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# Generalization of $\alpha$-$G$-Contractions to $j$-$G$-Contractions on Uniform Spaces

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ABSTRACT

This work introduces the concept of $j$-$G$-contractions on uniform spaces generated by a saturated collection of pseudo- metrics, extending the theoretical framework of $\alpha$-$G$-contractions. Two main theorems are presented, both of which establish that if the map satisfies certain conditions, then it is a Picard operator. The first theorem relies on conditions defined by the structure of the space and the graph, while the second theorem utilizes conditions related to orbital $G$-continuity. Criteria are provided for verifying $j$-$G$-contraction conditions, and it is shown that these maps satisfy the conditions of the main theorems. Some findings demonstrate that modifying the graph to satisfy $j$-$G$-contraction conditions can lead to an enlarged convergence set.

**KEYWORDS:** Uniform spaces, Fixed point theory, $j$-$G$-contractions, Picard operator

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## 1. INTRODUCTION

Over the past few decades, fixed point theory has emerged as a valuable tool in various branches of mathematics, particularly in the study of functional equations, differential equations, and optimization. One of its fundamental results is the Banach Contraction Principle, which states that every contraction mapping on a complete metric space has a unique fixed point. Furthermore, given any initial point in the space, the Picard iteration sequence generated by the mapping converges to this fixed point, ensuring uniqueness in iterative processes.

Many researchers have sought to generalize the Banach Contraction Principle within the framework of metric spaces, particularly by incorporating graph structures and partial orders. The study of contractions in partially ordered metric spaces was first initiated by Ran and Reurings (2004), who extended Banach's theorem to such settings. Building on this foundation, Nieto and Rodríguez-López (2007) introduced Picard operators within partially ordered sets, refining fixed point results under ordered conditions. Later, Jachymski (2008) extended these ideas by introducing graph-based contractions, proving that Banach's theorem remains valid in metric spaces endowed with a directed graph. Further advancements were made by Bojor (2012), who introduced $G$-Reich type contractions, a generalization that relaxes the Banach contraction condition while ensuring fixed points exist under suitable graph connectivity assumptions. These contributions have played a crucial role in extending fixed point results beyond classical contraction mappings, particularly in metric spaces incorporating graph structures and partial orders.

In parallel with these advancements, research on contractions within uniform spaces, which generalize metric spaces, has also progressed. Notably, uniform spaces permit the definition of Cauchy sequences and Cauchy filters, allowing for broader fixed point results. One direction of study focuses on the concept of E-distance, which provides a generalized way to measure distances in uniform spaces and has been widely used in contraction conditions. Another approach considers collections of saturated pseudo-metrics. These perspectives offer distinct formulations of contraction principles in uniform spaces, each contributing to the development of fixed point theory in different ways.

E-distance has been studied as a generalization of metric distance, allowing more general contraction conditions in uniform spaces. Several studies have explored its applications in fixed point theory. Olatinwo (2008) introduced integral-type contractions, incorporating E-distance and A-distance to define contractive conditions based on integral inequalities. Dhagat et al. (2009) established common fixed point theorems using E-distance, providing foundational results for contraction mappings in uniform spaces. Ali et al. (2014) applied E-distance to $\alpha-\psi$-contractive mappings, extending function-based contractions in uniform spaces, while Ali et al. (2017) introduced Es-distance, an extension of E-distance, to study FG-contractions and G-contractions in graph-based uniform spaces. Hosseini & Mirmostafaee (2020) developed common fixed point results in Hausdorff uniform spaces, incorporating A-distance and E-distance into contractive mappings with graph structures. Umudu & Adewale (2021) further extended these results by integrating E-distance into $\alpha$-admissible mappings and simulation functions, establishing fixed point results under more general contractive conditions.

An alternative approach in fixed point theory for uniform spaces involves using a collection of pseudo-metrics. Nashed (1971) was among the first to introduce $\gamma$-contractions in locally convex spaces, extending Banach contraction principle beyond standard metric spaces. Angelov (1987) built on this by proposing $\Phi$-contractions in Hausdorff uniform spaces, incorporating both contractions and $\gamma$-contractions, and proving fixed point existence under broader conditions. Later, Angelov (1991) further generalized $\Phi$-contractions to j-nonexpansive and j-contraction maps, which preserve non-expansiveness while allowing different pseudo-metrics before and after mapping. Angelov also established sufficient conditions for fixed points in uniform spaces under this framework. These contributions significantly expanded contraction principles beyond classical metric spaces, forming the foundation for graph-based contractions and multi-metric fixed point theory.

Building on this line of research, P. Chaoha and S. Songsa-ard (2014b) investigated functionally Lipschitzian (FL) mappings and functionally uniformly Lipschitzian (FUL) mappings in locally convex spaces, examining the weak topology on normed spaces and establishing criteria for FL and FUL mappings, showing that FL mappings are weakly continuous and the fixed point set of FUL mappings is a retract of their convergence set. Furthermore, they explored fixed points in uniform spaces (Chaoha & Songsa-ard, 2014a), focusing on sufficient conditions for the existence of fixed points of J-contractions in uniform spaces endowed with a collection of pseudo-metrics and investing the convergence set of J-nonexpansive mappings. As a result, they showed that the fixed point set of J-nonexpansive mappings is a retract of their convergence set and also provided examples of ordinary differential equations (ODEs) that utilize these fixed point results to guarantee the existence of solutions.

A further extension was developed by Songsa-ard (2024), who introduced $\alpha$-$G$-contractions on uniform spaces generated by a collection of pseudo-metrics, combining the concepts of $\gamma$-contractions and graph-based contractions. The study also provided corresponding criteria and illustrative examples. Under certain conditions, these $\alpha$-$G$-contractions are Picard operators on specific subsets with respect to the given graph.

The previously discussed work provides the motivation to extend the concept of $\alpha$-$G$-contractions to $j$-$G$-contractions, drawing inspiration from the framework of $j$-nonexpansive maps. Our investigation focuses on $j$-$G$-contractions within uniform spaces generated by a saturated collection of pseudo-metrics. This work not only introduces a new notion of $j$-$G$-contraction, which involves a somewhat intricate definition, but also provides criteria for determining whether a mapping satisfies the $j$-$G$-contraction condition.

In general, proving that a mapping is a contraction is challenging because it requires demonstrating that the distance between any two mapped points is at most their original distance scaled by a real number no greater than 1. However, the key advantage of $j$-$G$-contractions defined on a uniform space generated by a collection of pseudo-metrics is that the contraction condition has a higher likelihood of being satisfied. This is because the distances before and after applying the mapping may not need to be measured using the same pseudo-metric. Moreover, incorporating graph structure reduces the points requiring verification under the contraction condition, increasing the likelihood of a mapping satisfying the $j$-$G$-contraction condition. Consequently, the size of the convergence set also depends on the structure of the graph $G$.

## 2. PRELIMINARIES

We provide the necessary definitions and background, including uniform spaces generated by a saturated collection of pseudo metrics, graph-based contractions.

**Definition 2.1**. *A __pseudo-metric__ on a nonempty set $X$ is a mapping $p: X \times X \to \mathbb{R}$ having the following properties:*

1. $p(x,y) \geq 0$ *for all* $x, y \in X$.
2. $p(x,y) = p(y,x)$ *for all* $x, y \in X$.
3. $p(x,y) \leq p(x,z) + p(z,y)$ *for all* $x, y, z \in X$.

Next, we define the topology induced by a given collection of pseudo-metrics.

