. Over the centuries, its scope has expanded beyond the classical framework, giving rise to modern extensions that address new mathematical and scientific challenges. This article provides a comprehensive review of three significant branches: fractional calculus, multiplicative calculus, and difference calculus. Fractional calculus generalizes differentiation and integration to non-integer orders, offering powerful models for viscoelastic materials, anomalous diffusion, and memory effects in physical and biological systems. Multiplicative calculus, rooted in proportional change, reinterprets differentiation in terms of ratios, providing natural formulations for processes in finance, population dynamics, and biological growth. Difference calculus, along with its quantum variants, extends the principles of calculus to discrete settings, enabling applications in numerical methods, dynamical systems, and quantum models. By highlighting their theoretical foundations and practical applications, this paper illustrates how these non-Newtonian approaches enrich the classical view of calculus, bridging continuous and discrete analysis and offering versatile tools for modern science and engineering.

KEYWORDS: Fractional calculus; Multiplicative calculus; Difference calculus

Please cite this article as: J. Tariboon, "Modern Calculus," *Journal of Applied Science and Emerging Technology,* vol 24, no 2, pp. 1-**4**, ID. 900006, August 2025

1. INTRODUCTION

Calculus, developed by Isaac Newton and Gottfried Wilhelm Leibniz in the seventeenth century, has profoundly shaped mathematics and science by providing systematic methods to describe motion, change, and accumulation. Newton's approach, grounded in limits, derivatives, and integrals, defined fundamental physical concepts such as velocity, acceleration, and force. Leibniz, working independently, introduced a powerful notation and general framework for solving differential equations. Although many view the Newton–Leibniz era as the culmination of calculus, the field has continued to evolve over the past three centuries. These modern developments—often referred to as 'non-Newtonian calculus' or 'modern calculus' - extend differentiation and integration beyond the classical framework. This review focuses on three such extensions: fractional calculus, multiplicative calculus, and difference calculus, with emphasis on their definitions, theoretical foundations, and applications.

2. FRACTIONAL CALCULUS

The idea of fractional calculus originated in 1695, when Guillaume de l'Hôpital encountered Leibniz's notation for the nth derivative, D^n. He wrote to Leibniz asking what would happen if $n = 1/2$ Kilbas et al. (2006), Oldman & Spanier (1974), and Podlubny (1999). At that time, Leibniz could not give a precise answer, as the rigorous formulation required the Gamma function, which later appeared in the definitions of fractional derivatives and integrals:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau . \tag{1}$$

Although the development of fractional calculus was initially slow—mainly due to the difficulty of computing values of the Gamma function from cumbersome tables—modern computational software has made such evaluations straightforward. This has allowed fractional calculus to find widespread applications, particularly in materials science. In this field, solids are described by derivatives of order zero, fluids by order one, and viscoelastic materials by fractional orders between zero and one, providing a natural framework for describing non-Newtonian materials. The Riemann–Liouville fractional derivative of order α is defined as:

$$^{RL}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau,$$
$$n-1 < \alpha < n . \tag{2}$$

Later, Caputo proposed a modified definition, widely used in applied sciences due to its compatibility with classical initial conditions:

$$^{Ca}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau . \tag{3}$$

A classical application is the fractional Maxwell model in viscoelasticity, where fractional derivatives $(0 < \alpha \le 1)$ are used to encode memory effects:

$$D^{\alpha}\varepsilon(t) = \frac{1}{E} D^{\alpha}\sigma(t) + \frac{1}{\eta}\sigma(t) . \tag{4}$$

An equivalent form is:

$$\sigma(t) + \left(\frac{\eta}{E}\right)^{\alpha} D^{\alpha}\sigma(t) = \eta D^{\alpha}\varepsilon(t) . \tag{5}$$

Here, E denotes the elastic modulus, η the viscosity, $\sigma(t)$ the stress, and $\varepsilon(t)$ the strain. The parameter α interpolates material behavior between perfectly elastic solids $(\alpha \to 0)$ and Newtonian fluids $(\alpha = 1)$. Applications of fractional calculus span physics, rheology, materials science, and biology. Fractional models describe viscoelastic behavior, anomalous diffusion, and memory effects in complex systems.