**Definition 2.2**. *Suppose $A$ is an index set and $\mathscr{A} = \{p_\alpha : \alpha \in A\}$ where $p_\alpha$ is a pseudo-metric on $X$. Let $\varepsilon > 0$, $\alpha \in A$, $x \in X$ and define $B_{\alpha,x}(\varepsilon) = \{y \in X : p_\alpha(y,x) < \varepsilon\}$. Then the set $\mathscr{S} = \{B_{\alpha,x}(\varepsilon) : \varepsilon > 0, \alpha \in A, x \in X\}$ is a subbasis for topology on $X$. The space $(X, \mathscr{A})$, endowed with the topology generated by $\mathscr{S}$, is called a __uniform space generated by a collection $\mathscr{A}$ of pseudo-metrics__. A collection $\mathscr{A}$ is __saturated__ if and only if for all $x, y \in E$, if $\alpha \in A$, $p_\alpha(x,y) = 0$, then $x = y$.*

In a uniform space, we have the advantage of explicitly describing the notion of closeness between any two points, which naturally leads to the following definition of a Cauchy sequence.

**Definition 2.3.** *Let $(X, \mathscr{A})$ be a uniform space generated by a collection $\mathscr{A} = \{p_\alpha : \alpha \in A\}$ of pseudo-metrics indexed by $A$. Then $(x_n)$ is a __Cauchy sequence__ if and only if for all $\varepsilon > 0$, $\alpha \in A$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, $p_\alpha(x_n, x_m) < \varepsilon$. A uniform space $(X, \mathscr{A})$ is said to be __sequentially complete__ if every Cauchy sequence in $X$ converges to a point in $X$.*

Next, we introduce several graph-theoretic concepts used specifically in this paper.

**Definition 2.4.** *Let $(X, \mathscr{A})$ be a uniform space generated by a collection $\mathscr{A}$ of pseudo-metrics.* ***A __weighted directed graph associated with__*** $(X, \mathscr{A})$ *is defined as an ordered pair $G = (V(G), E(G))$ satisfying:*

1. $V(G) = X$.
2. *For any* $x \in V(G)$, $(x,x) \in E(G)$.
3. *Each edge $(x,y) \in E(G)$ is assigned a set of weights indexed by pseudo-metrics in $\mathscr{A}$ given by $\{p(x,y) : p \in \mathscr{A}\}$.*

We also introduce the notion of conversion graphs and the concept of undirected graphs.

**Definition 2.5.** *A __conversion graph__ of a graph $G$, denoted $G^{-1}$, is obtained by reversing all edges of $G$ and the vertex of $G^{-1}$ is the same as $G$, so $V(G^{-1}) = V(G)$, and the edge set is given by*

$$E(G^{-1}) = \{(x,y) : (y,x) \in E(G)\}.$$

*An __undirected graph__ of $G$, denoted by $\tilde{G}$, is constructed in such a way that its set of vertices is the vertices of $G$i.e., $V(\tilde{G}) = V(G)$, and its set of edges is taken as the union of the edges of $G$ and those of its conversion graph $G^{-1}$ ($E(\tilde{G}) = E(G) \cup G(G^{-1})$).*

Next, we recall the standard definitions related to connectivity in graphs, including the notions of a **path**, **connectedness**, and **weak connectedness**, which will be essential in our discussion.

**Definition 2.6.** *A __path__ of length $N \in \mathbb{N}$ in a graph $G$ from a vertex $x$ to a vertex $y$ is a sequence of $N+1$ vertices $(x_i)_{i=0}^N$ satisfying $x_0 = x$, $x_N = y$, and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, N$. The graph $G$ is called __connected__ if for every pair of $x$ and $y$ in $V(G)$, there is a path from $x$ to $y$ and a path from $y$ back to $x$. Moreover, $G$ is called __weakly connected__ if $\tilde{G}$ is connected.*

We now introduce two additional concepts specific to the work of Jachymski (2008): the definition of a

**component** and the notion of **orbital $G$-continuity**.

**Definition 2.7.** (Jachymski, 2008). ***The equivalence class of*** $x$ *in a graph* $G$, *denoted by* $[x]_G$, *is defined as the set of vertices of* $G$ *that are linked to* $x$ *by a path on* $G$. *Moreover, the* ***component*** *of* $G$ ***containing*** $x$, *denoted by* $G_x$, *is the largest connected subgraph of* $G$ *that includes* $x$ *among its vertices. The vertices of this connected subgraph form the equivalence class* $[x]_G$.

**Definition 2.8.** (Jachymski, 2008). *A map* $T:X\to X$ *is said to be an* ***orbitally G-continuous*** *for all* $x,y\in X$ *and sequence* $(k_n)$ *of positive integers such that*

$$T^{k_n}x\to y \text{ and } (T^{k_n}x, T^{k_{n+1}}x)\in E(G) \text{ for any } n\in\mathbb{N},$$

*imply* $T^{k_n+1}x=T(T^{k_n}x)\to Ty$.

The notion of **orbital G-continuity** is presented as a sufficient condition ensuring that the mapping under consideration is a **Picard operator**.

**Definition. 2.9.** (Nieto & Rodríguez-López, 2007) *Let* $X$ *be a nonempty subset of a Hausdroff space* $E$. *A map* $T:X\to X$ *is said to be a* ***Picard operator*** *if there exists a unique fixed point* $x_0\in X$ *and* $T^n x\to x_0$ as $n\to\infty$ *for any* $y\in X$.

Next, we introduce the recent definitions given by Songsa-ard (2024), including the concept of an **$\alpha$-$G$-contraction** and the notion of **Cauchy equivalence** between two sequences in uniform spaces.

**Definition 2.10.** (Songsa-ard, 2024). *Let* $(E,\mathscr{A})$ *be a uniform space generated by a saturated collection of pseudo metrics* $\mathscr{A}$ *with an indexed set* $A$. *Suppose* $X\subseteq E$ *is nonempty. A map* $T:X\to X$ *is called a* $\alpha$-$G$***-contraction*** *if there exists a graph* $G$ *satisfying the following two conditions:*

1. *For each* $(x,y)\in E(G)$, $(Tx,Ty)\in E(G)$.
2. *For each* $\alpha\in A$, *there exists a constant* $c_\alpha\in(0,1)$ *such that*

$$p_\alpha(Tx,Ty)\le c_\alpha p_\alpha(x,y)$$

*for any* $(x,y)\in E(G)$.

**Definition 2.11.** (Songsa-ard, 2024). *Sequences* $(x_n)$ *and* $(y_n)$ *are said to be* ***Cauchy equivalent*** *in a uniform space* $(E,\mathscr{A})$ *generated by a saturated collection* $\mathscr{A}$ *of pseudo metrics with* $A$ *as the index set, if* $(x_n)$, and $(y_n)$ *are Cauchy sequences, and for all* $\alpha\in A$, $p_\alpha(x_n,y_n)$ *converges to 0 as* $n\to\infty$.

Moreover, Songsa-ard presented the following theorem as a criterion for determining $\alpha$-$G$-contractions.

**Theorem 2.12.** (Songsa-ard, 2024). *Let* $1<p<\infty$ *and let* $f_1, f_2, \ldots, f_N$ *be non-decreasing contraction functions with respective contraction constants* $k_1, k_2, \ldots, k_N$. *Let* $c\in(0,1)$. *Suppose* $T:\ell_p\to\ell_p$ *is defined by*

$$T(x_m)=(f_1x_1, f_2x_2, \ldots, f_Nx_N, cx_{N+1}, cx_{N+2}, \ldots).$$

*Let* $G$ *be a graph with* $V(G)=\ell_p$ *and*

$$E(G)=\{((x_m),(y_m)) : x_i=y_i \text{ for all } i=1,\ldots,N\}.$$

*Then* $T$ *is a* $\alpha$-$G$*-contraction on* $\ell_p$ *endowed with the weak topology.*

Next, we briefly recall the definitions and essential properties of certain sequence spaces along with their associated topologies, as these will be used to illustrate examples later in this work.

**Definition 2.13.** *Let* $1<p<\infty$. ***The sequence space*** $\ell_p$ *is defined by* $\ell_p=\left\{(x_n)\subseteq\mathbb{R} : \sum_{i=1}^{\infty}|x_i|^p<\infty\right\}$ *and* $\|\cdot\|_p:\ell_p\to\mathbb{R}$ *is defined by* $\|(x_n)\|_p=\left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}}$.

*Thus* $\ell_p$ *is a vector space (over* $\mathbb{R}$*) and* $\|\cdot\|_p$ *is a norm on* $\ell_p$. *It is well-known that the dual space* $\ell_p^*$ *can be identified with the sequence space* $\ell_q$, *where* $\frac{1}{p}+\frac{1}{q}=1$. *Precisely, every bounded linear functional* $f\in\ell_p^*$ *is uniquely represented by a sequence* $(y_n)\in\ell_q$ *such that* $f(x_n)=\sum_{i=1}^{\infty}x_iy_i$ *for all* $(x_n)\in\ell_p$. ***The weak topology*** *on* $\ell_p$ *is the coarsest topology on* $\ell_p$ *in which each bounded linear functional remains continuous. A sequence*

*$(w_n)$ in $\ell_p$ **<u>converges weakly</u>** to a point $w \in \ell_p$, denoted by $w_n \xrightarrow{w} w$ as $n \to \infty$, if for each $f \in \ell_p{}^*$, $\left|f\left(w_n - w\right)\right| \to 0$ as $n \to \infty$.*

**Definition 2.14.** *<u>The standard Schauder basis for</u> $\ell_p$ is given by the canonical unit vectors $e_n = (\delta_{n,k})_{k=1}^{\infty}$ in $\ell_p$ where $\delta_{n,k}$ denotes the Kronecker delta such that for any $(x_m) \in \ell_p$, there exists a unique sequence of real numbers $(a_m)$, $(x_m) = \sum_{n=1}^{\infty} a_n e_n$ (in the weak topology). The associated **<u>coordinate functionals</u>** $e_n{}^*$ are given explicitly by $e_n{}^*(x_k) = x_n$ for any $(x_k) \in \ell_p$.*

The space $\ell_p$ , endowed with the weak topology induced by its dual $\ell_q$ , provides a canonical example of a locally convex space. As is well known in function analysis, the weak topology on $\ell_p$ is **Hausdorff** and it is **sequentially complete**. Moreover, it is generated by a collection of seminorms given by $s_{(\alpha_n)}(x_n) = \left|\sum_{i=1}^{\infty} \alpha_i x_i\right|$ where $(\alpha_n) \in \ell_q$ and $(x_n) \in \ell_p$ and each seminorm $s_{(\alpha_n)}$ corresponds naturally to a pseudo-metric $p_{(\alpha_n)}$, defined by

$$p_{(\alpha_n)}(x, y) = s_{(\alpha_n)}(x - y) = \left|\sum_{i=1}^{\infty} \alpha_i\left(x_i - y_i\right)\right|,\ x, y \in \ell_p,$$

explicitly connecting the locally convex structure to the uniform structure. This viewpoint provides the conceptual background required for constructing examples considered later in this paper.

Finally, we introduce the definition of the **convergence set** in the general topology on $X$.

**Definition 2.15.** *Let $T{:}X \to X$ be a mapping on a Hausdroff topological space $X$. <u>The convergence set of</u> $T$, denoted by $C(T)$, is given by*

$$C(T) = \{x \in X : \textit{the sequence } (T^n x) \textit{ converges}\}.$$

## 3. MAIN RESULTS

In this section, we formally introduce the concept of $j$-$G$-contractions and provide explicit criteria for verifying whether a mapping satisfies this condition. We present illustrative examples, including cases where a mapping is a $j$-$G$-contraction but not an $\alpha$-$G$-contraction, as well as examples demonstrating the crucial role of graph structure in establishing $j$-$G$-contractions. Following this, we provide key lemmas essential for proving the main theorems, which are structured as follows: (1) conditions imposed on the space and the graph (Main Theorem 1) and (2) conditions imposed on the mapping (Main Theorem 2). Finally, we show that the proposed criteria and examples align with these conditions, leading to the conclusion that the mapping is a Picard operator on specific subsets with respect to the given graph. Moreover, we establish that the size of the convergence set depends on the underlying graph structure.

This definition is inspired by $\alpha$-$G$-contractions, allowing the distance measurements in the contraction condition to be taken with different pseudo-metrics.

**Definition 3.1.** *Let $(E, \mathscr{A})$ be a uniform space generated by a saturated collection of pseudo metrics $\mathscr{A}$ with an indexed set $A$. Suppose $X \subseteq E$ is nonempty. A map $T{:}X \to X$ is called a $j$-$G$-<u>contraction</u> if there exists a mapping $j{:}A \to A$ and a weighted directed graph $G$ associated with $(X, \mathscr{A})$ that satisfy the following two conditions:*

1. ***Edge preservation:*** *For each $(x, y) \in E(G)$, we have $(Tx, Ty) \in E(G)$.*
2. ***Contraction condition:*** *For each $\alpha \in A$, there exists a constant $c_\alpha \in (0, 1)$ such that*
$$p_\alpha(Tx, Ty) \le c_\alpha p_{j(\alpha)}(x, y)$$
*for any $(x, y) \in E(G)$.*

**Example 3.2.** *Every $\alpha$-$G$-contraction is a $j$-$G$-contraction when $j$ is the identity on the index set of a collection of pseudo-metrics.*

Moreover, in this work, a criterion is given for checking whether a map $T{:}\ell_p \to \ell_p$ is a $j$-$G$-contraction on $\ell_p$ endowed with the weak topology, for $1<p<\infty$. This is presented in the following the theorem.

**Theorem 3.3.** *Let* $1<p<\infty$ *and let* $f_1, f_2, \ldots, f_N$ *be non-decreasing contraction functions with respective contraction constants* $k_1, k_2, \ldots, k_N$. *Let* $c\in(0,1)$. *Suppose* $T:\ell_p\to\ell_p$ *is defined by*

$$T(x_m)=\left(f_1x_1, f_2x_2, \ldots, f_Nx_N, cx_{N+1}, cx_{N+2}, \ldots\right).$$

*Let* $G$ *be a graph with* $V(G)=\ell_p$ *and*

$$E(G)=\left\{\left((x_m),(y_m)\right): x_i\geq y_i \text{ for all } i\in\mathbb{N}\right\}.$$

*Then* $T$ *is a j-G-contraction on* $\ell_p$ *endowed with the weak topology.*

**Proof.** Choose $j((\alpha_i))=(|\alpha_i|)_{i\in\mathbb{N}}$. To prove edge preservation, let $((x_m),(y_m))\in E(G)$. By the definition of $E(G)$, for all $i=1, \ldots, N$, $f_ix_i\geq f_iy_i$ and for all $i\in\mathbb{N}$, $cx_{N+i}\geq cy_{N+i}$, so $\left(T((x_m)),T((y_m))\right)\in E(G)$.

Finally, to prove a contraction condition, let $(\alpha_m)\in\ell_q$ with $\frac{1}{p}+\frac{1}{q}=1$, $((x_m),(y_m))\in E(G)$.

Define $K=max\left\{\{k_i : i\in\mathbb{N}\}\cup\{c\}\right\}\in(0,1)$.

Then $\left|\sum_{m=1}^{\infty}\alpha_m e^*_m\left(T((x_m))-T((y_m))\right)\right|\leq A+B$ where $A=\left|\sum_{i=1}^{N}\alpha_i\left(f_ix_i-f_iy_i\right)\right|$ and $B=c\left|\sum_{i=N+1}^{\infty}\alpha_i\left(x_i-y_i\right)\right|$.