3. MULTIPLICATIVE CALCULUS

Multiplicative calculus offers a framework for proportional rather than additive change. It is naturally motivated by compound interest. Suppose depositing $\$a$ yields $\$b$ after one year. The growth factor is b/a. With monthly compounding, the monthly growth becomes $(b/a)^{\frac{1}{12}}$. As compounding frequency increases, the limiting process gives rise to the multiplicative derivative Grossman & Katz (1972).

The multiplicative derivative of a positive function f at point x is defined as:

$$D_* f(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}. \qquad (6)$$

Compared with the Newtonian derivative

$$f'(x) = \lim_{h\to 0}\big(f(x+h) - f(x)\big)/h, \qquad (7)$$

the difference is replaced by a ratio, and the division by h is replaced by raising to the reciprocal power $1/h$. This formulation directly links multiplicative calculus to continuous compounding.

The exponential growth law $f(t) = f(0)e^{rt}$ emerges seamlessly from multiplicative calculus, where r represents the relative growth rate. This shows how multiplicative differentiation provides a natural setting for proportional processes such as finance, population dynamics, and biological growth.

Beyond finance, multiplicative calculus has been applied to signal processing, where relative changes are more meaningful than absolute ones, and to models of economic and population growth, offering insights into systems governed by proportional change.

4. DIFFERENCE CALCULUS

Difference calculus addresses the discrete analog of classical calculus, where functions are defined on discrete sets rather than continuous intervals Jordan (1956). The forward difference operator is defined as $\Delta f(x) = f(x+1) - f(x)$, while the backward difference operator is $\nabla f(x) = f(x) - f(x-1)$. These operators provide the basis for numerical methods and discrete modeling.

A well-known tool in this context is the Newton–Gregory forward interpolation formula, used to approximate functions from equally spaced data points:

$$f(x) \approx f(x_0) + \frac{(x-x_0)}{h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f(x_0) + \cdots, \qquad (8)$$

where h is the interval between data points and $\Delta^n f(x_0)$ is the nth forward difference at x_0. This formula underlies many interpolation and numerical computation techniques.

Difference equations, the discrete counterparts of differential equations, are fundamental in modeling population growth, economic cycles, and computer simulations. For example, the logistic map $x_{n+1} = r x_n(1-x_n)$ demonstrates how discrete dynamics connect to chaos theory, highlighting the importance of difference calculus in modern dynamical systems.

Further extensions include the Hahn difference calculus, which blends the h- and q-calculus into a unified operator, and fractional versions of q- and Hahn calculus, which extend differentiation to non-integer orders. These advanced forms have been employed in boundary value problems, stability analysis, and the construction of orthogonal polynomials. Thus, quantum calculus enriches the landscape of discrete analysis, providing bridges between classical calculus, difference equations, and modern applications in science and engineering.

The q-difference operator, for $q \in (0,1)$, is defined for a function f as:

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad t \neq 0 \,. \tag{9}$$

Its inverse leads to the q -integral, often expressed via the Jackson sum. As $q \to 1$, the q -difference operator recovers the classical derivative. This makes q -calculus a natural framework to interpolate between discrete and continuous models. Applications of quantum calculus can be found in number theory, combinatorics, special functions, and even physics, such as models of quantum mechanics, relativity, and cosmology.

An important extension of difference calculus is Quantum Calculus, often described as 'calculus without limits.' In this framework, derivatives and integrals are replaced by difference operators that do not rely on the notion of limits. Quantum calculus includes several variants such as the h-difference calculus, q-difference calculus, and the Hahn difference calculus. These generalizations provide powerful tools for analyzing systems where standard differentiation is not possible.

5. CONCLUSION

From Newton and Leibniz to contemporary extensions, calculus remains a dynamic and evolving discipline. Fractional calculus extends differentiation and integration to arbitrary orders, multiplicative calculus reframes change in terms of ratios, and difference calculus adapts methods for discrete settings. Together, these approaches expand the modeling capabilities of calculus, enhancing its relevance across physics, engineering, finance, and computational sciences.

ACKNOWLEDGMENT

The editorial board would like to thank author(s) for your contribution to Journal of Applied Science.

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