Therefore

$$\left|\sum_{m=1}^{\infty}\alpha_m e^*_m\left(T((x_m))-T((y_m))\right)\right|\leq K\left|\sum_{i=1}^{\infty}|\alpha_i|\left(x_i-y_i\right)\right|.$$

Note that a constant $K$ dose not depended on $(\alpha_m)\in\ell_q$.

#

**Example 3.4.** *Let* $T:\ell_2\to\ell_2$ *is defined by*

$$T(x_m)=\left(f_1x_1, f_2x_2, f_3x_3, \frac{1}{3}x_4, \frac{1}{3}x_5, \ldots\right),$$

*where* $f_1(x)=f_2(x)=\dfrac{\sin x+2x+1}{5}$ *and* $f_3(x)=\dfrac{2}{3}x$.

*Let* $G$ *be a graph with* $V(G)=\ell_2$ *and*

$$E(G)=\left\{\left((x_m),(y_m)\right): x_i\geq y_i \text{ for all } i\in\mathbb{N}\right\}.$$

*Then* $T$ *is a j-G-contraction but it is not* $\alpha$*-G-contraction.*

**Proof.** Since $f_1, f_2$ and $f_3$ are *non-decreasing contraction functions with respective contraction constants* $k_1=k_2=\frac{3}{5}$ and $k_3=\frac{2}{3}$, by theorem 3.3, $T$ *is a j-G-contraction*.

To show that $T$ is not an $\alpha$-$G$-contraction, let $(\alpha_m)=(0,0,-1,1,0,\ldots)$ be a sequence in $\ell_2$ and $b\in(0,1)$. Choose $(x_m)=(1,1,9,9,1,0,\ldots)$ and $(y_m)=(0,0,\ldots)$ are in $\ell_2$. Then

$$\left|\sum_{m=1}^{\infty}\alpha_m e^*_m\left(T((x_m))-T((y_m))\right)\right|=\left|(-1)f_3(9)+\frac{1}{3}(9)\right|=3.$$

However,

$$b\left|\sum_{m=1}^{\infty}\alpha_m\left(x_m-y_m\right)\right|=b|-9+9|=0.$$

Therefore $T$ is not $\alpha$-$G$-contraction. #

This example illustrates that the class of $j$-$G$-contraction is more general than the class of *not* $\alpha$-$G$-contraction. We present an additional example to further illustrate the flexibility of $j$-$G$-contraction in selecting an appropriate graph structure.

The next example demonstrates how the choice of a graph can be utilized to enforce a mapping to satisfy the $j$-$G$-contraction condition.

**Example 3.5.** *Let* $T:\ell_2\to\ell_2$ *is defined by*

$$T(x_m)=\left(f_1x_1, f_2x_2, f_3x_3, \frac{1}{3}x_4, \frac{1}{3}x_5, \ldots\right),$$

*where* $f_1(x)=f_2(x)=\dfrac{\sin x+x}{2}$ *and* $f_3(x)=\dfrac{2}{3}x$.

*Let* $G$ *be a graph with* $V(G)=\ell_2$ *and*

$$E(G)=\left\{\left((x_m),(y_m)\right): x_1=y_1, x_2=y_2, x_i\geq y_i \text{ for all } i\geq 3\right\}.$$

*Then* $T$ *is a j-G-contraction.*

**Proof.** The proof follows a reasoning similar to that of Theorem 3.3. Since $x_1=y_1$, $x_2=y_2$,

$$\left|\sum_{m=1}^{\infty}\alpha_m e^*_m\left(T((x_m))-T((y_m))\right)\right|\leq A+B \text{ where}$$

$A=\left|\alpha_3\left(f_3x_3-f_3y_3\right)\right|$ and $B=\frac{1}{3}\left|\sum_{i=4}^{\infty}\alpha_i\left(x_i-y_i\right)\right|$.

Therefore

$$\left|\sum_{m=1}^{\infty}\alpha_m e^*_m\left(T((x_m))-T((y_m))\right)\right|\leq\frac{2}{3}\left|\sum_{i=1}^{\infty}|\alpha_i|\left(x_i-y_i\right)\right|.$$

Then $T$ *is a* $j$-$G$-contraction. #

From the above example, it can be observed that the key factor enabling $T$ to be *a* $j$-$G$-contraction lies in the graph $G$, where the definition of $E(G)$ ensures that $x_1=y_1$ and $x_2=y_2$. This structure removes the need to consider $f_1$ and $f_2$, which create difficulties in establishing bounds.

According to the definition of a *j-G-contraction*, we observe the following interesting relationship among the graphs $G$, $G^{-1}$, and $\widetilde{G}$.

**Lemma 3.6.** *Let $(E,\mathscr{A})$ be a uniform space generated by a collection of pseudo metrics $\mathscr{A}$ with an indexed set $A$, where $\mathscr{A}$ is saturated. Suppose $X\subseteq E$ is nonempty and $T:X\to X$.*

*If $T$ is a $j$-$G$-contraction, then $T$ is also a $j$-$G^{-1}$-contraction and a $j$-$\widetilde{G}$-contraction.*

**Proof.** Since $T$ is a $j$-$G$-contraction, there exist a map $j:A\to A$ and a graph $G$. To show that $T$ is a $j$-$G^{-1}$-contraction, since $(y,x)\in E(G)$, $(Tx,Ty)\in E(G^{-1})$. Hence, $T$ preserves the edges of $G^{-1}$.

Let $\alpha\in A$ and $(x,y)\in E(G^{-1})$. Then $(y,x)\in E(G)$, so

$$p_\alpha(Tx,Ty)\le c_\alpha p_{j(\alpha)}(x,y).$$

To show that $T$ is a $j$-$\widetilde{G}$-contraction, by the definition of $\widetilde{G}$, $E(\widetilde{G})=E(G)\cup E(G^{-1})$. Since $T$ is a $j$-$G$-contraction and a $j$-$G^{-1}$ contraction, $T$ is also a $j$-$\widetilde{G}$-contraction. #

To prove the main theorem, we need the following lemma and additional supporting results:

**Lemma 3.7.** *Let $X$ be a nonempty subset of a uniform space generated by a saturated collection of pseudo metrics $\mathscr{A}$, where $\mathscr{A}$ is indexed by a set $A$. Suppose $T:X\to X$ is a $j$-$G$-contraction with constants $c_\alpha$. Assume further that for every $\alpha\in A$, there exists $n_0\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ and all $x,y\in X$,*

$$p_{j^n(\alpha)}(x,y)\le p_{j^{n_0}(\alpha)}(x,y).$$

*Then for every $x\in X$ and $y\in[x]_{\widetilde{G}}$ there exists $r_\alpha(x,y)\ge 0$ such that*

$$p_{j^n(\alpha)}(x,y)\le\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)r_\alpha(x,y)$$

*For all $\alpha\in A$ and $n\in\mathbb{N}$.*

**Proof.** Let $x\in X$ and $y\in[x]_{\widetilde{G}}$. Then there is a path $\{x_i\}_{i=0}^N$ in $\widetilde{G}$ from $x$ to $y$ for some $N\in\mathbb{N}$ where $x_0=x$, $x_N=y$ and $(x_{i-1},x_i)\in E(\widetilde{G})$ for all $i=1,2,\ldots,N$.

Let $\alpha\in A$. Define $r_\alpha(x.y)=\sum_{i=1}^{N}p_{j^{n_0}(\alpha)}(x_{i-1},x_i)\ge 0$.

Since $T$ is a $j$-$G$-contraction, there is a constant $c_\alpha\in(0,1)$ such that

$$p_\alpha(T^nx_{i-1},T^nx_i)\le\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)p_{j^n(\alpha)}(x_{i-1},x_i)$$

For all $i=1,2,\ldots,N$. Then

$$\begin{aligned}p_\alpha(T^nx,T^ny)&\le\sum_{i=1}^{N}p_\alpha(T^nx_{i-1},T^nx_i)\\&\le\sum_{i=1}^{N}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)p_{j^n(\alpha)}(x_{i-1},x_i)\\&\le\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)\sum_{i=1}^{N}p_{j^{n_0}(\alpha)}(x_{i-1},x_i)\\&=\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)r_\alpha(x,y).\end{aligned}$$

#

**Theorem 3.8.** *A graph G is weakly connected if and only if for every $j$-$G$-contractions $T:X\to X$ such that $\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)$ converges, the sequences $(T^nx)$ and $(T^ny)$ are Cauchy equivalent for all $x,y\in X$.*

**Proof.** Suppose $G$ is weakly connected. Let $x\in X$. Then $[x]_{\widetilde{G}}=X$, so $Tx\in[x]_{\widetilde{G}}$.

First, to show that $(T^nx)$ is a Cauchy sequence, let $\alpha\in A$ and $\varepsilon>0$. By Lemma 3.7 and Definition 3.1, there exists $r_\alpha(x,Tx)>0$ such that

$$p_{j^n(\alpha)}(x,y)\le\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)r_\alpha(x,y).$$

Since $\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)$ converges, it is Cauchy. Then

$$\begin{aligned}p_\alpha(T^mx,T^nx)&\le\sum_{i=m}^{n-1}p_\alpha(T^ix,T^{i+1}x)\\&\le\sum_{i=m}^{n-1}\left(\prod_{k=0}^{i-1}c_{j^k(\alpha)}\right)r_\alpha(x,Tx)\\&<\varepsilon.\end{aligned}$$

By the same argument, $(T^ny)$ is also Cauchy.

Next, to show that $(T^nx)$ and $(T^ny)$ are Cauchy equivalent, let $\alpha\in A$. Since $y\in[x]_{\widetilde{G}}$, by Lemma 3.5,

$$\lim_{n\to\infty}p_\alpha(T^nx,T^ny)\le\lim_{n\to\infty}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)r_\alpha(x,y).$$

By the assumption, then $\lim_{n\to\infty}p_\alpha(T^nx,T^ny)=0$.

Hence, $(T^nx)$ and $(T^ny)$ are Cauchy equivalent.

Conversely, suppose that $G$ is not weakly connected. That is, $\widetilde{G}$ is not a connected graph. Then there exist $x_0$, $y_0\in X$ such that there is no path from $x_0$ to $y_0$; in other words, $y_0\notin[x_0]_{\widetilde{G}}$. Define a map $T:X\to X$ by

$$Tx = \begin{cases} x_0, & \text{if } x \in [x_0]_{\tilde{G}}, \\ y_0, & \text{if } x \notin [x_0]_{\tilde{G}}. \end{cases}$$

Then $(T^n x_0) = (x_0)$ and $(T^n y_0) = (y_0)$ are constant sequences with $x_0 \neq y_0$. Since $\mathscr{A}$ is saturated, there exists $\alpha_0 \in A$ such that $p_{\alpha_0}(x_0, y_0) \neq 0$, and hence, $(T^n x)$ and $(T^n y)$ are not Cauchy equivalent.

Finally, it remains to show that $T$ is a $j$-$G$-contraction by choosing $j$ to be the identity function on $A$. If $(x, y) \in E(G)$ then $y \in [x]_{\tilde{G}}$, so $(Tx, Ty)$ is either $(x_0, x_0)$ or $(y_0, y_0)$, both of which lie in $E(G)$. Therefore, $T$ preserves edges of $G$. Let $\alpha \in A$ and choose $c_\alpha = \frac{1}{3}$. If $(x, y) \in E(G)$, then

$$p_\alpha(Tx, Ty) = 0 \leq c_\alpha p_{j(\alpha)}(x, y).$$

Hence, $T$ is a $j$-$G$-contraction. #

**Theorem 3.9**. *Let $X$ be a nonempty subset of a uniform space generated by a saturated collection of pseudo metrics $\mathscr{A}$, where $\mathscr{A}$ is indexed by a set $A$. Suppose $T{:}X \to X$ is a $j$-$G$-contraction, and there is a point $x_0 \in X$ such that $Tx_0 \in [x_0]_{\tilde{G}}$.*

*Let $\tilde{G}_{x_0}$ be the connected component of $\tilde{G}$ that has $x_0$ as a vertex. Then $[x_0]_{\tilde{G}}$ is $T$-invariant, and $T|_{[x_0]_{\tilde{G}}}$ is a $j$-$\tilde{G}_{x_0}$-contraction.*

**Proof.** Let $x \in [x_0]_{\tilde{G}}$. Then there is a path $\{x_i\}_{i=0}^N$ in $\tilde{G}$ from $x_0$ to $x$, where $x_N = x$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for all $i = 1, 2, \ldots, N$. Since $T$ preserves edges of $\tilde{G}$, we have $(Tx_{i-1}, Tx_i) \in E(\tilde{G})$ for all $i = 1, 2, \ldots, N$. Consequently, $\{Tx_i\}_{i=0}^N$ is a path in $\tilde{G}$ from $Tx_0$ to $Tx$. Hence, $Tx \in [Tx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$.

To show that $T|_{[x_0]_{\tilde{G}}}$ is a $j$-$\tilde{G}_{x_0}$-contraction, let $(x, y) \in E(\tilde{G}_{x_0})$. Then there is a path $\{x_i\}_{i=0}^N$ in $\tilde{G}_{x_0}$ from $x_0$ to $y$ with $x_{N-1} = x$ and $x_N = y$, and each $(x_{i-1}, x_i) \in E(\tilde{G}_{x_0})$.

Since $(x_{N-1}, x_N) = (x, y) \in E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$, it follows that $(Tx_{N-1}, Tx_N) = (Tx, Ty) \in E(\tilde{G})$.

Moreover, since $Tx_0 \in [x_0]_{\tilde{G}}$ there is a path $\{y_i\}_{i=0}^M$ in $\tilde{G}_{x_0}$ from $x_0$ to $Tx_0$. Consequently, $T$ being a $j$-$\tilde{G}$-contraction implies that $\{Tx_i\}_{i=0}^N$ is a path in $\tilde{G}$ from $Tx_0$ to $Ty$. Then the sequence

$$(x_0 = y_0, y_1, \ldots, y_M = Tx_0, Tx_0, Tx_1, \ldots, Tx_{N-2}, Tx, Ty)$$

is a path in $\tilde{G}$ from $x_0$ to $Ty$. Hence $(Tx, Ty) \in E(\tilde{G}_{x_0})$, so $T|_{[x_0]_{\tilde{G}}}$ preserves edges in $\tilde{G}_{x_0}$.

Since $T$ is a $j$-$\tilde{G}$-contraction, for each $\alpha \in A$ there is $c_\alpha \in (0, 1)$ such that for every $(x, y) \in E(\tilde{G})$,

$$p_\alpha(Tx, Ty) \leq c_\alpha p_{j(\alpha)}(x, y).$$

If $(x, y) \in E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$, then

$$p_\alpha\left(T|_{[x_0]_{\tilde{G}}} x, T|_{[x_0]_{\tilde{G}}} y\right) = p_\alpha(Tx, Ty) \leq c_\alpha p_{j(\alpha)}(x, y).$$

#

Next, we will present the first main theorem, which utilizes certain properties of these spaces.

**Theorem 3.10.** (Main Theorem 1). *Let $X$ be a nonempty subset of a uniform space generated by a saturated collection of pseudo metrics $\mathscr{A}$, where $\mathscr{A}$ is indexed by a set A. Suppose $(X, \mathscr{A}, G)$ satisfies the following property:*

***Condition (*).*** *For every sequence $(x_n)$ in X, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

*Let $T{:}X \to X$ be a $j$-$G$-contraction, and define*

$$X_T = \{x \in X : (x, Tx) \in E(G)\}.$$

*If $x \in X_T$ and $\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1} c_{j^k(\alpha)}\right)$ converges then $T|_{[x]_{\tilde{G}}}$ is a Picard operator.*

**Proof.** Let $x \in X_T$. By Theorem3.7, the set $[x]_{\tilde{G}}$ is invariant under $T$, and $T|_{[x]_{\tilde{G}}}$ *is a $j$-$\tilde{G}_x$-contraction.* Since $\tilde{G}_x$ is weakly connected, Theorem 3.6, implies that for any $x, y \in [x]_{\tilde{G}}$, the sequences $(T^n x)$ and $(T^n y)$ are Cauchy equivalent.

Since $X$ is sequentially complete, there exists some $x_0 \in X$ such that $T^n x \to x_0$ and $T^n y \to x_0$ as $n \to \infty$. Moreover, since $(x, Tx) \in E(G)$, it follows that $(T^n x, T^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$. By Condition (*), there is a subsequence $(T^{n_k} x)$ of $(T^n x)$ such that $(T^{n_k} x, x_0) \in E(G)$ for every $k \in \mathbb{N}$. Consequently, $(T^{n_k+1} x, Tx_0) \in E(G)$ as well.

Hence, $\left(x, Tx, T^2x, \ldots, T^{n-1}x, x_0\right)$ forms a path in $\widetilde{G}$ from $x$ to $x_0$, so $x_0 \in [x]_{\widetilde{G}}$. Since $T$ is a $j$-$\widetilde{G}_x$-contraction, for all $\alpha \in A$ and $k \in \mathbb{N}$,

$$p_\alpha\left(T^{n_k+1}x, Tx_0\right) \le c_\alpha p_{j(\alpha)}\left(T^{n_k}x, x_0\right).$$

Since $T^nx \to x_0$, it follows that $T^{n_k+1}x \to Tx_0$. Since $\mathscr{A}$ is saturated, we get $Tx_0 = x_0$. Therefore, $x_0$ is a fixed point of $T|_{[x]_{\widetilde{G}}}$.

Suppose that $x, y$ are fixed points of $T|_{[x]_{\widetilde{G}}}$. By Lemma 3.7, for each $\alpha \in A$,

$$p_\alpha(x, y) = \lim_{n\to\infty} p_\alpha(T^nx, T^ny)$$
$$\le \lim_{n\to\infty}\left(\prod_{k=0}^{n-1} c_{j^k(\alpha)}\right) r_\alpha(x, y).$$

Therefore, $x = y$ since $\lim_{n\to\infty}\left(\prod_{k=0}^{n-1} c_{j^k(\alpha)}\right) = 0$, so $T|_{[x]_{\widetilde{G}}}$ has a unique fixed point.

Hence $T|_{[x]_{\widetilde{G}}}$ is a Picard operator. #

**Corollary 3.11.** *Let $T: X \to X$ be a j-G-contraction such that $\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1} c_{j^k(\alpha)}\right)$ converges and suppose $G$ is weakly connected. Then $T$ is a Picard operator.*

**Proof.** Since $G$ is weakly connected, we have $X_T = X$. Consequently, for every $x \in X$, $[x]_{\widetilde{G}} = X$. By Theorem 3.10, it follows that $T$ is a Picard operator. #

The following results focus on the map from the sequence spaces in Theorem 3.3.

**Theorem 3.12.** *Let $1 < p < \infty$ and $G$ be a graph with $V(G) = \ell_p$ and the edges of $G$ by*

$$E(G) = \left\{\left((x_m), (y_m)\right) : x_i \ge y_i \text{ for all } i \in \mathbb{N}\right\}$$

*Then $\left(\ell_p, \ell_p^*, G\right)$ satisfies **Condition (*)** from Theorem 3.10.*

**Proof.** Let $\left(x_n^k\right)_{k\in\mathbb{N}}$ be a sequence in $\ell_p$.

Suppose $\left(x_n^k\right)$ converges weakly to $\left(x_n^0\right)$ as $k \to \infty$, and that $\left(\left(x_n^k\right), \left(x_n^{k+1}\right)\right) \in E(G)$ for all $k \in \mathbb{N}$.

Then, for each $k$, and for every $\left(\alpha_n\right) \in \ell_q$, we have

$$\sum_{n=1}^{\infty} \alpha_n x_n^k \to \sum_{n=1}^{\infty} \alpha_n x_n^0 \text{ as } k \to \infty.$$

Since $\left(\left(x_n^k\right), \left(x_n^{k+1}\right)\right) \in E(G)$ for all $k \in \mathbb{N}$, it follows that for each $i \in \mathbb{N}$,

$$x_i^k = e_i^*\left(x_n^k\right) \ge e_i^*\left(x_n^{k+1}\right) = x_i^{k+1} \text{ for all } k \in \mathbb{N}.$$

Since $e_i \in \ell_q$ for all $i \in \mathbb{N}$, we also obtain

$$\sum_{n=1}^{\infty} e_i^*(e_i) \cdot x_n^k \to \sum_{n=1}^{\infty} e_i^*(e_i) \cdot x_n^0 \text{ as } k \to \infty$$

Hence $x_i^k \to x_i^0$ as $k \to \infty$, which implies $x_i^k \ge x_i^0$.

Therefore, $\left(x_n^k, x_n^0\right) \in E(G)$, which shows that $\left(\ell_p, \ell_p^*, G\right)$ *satisfies **Condition (*)** from Theorem 3.10.* #

Next, we introduce another important lemma used to prove that the mapping in Theorem 3.3 is a Picard operator on specific subsets with respect to the given graph.

**Lemma 3.13.** *Let $1 < p < \infty$ and let $f_1, f_2, \ldots, f_N$ be non-decreasing contraction functions with respective contraction constants $k_1, k_2, \ldots, k_N$. Let $c \in (0, 1)$. Suppose $T: \ell_p \to \ell_p$ is defined by*

$$T(x_m) = \left(f_1x_1, f_2x_2, \ldots, f_Nx_N, cx_{N+1}, cx_{N+2}, \ldots\right).$$

*Let $G$ be a graph with $V(G) = \ell_p$ and*

$$E(G) = \left\{\left((x_m), (y_m)\right) : x_i \ge y_i \text{ for all } i \in \mathbb{N}\right\}.$$

*Define*

$$\left(\ell_p\right)_T = \left\{(x_m) \in \ell_p : \left((x_m), T(x_m)\right) \in E(G)\right\}.$$

*Then the following statements hold:*

*(1) If each $f_i$ has a fixed point, then $\left(\ell_p\right)_T \ne \varnothing$.*

*(2) $\left[(x_m)\right]_{\widetilde{G}}$ is a closed set.*

**Proof.** Assume $f_1, f_2, \ldots, f_N$ each admit a fixed point. That is, for each $i = 1, 2, \ldots, N$ there is $x_i$ such that $f_ix_i = x_i$. Then

$$T\left((x_m)\right) = \left(x_1, x_2, \ldots, x_N, cx_{N+1}, cx_{N+2}, \ldots\right).$$

Hence, $\left((x_m), T(x_m)\right) \in E(G)$, so $\left(\ell_p\right)_T \ne \varnothing$.

We next show that $\left[(x_m)\right]_{\widetilde{G}}$ is closed.

Let $\left((w_m^\alpha)\right)_{\alpha\in\Lambda}$ be a net in $\left[(x_m)\right]_{\widetilde{G}}$, and let $(y_m) \in \ell_p$. Suppose

$$\left(w_m^\alpha\right) \xrightarrow{w} (y_m) \text{ as } \alpha \to \infty.$$

Since each $\left(w_m^\alpha\right)$ lies in $\left[(x_m)\right]_{\widetilde{G}}$, we have

$$w_i^\alpha \ge x_i \text{ for all } i = 1, \ldots, N.$$

Since $e_i^* \in \ell_p^*$ for each $i \in \mathbb{N}$, and $\left(w_m^\alpha\right)$ converges weakly to $(y_m)$, it follows that

$$w_i^\alpha = e_i^*\left(w_m^\alpha\right) \to e_i^*\left(y_m\right) = y_i \text{ for all } i \in \mathbb{N}.$$

By the comparison theorem, we get $y_i \ge x_i$ for all $i = 1, \ldots, N$. Therefore, $(y_m) \in \left[(x_m)\right]_{\widetilde{G}}$ which proves that $\left[(x_m)\right]_{\widetilde{G}}$ is closed. #

The next theorem shows that the map in Theorem 3.3 is a Picard operator on specific subsets with respect to the given graph.

**Theorem 3.14.** *Let $1<p<\infty$ and let $f_1, f_2, \ldots, f_N$ be non-decreasing contraction functions with respective contraction constants $k_1, k_2, \ldots, k_N$. Let $c\in(0,1)$. Suppose $T:\ell_p\to\ell_p$ is defined by*

$$T(x_m)=(f_1x_1, f_2x_2, \ldots, f_Nx_N, cx_{N+1}, cx_{N+2}, \ldots).$$

*Let $G$ be a graph with $V(G)=\ell_p$ and*

$$E(G)=\{((x_m),(y_m)): x_i\geq y_i \text{ for all } i\in\mathbb{N}\}.$$

*Define*

$$(\ell_p)_T=\{(x_m)\in\ell_p:((x_m),T(x_m))\in E(G)\}.$$

*Then there exists $(x_m)\in(\ell_p)_T$ such that $T|_{[(x_m)]_{\tilde{G}}}$ is a Picard operator.*

**Proof.** By Lemma 3.13 (1), we have $(\ell_p)_T\neq\varnothing$. That is, there exists $(x_m)\in(\ell_p)_T$. By Theorem 3.2, $T$ is a $j$-$G$-contraction with the constant $K\in(0,1)$ on $\ell_p$ endowed with the weak topology. By Theorem 3.12, $(\ell_p,\ell_p^*,G)$ satisfies **Condition (*).** Using the main Theorem 3.10, since $K$ does not depended on $(\alpha_m)\in\ell_p^*$, $\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)=\sum_{n=1}^{\infty}K^n$ converges, so we have $T|_{[(x_m)]_{\tilde{G}}}$ is a Picard operator. #

Next, we present the second main theorem, which relies on certain properties of the maps.

**Theorem 3.15. (Main Theorem 2).** *Let $X$ be a nonempty subset of a uniform space generated by a saturated collection of pseudo metrics $\mathscr{A}$, where $\mathscr{A}$ is indexed by a set $A$. Suppose $T:X\to X$ is a $j$-$G$-contraction that is also orbitally $G$-continuous. If $x\in X_T$, $\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)$ converges, and $[x]_{\tilde{G}}$ is closed, then $T|_{[x]_{\tilde{G}}}$ is a Picard operator.*

**Proof.** Let $x\in X_T$, and assume $[x]_{\tilde{G}}$ is closed. Pick any $y\in[x]_{\tilde{G}}$. We aim to show that the sequence $(T^ny)$ converges to some $x_0\in[x]_{\tilde{G}}$.

Since $x\in X_T$, we have $(x,Tx)\in E(G)$. By Theorem 3.9, the set $[x]_{\tilde{G}}$ is $T$-invariant, and $T|_{[x]_{\tilde{G}}}$ is a $j$-$G$-contraction.

Since $\tilde{G}_x$ is a connected graph, Theorem 3.9 implies that for any $j$-$\tilde{G}_x$-contraction and all $u,x\in[x]_{\tilde{G}}$, the sequences $(T^nu)$ and $(T^nv)$ are Cauchy equivalent. In particular, $(T^nx)$ and $(T^ny)$ are both Cauchy sequences.

Since $X$ is sequentially complete, there exists $x_0\in X$ such that $(T^nx)$ and $(T^ny)$ converge to $x_0$.

Since $T^ny\in[x]_{\tilde{G}}$ for all $n\in\mathbb{N}$, and $[x]_{\tilde{G}}$ is closed, it follows that $x_0\in[x]_{\tilde{G}}$.

It remains to show that $x_0$ is a fixed point of $T|_{[x]_{\tilde{G}}}$.

Since $(x,Tx)\in E(G)$ ,we have $(T^nx,T^{n+1}x)\in E(G)$ for each $n\in\mathbb{N}$.

By the orbitally $G$-continuous property of $T$, $T(T^nx)\to Tx_0$ but also $T^{n+1}x\to x_0$. Since $\mathscr{A}$ is saturated, convergence under all pseudo metrics in $\mathscr{A}$ forces $Tx_0=x_0$.

Suppose that $x$, $y$ are fixed points of $T|_{[x]_{\tilde{G}}}$. By Lemma 3.7, for each $\alpha\in A$,

$$p_\alpha(x,y)=\lim_{n\to\infty}p_\alpha(T^nx,T^ny)$$
$$\leq\lim_{n\to\infty}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)r_\alpha(x,y).$$

Therefore, $x=y$ since $\lim_{n\to\infty}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)=0$, so $T|_{[x]_{\tilde{G}}}$ has a unique fixed point.

Therefore, $T|_{[x]_{\tilde{G}}}$ is a Picard operator. #

We will now show that the map $T$ defined in Theorem 3.3 is a $j$-$G$-contraction, and moreover, that $T$ is orbitally continuous, as stated in the following theorem.

**Theorem 3.16.** *Let $1<p<\infty$ and let $f_1, f_2, \ldots, f_N$ be non-decreasing contraction functions with respective contraction constants $k_1, k_2, \ldots, k_N$. Let $c\in(0,1)$. Suppose $T:\ell_p\to\ell_p$ is defined by*

$$T(x_m)=(f_1x_1, f_2x_2, \ldots, f_Nx_N, cx_{N+1}, cx_{N+2}, \ldots).$$

*Let $G$ be a graph with $V(G)=\ell_p$ and*

$$E(G)=\{((x_m),(y_m)): x_i\geq y_i \text{ for all } i\in\mathbb{N}\}.$$

*Then $T$ is $G$-orbitally continuous.*

**Proof.** Let $(x_m),(y_m)\in\ell_p$ and let $(k_a)$ be a sequence of positive integers, with $a\in\mathbb{N}$.

Suppose

$$T^{k_a}((x_m)) \xrightarrow{w} (y_m) \text{ as } a\to\infty,$$

and $\left(T^{k_a}((x_m)), T^{k_{a+1}}((x_m))\right) \in E(G)$ for all $a\in\mathbb{N}$. Then for every $(\alpha_m)\in\ell_q$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\sum_{n=1}^{\infty}\alpha_n e_n^*\left(T^{k_a}(x_m)\right)\to\sum_{n=1}^{\infty}\alpha_n y_n \text{ as } a\to\infty.$$

Then

$$\sum_{n=1}^{\infty}\alpha_n e_n^*\left(T^{k_a}(x_m)\right)=\sum_{i=1}^{N}\alpha_i\left(f_i^{k_a}x_i\right)+\sum_{i=N+1}^{\infty}\alpha_i\left(c^{k_a}x_i\right).$$

Let $(\beta_m)\in\ell_q$We have:

$$\sum_{n=1}^{\infty}\beta_n e_n^*\left(T^{k_a+1}(x_m)\right)=\sum_{i=1}^{N}\beta_i\left(f_i^{k_a+1}x_i\right)+\sum_{i=N+1}^{\infty}\beta_i\left(c^{k_a+1}x_i\right),$$

and

$$\sum_{n=1}^{\infty}\beta_n e_n^*\left(T(y_m)\right)=\sum_{i=1}^{N}\beta_i\left(f_i y_i\right)+\sum_{i=N+1}^{\infty}\beta_i\left(cy_i\right).$$

Define

$$(\alpha_m)=\left(\beta_1,\beta_2,\dots,\beta_N,c\beta_{N+1},c\beta_{N+2},\dots\right)\in\ell_q.$$

Since $e_i^*$ is continuous for every $i\in\mathbb{N}$, each $f_i$ is continuous for $i=1,2,\dots,N$ and $f_i^{k_a}x_i\to y_i$, it follows that $\alpha_i f_i\left(f_i^{k_a}x_i\right)\to\alpha_i f_i(y_i)$ as $a\to\infty$.

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\beta_n e_n^*\left(T^{k_a+1}(x_m)\right)&=\sum_{i=1}^{N}\beta_i\left(f_i^{k_a+1}x_i\right)+\sum_{i=N+1}^{\infty}\beta_i\left(c^{k_a+1}x_i\right)\\&=\sum_{i=1}^{N}\alpha_i\left(f_i^{k_a+1}x_i\right)+\sum_{i=N+1}^{\infty}\alpha_i\left(c^{k_a}x_i\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\beta_n e_n^*\left(T(y_m)\right)&=\sum_{i=1}^{N}\beta_i\left(f_i y_i\right)+\sum_{i=N+1}^{\infty}\beta_i\left(cy_i\right)\\&=\sum_{i=1}^{N}\alpha_i\left(f_i y_i\right)+\sum_{i=N+1}^{\infty}\alpha_i y_i.\end{aligned}$$

Since $\sum_{i=1}^{N}\alpha_i\left(f_i^{k_a+1}x_i\right)+\sum_{i=N+1}^{\infty}\alpha_i\left(c^{k_a}x_i\right)$ converges to $\sum_{i=1}^{N}\alpha_i f_i(y_i)+\sum_{i=N+1}^{\infty}\alpha_i y_i$ as $a\to\infty$, it follows that $\sum_{n=1}^{\infty}\beta_n e_n^*\left(T^{k_a+1}(x_m)\right)\to\sum_{n=1}^{\infty}\beta_n e_n^*\left(T(y_m)\right)$ as $a\to\infty$.

Therefore,

$$T\left(T^{k_a}(x_m)\right)\xrightarrow{w}T(y_m) \text{ as } a\to\infty. \qquad \#$$

Next, we will show that the map defined by Theorem 3.3 satisfies all the conditions of the main Theorem 3.15. This is illustrated by the following theorem:

**Theorem 3.17.** *Let* $1<p<\infty$ *and let* $f_1, f_2, \dots, f_N$ *be non-decreasing contraction functions with respective contraction constants* $k_1, k_2, \dots, k_N$. *Let* $c\in(0,1)$ . *Suppose* $T:\ell_p\to\ell_p$ *is defined by*

$$T(x_m)=\left(f_1x_1, f_2x_2, \dots, f_Nx_N, cx_{N+1}, cx_{N+2}, \dots\right).$$

*Let* $G$ *be a graph with* $V(G)=\ell_p$ *and*

$$E(G)=\left\{\left((x_m),(y_m)\right): x_i\ge y_i \text{ for all } i\in\mathbb{N}\right\}.$$

*Then there exists* $(x_m)\in(\ell_p)_T$ *such that* $T|_{[(x_m)]_{\tilde{G}}}$ *is a Picard operator.*

**Proof.** By Lemma 3.11 (1), we have $(\ell_p)_T\neq\varnothing$ . That is, there exists $(x_m)\in(\ell_p)_T$. By Theorem 3.3, $T$ is a $j$-$G$-contraction with the constant $K\in(0,1)$ on $\ell_p$ endowed with the weak topology. By Theorem 3.12, $T$ is orbitally continuous. Using the main Theorem 3.15, there exists $(x_m)\in(\ell_p)_T$ and since $K$ does not depended on $(\alpha_m)\in\ell_p^*$, $\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1}c_{j^k(\alpha)}\right)=\sum_{n=1}^{\infty}K^n$ *converges. Then* $T|_{[(x_m)]_{\tilde{G}}}$ is a Picard operator. #

By the result of Theorem 3.14 and 3.17, we conclude that $T|_{[(x_m)]_{\tilde{G}}}$ is a Picard operator where $T$ is the map on $\ell_p$. In particular, mappings constructed from Theorem 3.3 remain Picard operators on specific subsets with respect to the given graph.

**Example 3.18.** *Let* $T:\ell_2\to\ell_2$ *is defined by*

$$T(x_m)=\left(f_1x_1, f_2x_2, f_3x_3, \frac{1}{3}x_4, \frac{1}{3}x_5, \dots\right),$$

*where* $f_1(x)=f_2(x)=\dfrac{\sin x+2x+1}{5}$ *and* $f_3(x)=\dfrac{2}{3}x.$

*Let* $G$ *be a graph with* $V(G)=\ell_2$ *and*

$$E(G)=\left\{\left((x_m),(y_m)\right): x_i\ge y_i \text{ for all } i\in\mathbb{N}\right\}.$$

*Then* $T|_{[(0)]_{\tilde{G}}}$ *is Picard operator.*

**Example 3.19.** *Let* $T:\ell_2\to\ell_2$ *is defined by*

$$T(x_m)=\left(f_1x_1, f_2x_2, f_3x_3, \frac{1}{3}x_4, \frac{1}{3}x_5, \dots\right),$$

*where* $f_1(x)=f_2(x)=\dfrac{\sin x+x}{2}$ *and* $f_3(x)=\dfrac{2}{3}x.$

*Let $G$ be a graph with $V(G)=\ell_2$ and*
$E(G)=\{((x_m),(y_m)) : x_1=y_1,\ x_2=y_2,\ x_i \geq y_i \text{ for all } i \geq 3\}$.
*Then $(\ell_2, \ell_2^*, G)$ satisfies **Condition (*)** from Theorem 3.10, and hence $T|_{[(0)]_{\tilde{G}}}$ is a Picard operator.*

**Proof.** By Theorem 3.12, $(\ell_2, \ell_2^*, G)$ satisfies **Condition (*)**. Since $f_1(0)=0=f_2(0)$ and $f_1, f_2$ are nondecreasing, $T$ and $G$ satisfy all the conditions in Lemma 3.13. Therefore $T|_{[(0)]_{\tilde{G}}}$ is a Picard operator for the same reasons as in the proof of Theorem 3.14. #

## 4. CONCLUSION

This work introduces the concept of $j$-$G$-contractions on uniform generated by a saturated collection of pseudo metrics and establishes criteria for a map on $\ell_p$ endowed with the weak topology. Two main theorems are presented. The first theorem provides conditions on the space and a graph referred to as **condition (*)**, for the map to be a Picard operator on specific subsets with respect to the given graph. The second theorem establishes conditions on the map, specifically orbital $G$-continuity. Finally, we demonstrate that our examples satisfy all conditions in main theorems, thereby confirming that they are Picard operators with respect to $[x]_{\tilde{G}}$. Furthermore, in future work, we will explore the topological properties of the convergence set of $j$-$G$-contractions. Since the graph satisfying the conditions of $j$-$G$-contractions can potentially be modified to enlarge the convergence set, this provides a promising direction for studying their topological properties.

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